SOFT GLUON FACTORIZATION AT TWO LOOPS IN FULL COLOR

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based on Dixon, Herrmann, Yan, Zhu. 1912.09370

When one or more external particles are unresolved, gauge theory amplitudes factorize into lower-point amplitudes multiplied by a universal emission factor.

$$M(\{p_i\};\ X) \sim M(\{p_i\}) \times F(X;\ \{p_i\})$$

Soft factor: X contains a single soft gluon

The emission factor is usually simple and nice, can be obtained in alternative ways more efficiently.

It contains rich information about the infrared divergence and analytic properties of the amplitudes and helps resolving conceptual issues with factorization violation.

From an effective theory point of view, soft emission factor can be computed from Wilson-line matrix element.

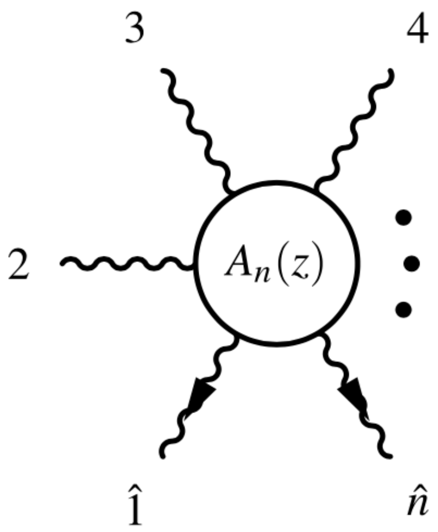
We use this method to obtain the two-loop emission factor with a single soft gluon for generic multi-point scattering amplitudes.

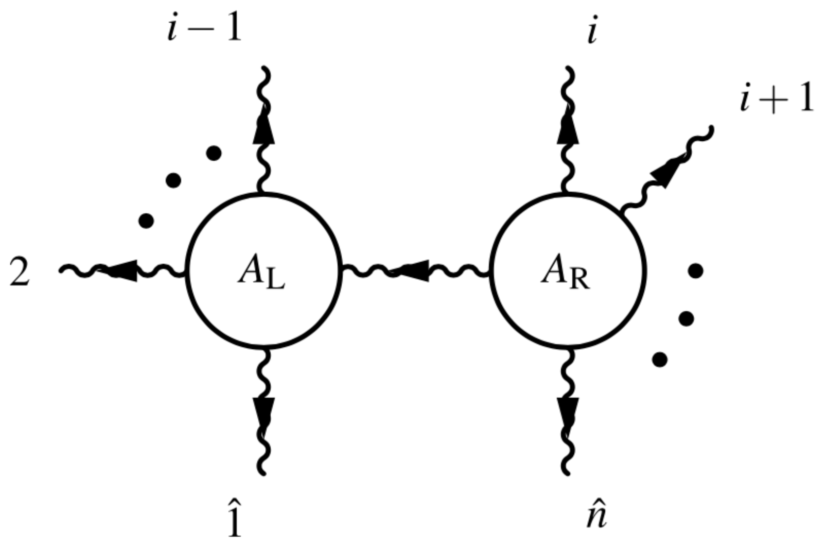
FACTORIZATION

Factorization of scattering amplitudes on multi-particle poles

Color-ordered amplitudes can have poles when region momenta $P_{i,j} \coloneqq p_i + p_{i+1} + \cdots + p_j$ go on shell. At leading power as $P_{i,j}^2 \to 0$, they factorize into product of lower-point amplitudes.

$$A_{tree}(1,\dots,n) \sim \sum_{\lambda} A_{tree}\left(i,\dots,j,P^{\lambda}\right)\frac{1}{P_{i,j}^2}\,A_{tree}\left(P^{\lambda},j+1,\dots,i-1\right)$$

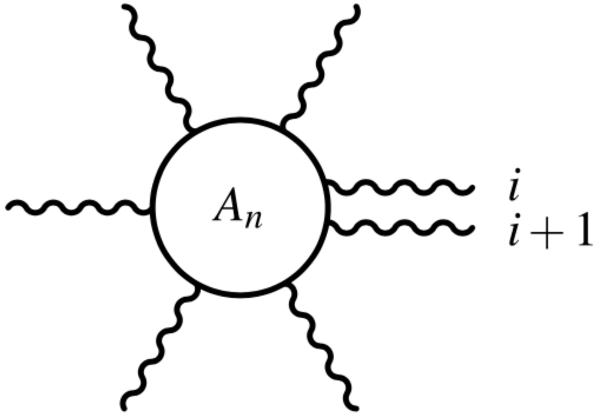




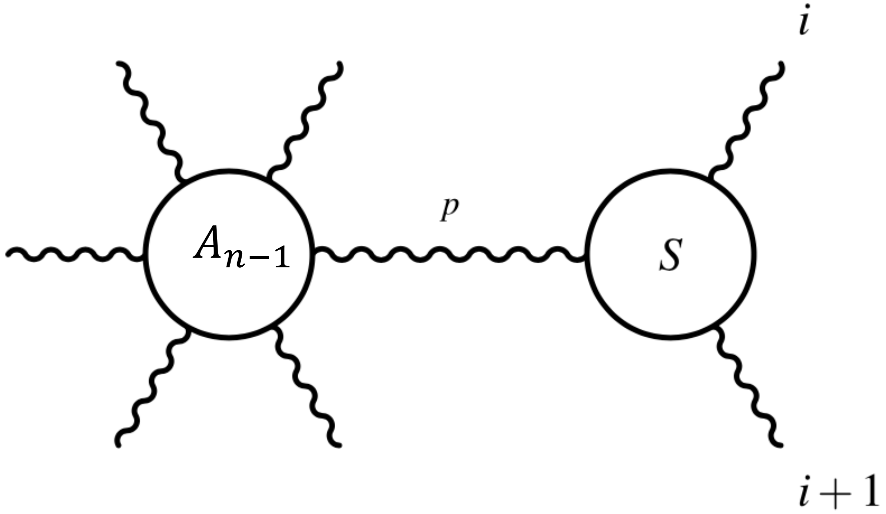
Collinear Factorization

On the two-particle pole $P_{i,i+1} = 0$, two adjacent external momenta are collinear.

$$A_n(\dots, i, i+1, \dots) \xrightarrow{i \parallel i+1} \sum_{\lambda} Split_{-\lambda}(z;\ i, i+1)\ A_{n-1}(\dots, P^{\lambda}, \dots)$$



$$\lambda_i = \sqrt{z}\,\lambda_P, \qquad \lambda_{i+1} = \sqrt{1-z}\,\lambda_P$$

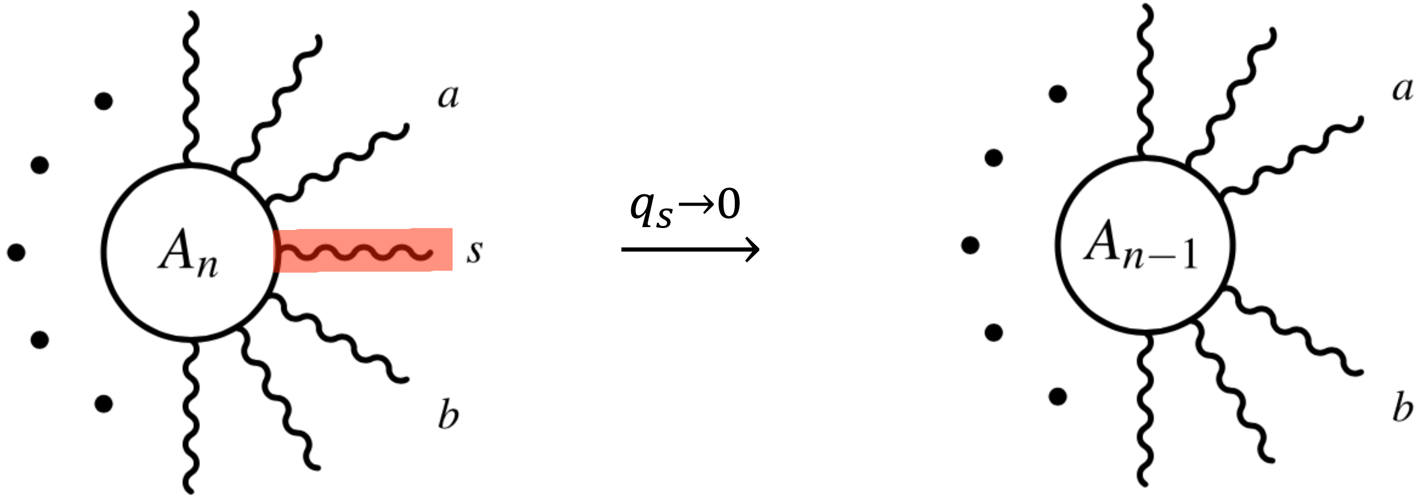


$$Split_{-}^{tree}(z, a^-, b^-) = 0\,, \qquad Split_{-}^{tree}(z, a^+, b^-) = -\frac{z^2}{\sqrt{z(1-z)}[ab]}\,,$$

$$Split_{-}^{tree}(z, a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\langle ab\rangle}, \quad Split_{-}^{tree}(z, a^-, b^+) = -\frac{(1-z)^2}{\sqrt{z(1-z)}[ab]}\,.$$

Independent of non-collinear external legs

Soft Factorization



$\times\ Soft^{tree}(s^{\pm}; a, b\,)$

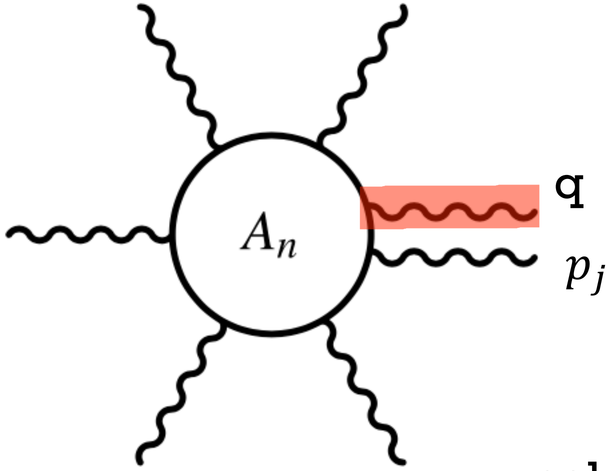
$$Soft^{tree}(s^{+}; a, b\,) = \frac{\langle a\ b\rangle}{\langle a\ q\rangle\langle q\ b\rangle} \qquad Soft^{tree}(s^{-}; a, b\,) = -\frac{[a\ b]}{[a\ q][q\ b]}$$

depend on the momenta and helicities of the soft gluon and the momenta of the color-ordered neighbors a and b, independent of the helicities and particle types of the neighboring legs

(Tree-level) soft emission factor is a sum of gauge invariant dipoles

$$\left|M_{n+1}^{(0)}\right\rangle = S_{\pm}^{(0)}(q;\ \{\,p_i\})\ \left|\ M_n^{(0)}\right\rangle$$

Soft-collinear Factorization



q collinear to p_j , {p_i} generic

$$\sqrt{z_q} \sim \frac{\langle i\, q\, \rangle}{\langle\, i\, j \rangle}\, , \forall\, i \neq j$$

$$S_a\,(q^+, \{p_i\}) \;\rightarrow -\, T_j \frac{\langle ij \rangle}{\langle iq\, \rangle \langle qj \rangle} \qquad \sum_{i\neq j}(T_i - T_j) = -2\; T_j$$

Same limit applies to q being wide angle, {p_i} collinear

color coherence : when certain hard partons are collinear, the soft gluon cannot resolve the angle between them and sees the total color charge.
The emission is dipole-like.

Agees with the soft limit of splitting function

$$Sp_-\big(z_q, q^+, j^+\,\big) \rightarrow -T_j \frac{1}{\sqrt{z_q}} \frac{1}{\langle\, qj \rangle}$$

Generalization to higher loop order

All-order factorization formula $\qquad |M_{n+1}\rangle = S_{\pm}(q;\,\{\,p_i\})\,|\;M_n\rangle$

$$M_n \;:= \sum_i a^i M_n^{(i)}, \qquad \mathrm{S}_{\pm} \;:= \sum_i a^i S_{\pm}{}^{(i)}$$

At two loops, the dipole soft factor has been known for collision processes with two hard colored external states ; as well as soft emission in the (planar) large Nc limit.

Evidence that dipole emission formula needs to be modified, for multi-parton scattering processes

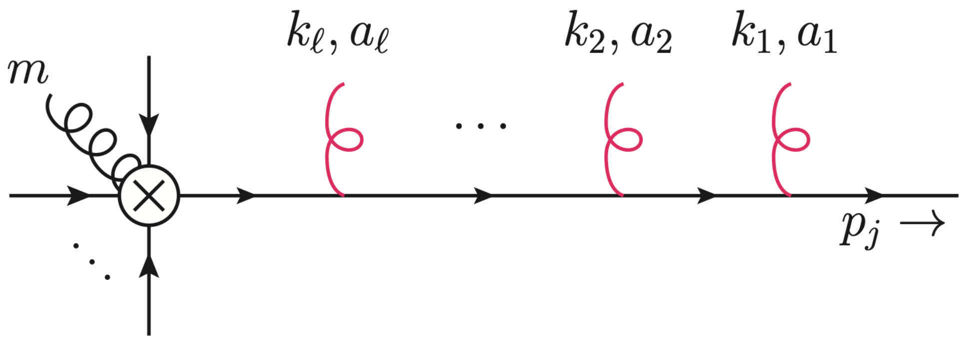
Quadruple correlation in three loop soft anomalous dimension
Almelid, Duhr, Gardi, 1507.00047

Collinear factorization violation with initial-state collinear splitting.
Catani, de Florian, Rodrigo 1112.4405

METHODS

Effective theory : soft gluon emissions from Wilson lines

(e.g. HQET, SCET)



$$Y_j(x) := \mathrm{P}\exp ig \int_0^\infty n_j \cdot A^a T^a \left(x + s\, n_j\right) ds$$

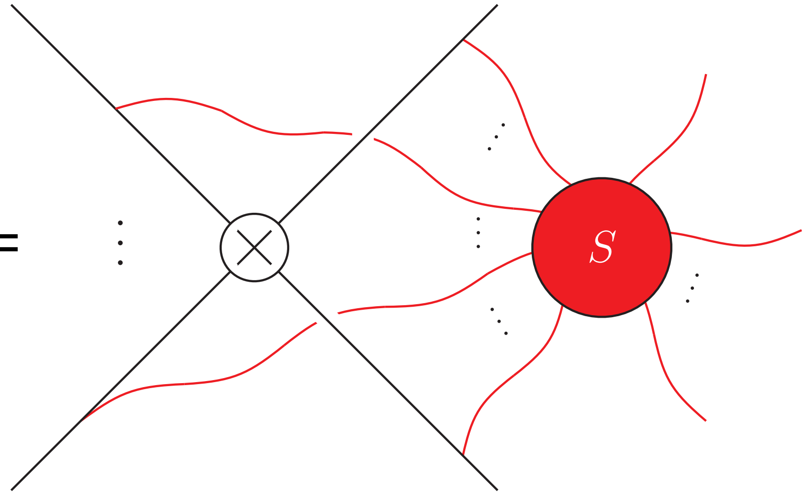
$$\langle\, q; a;\ \pm |\, Y_1 \cdots Y_n |\, 0 \rangle = S_a^{\pm}\left(q, \{\, n_i\}\right) \langle\, 0 |\, Y_1 \cdots Y_n |\, 0 \rangle$$

Represent classical sources traveling in direction $\vec{n}_j := \frac{\vec{p}_j}{p^0}$

In pure dimReg $\langle\, 0 |\, Y_1 \cdots Y_n |\, 0 \rangle$ vanishes for lightlike Wilson lines

Multiple soft-gluon emission factor from n-point scattering amplitude

$$S\,(X_s, \{n_i\}) =$$



$$= \langle X_s | \overline{Y}_1 \overline{Y}_2 \cdots Y_n | 0 \rangle$$

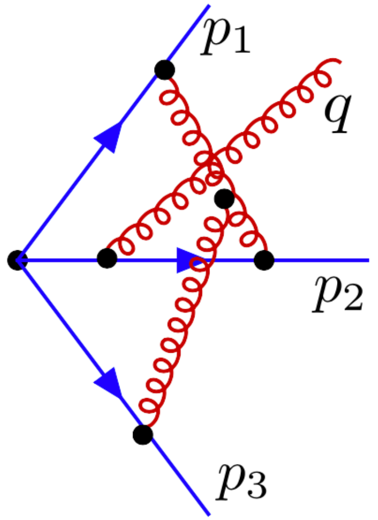
IR regularization for Wilson-line matrix elements

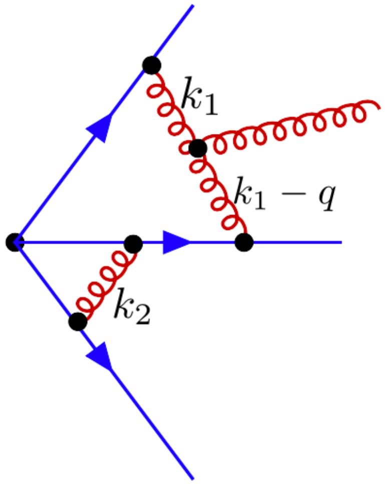
Integrate along closed contour (cusp singularities of light-like Wilson loop)
Offsheness (matter-dependent cusp anomalous dimensions, soft anomalous dimensions)

Light-like semi-infinite Wilson lines, no need to introduce offshellness.
IR divergence regulated by Dim-Reg.

$$x_{ij} := \frac{(-s_{ij})}{(-s_{iq})(-s_{qj})}$$

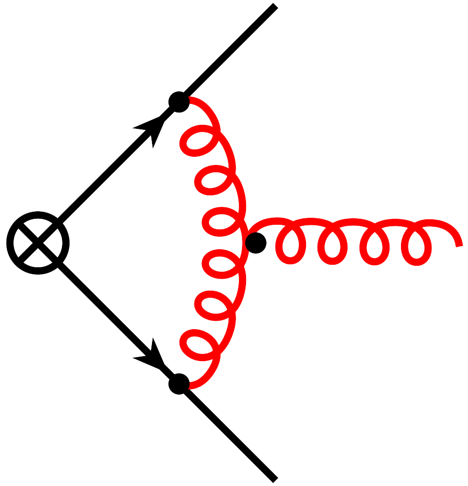
Vanishing diagrams





One loop emission factor

At one-loop, soft gluon can couple to two Wilson lines.
Emission factor is dipole like.



$$S_{a,+}^{(1)}(q) = \frac{1}{2}\sum_{i\neq j} V_{ij}^{q} f_{abc} T_i^b T_j^c \, C_1(\epsilon)\frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$

$$V_{ij}^{q} := \left[\frac{\mu^2(-s_{ij})}{(-s_{iq})(-s_{qj})}\right]^{\epsilon}, \quad s_{ab} = \langle ab\rangle[ba] = -|p_a\cdot p_b|e^{-i\pi\lambda_{ab}}$$

$$C_1(\epsilon) = -\frac{1}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon^2} - \frac{\zeta_2}{2} + \epsilon\frac{7}{3}\zeta_3 + \ldots$$

λ_{ab}=1 both incoming/outgoing
λ_{ab}=0, otherwise

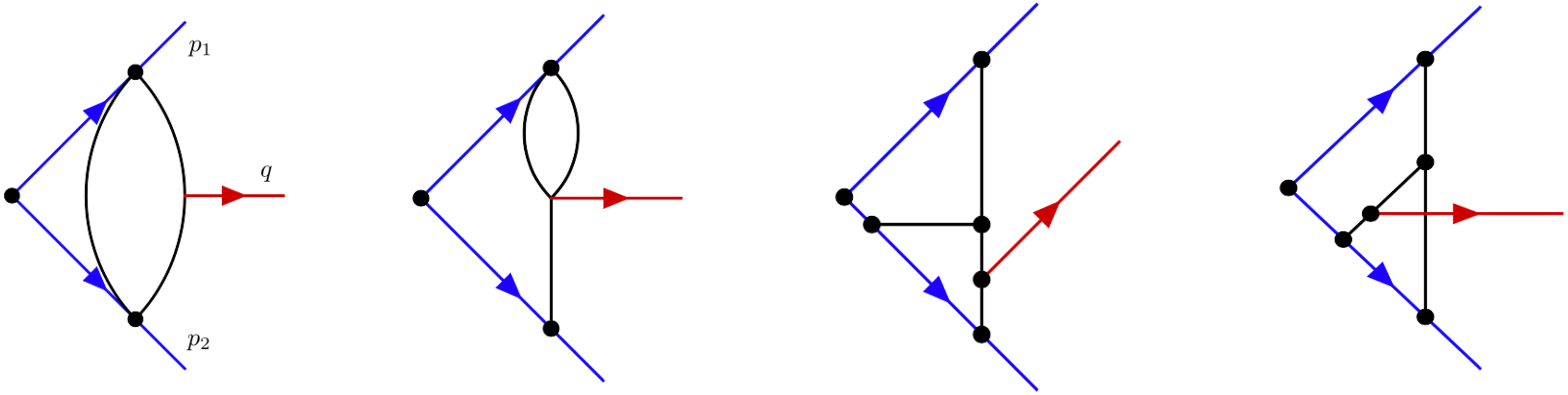
Uniform transcendental weight

TWO-LOOP SOFT FACTOR IN AN EUCLIDEAN REGION

Two-loop dipole

Two hard external partons $$S^{(2)}_{a,+}(q) = \left(V^{q}_{ij}\right)^2 f_{abc} T^b_i T^c_j \, C_2(\epsilon) \frac{\langle ij \rangle}{\langle iq \rangle \langle qj \rangle}$$

Master integrals



I1 I2 I3 I4

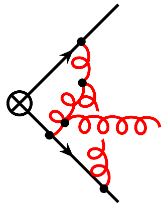
1309.4941

$$C_2(\epsilon) = C_A^2 \, B_1 \; + C_A N_s \, B_2 \; + C_A \, N_f \, B_3$$

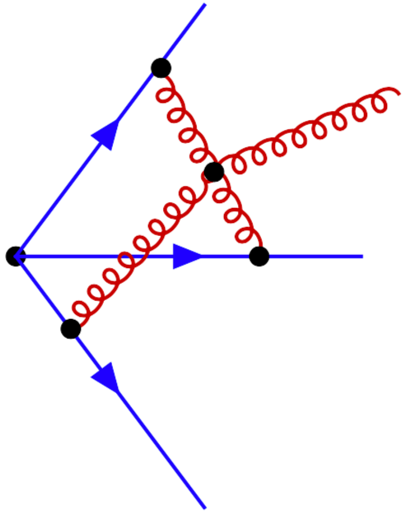
$B_{1,2,3}$: linear combinations of $I_{1,2,3}$

Vanish upon taking color trace

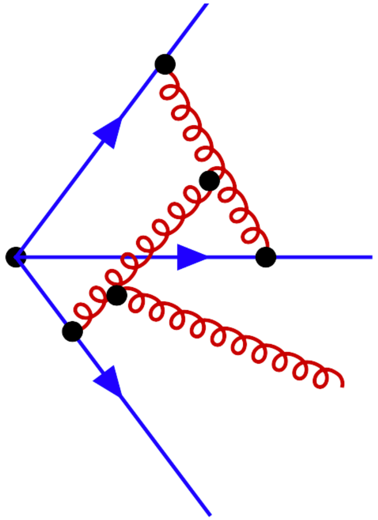
Two-loop tripole

The contribution from non-planar dipole emission diagrams 

must cancel with non-vanishing diagrams with three parton correlations



A



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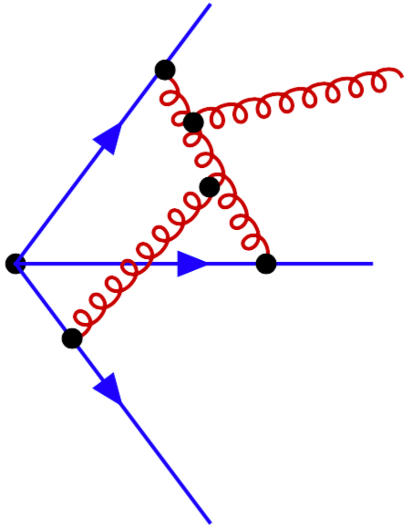

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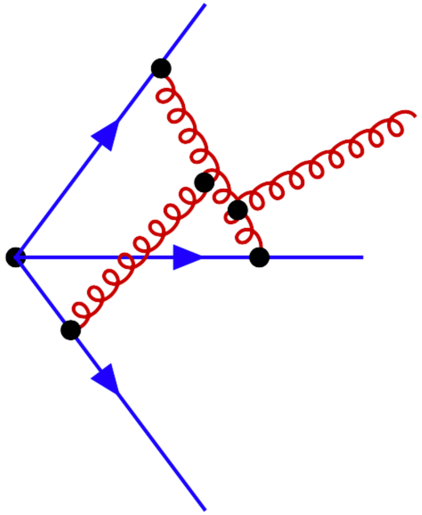
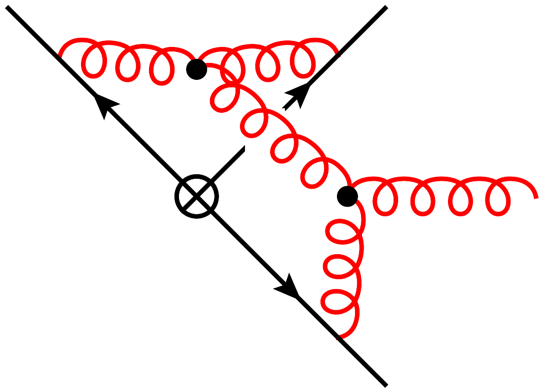


D

Definition of the integral family:

Topo(1,1,1,1,1,1,1,0,0,0,0)



$$D_1 = k_1^2, \quad D_2 = k_2^2, \quad D_3 = (k_1 - p_1)^2,$$
$$D_4 = (k_1 + k_2 - q)^2, \quad D_5 = k_1 \cdot p_1, \quad D_6 = k_2 \cdot p_2, \quad D_7 = (k_1 + k_2 - p_q) \cdot p_3,$$
$$D_8 = k_1, . p_2 \quad D_9 = k_1 . k_2, D_{10} = k_2 . p_1, D_{11} = k_1 . p_3$$

External kinematics $\quad \frac{(-s_{ij})}{(-s_{iq})(-s_{qj})} := 1, \quad \frac{s_{ik} s_{qj}}{s_{ij} s_{qk}} := u, \quad \frac{s_{jk} s_{iq}}{s_{ij} s_{qk}} := v.$

8 Master integrals $\quad d\,\vec{f} = d\,A\,(u, v)\,\vec{f}$

Differential equation contains singularities at $u = 0, v = 0,\ \Delta := 1 - 2\,u - 2\,v + (u - v)^2 = 0$

Switch to variables

$$z_k^{ij} := \frac{\langle iq\rangle\langle kj\rangle}{\langle ij\rangle\langle kq\rangle}\ , \qquad \bar{z}_k^{ij} := \frac{[iq][kj]}{[ij][kq]}$$

$$u = (1 - z_k^{ij})(\,1 - \ \bar{z}_k^{ij}), \qquad v = z_k^{ij}\ \bar{z}_k^{ij}.$$

$$\sqrt{\Delta} = z - \bar{z} = 4i\frac{\epsilon\big(p_i, p_j, p_k, q\big)}{s_{ij}\ s_{kq}}$$

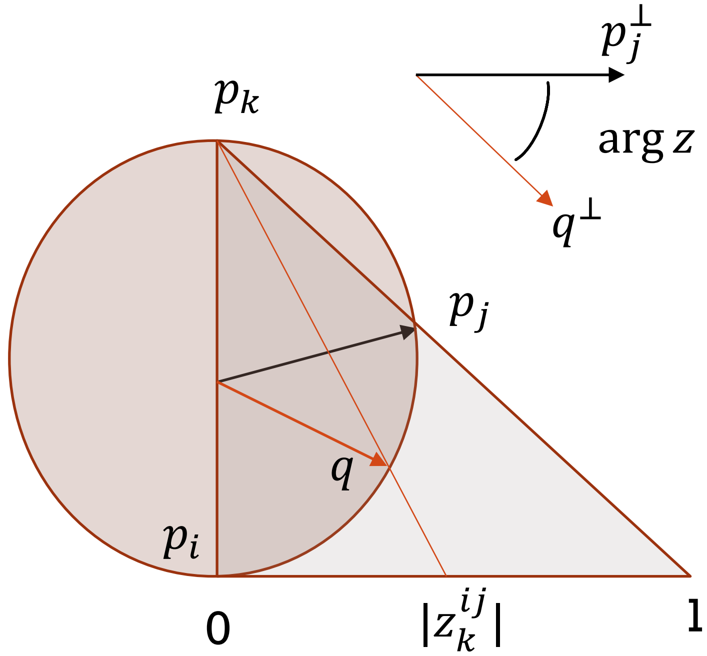
DE can be brought into canonical form with rational letters

$$d\ \vec{g} = \epsilon \textstyle\sum_i d\ln\alpha_i\ (z,\bar{z})\ \ m_i\ \vec{g}, \qquad \alpha = \{z, 1 - z, \bar{z}, 1 - \bar{z}\ , z - \bar{z}\}$$

In Euclidean region (i.e. $\frac{(-s_{ij})}{(-s_{iq})(-s_{qj})}$ >0), the integrals are real and analytic.

Only contains logarithms in $z\ \bar{z},\ (1 - z)(1 - \bar{z}),$ no $\ln(z - \bar{z})$

Stereographic projection



RESULTS AND CROSS CHECK

Summing over dipole and tripole contributions, using color conservation, non planar dipole contribution cancels out.

$$\boldsymbol{S}_a^{+,(2)} = \frac{1}{2}\sum_{i\neq j} \boldsymbol{S}_{a,ij}^{+,(2)} - \frac{1}{4}\sum_{i\neq k\neq j} \boldsymbol{S}_{a,ikj}^{+,(2)}$$

$$S_{a,ijk}^{+(2)} = V_{q,ij}^{2}\, f_{aa_k b}\, f_{ba_i a_j}\, T_i^{a_i}\, T_j^{a_j}\, T_k^{a_k} \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F\left(z_k^{ij},\epsilon\right) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F\left(z_k^{ji},\epsilon\right) \right]$$

The symbol level cross check :
two-loop five-point amplitudes in N=4 SYM

$$s_{12} = x[1];\ s_{23} = x[2]\,x[4];$$

$$s_{34} = x[1]\left(x[4] - \frac{x[3](1 - x[4])}{x[2]}\right) + x[3]\,(x[4] - x[5]);$$

$$s_{45} = x[2](x[4] - x[5]);\ s_{15} = x[3]\,(1 - x[5]);$$

In the soft limit p5-> 0, d-> 0,

$$x[1] \to s,\quad x[2] \to s\,x,\quad x[3] \to -s\,x/(1 - z),$$

$$x[4] \to 1 + d\left(\frac{x + \bar{z}}{1 - \bar{z}}\right),\quad x[5] \to 1 + d\,\left(1 + \frac{x + \bar{z}}{1 - \bar{z}}\right)$$

$$S^{+(2)}_{a,ijk} = V^2_{q,ij}\, f_{aa_kb}\, f_{ba_ia_j}\; T_i^{a_i}\, T_j^{a_j}\, T_k^{a_k} \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F\left(z_k^{ij},\epsilon\right) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F\left(z_k^{ji},\epsilon\right) \right]$$

$$F(z,\bar z,\varepsilon) = \frac{1}{\epsilon^2} L_0 L_1 + \frac{1}{3\,\epsilon}\left(L_1^2 L_0 - 2\, L_0 L_1^2\right)$$

$$- L_1\left(\frac{2}{9} L_0 L_1 + \frac{1}{3} L_0^2 L_1 + \frac{13}{18} L_0 L_1^2 + \frac{7}{12} L_1^3\right) +$$

]

$$+\zeta_2(2L_{0,1} - L_0 L_1) + \frac{40}{3}\zeta_3 L_1 + O(\varepsilon)$$

Symmetric under z<->1-z

Simple-valued Harmonic Polylogarith-ms

$$\partial_z L_{w_0,\vec{w}} := (-1)^{w_0} \frac{1}{z-w_0} L_{\vec{w}},$$

$$L_{0^n} := \frac{1}{n!}\log^n(z\,\bar z), \quad L_{\vec{w}} = 0,\ \forall\, \vec{w} \neq \vec{0},\ z = 0.$$

Alternate definition of of the tripole term:

$$-\frac{1}{4}\sum_{i\neq k\neq j} \boldsymbol{S}^{+,(2)}_{a,ikj} = -\frac{1}{4}\sum_{\substack{\text{tripoles}\\ \{i,j,k\}}} \boldsymbol{S}^{+,(2)}_{a,\{i,j,k\}}$$

Sum over permutations among the three Wilson lines, project onto independent color and kinematic basis

$$\begin{aligned}\boldsymbol{S}^{+,(2)}_{a,\{i,j,k\}} &= 2\Big(\boldsymbol{S}^{+,(2)}_{a,ikj} + \boldsymbol{S}^{+,(2)}_{a,kji} + \boldsymbol{S}^{+,(2)}_{a,jik}\Big)\\ &= 2\,\boldsymbol{T}_i^{a_i}\boldsymbol{T}_j^{a_j}\boldsymbol{T}_k^{a_k}\left\{\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle}(V^q_{ik})^2\Big[f^{aa_jb}f^{ba_ia_k}D_1(z,\bar{z}) + f^{aa_ib}f^{ba_ka_j}D_2(z,\bar{z})\Big]\right.\\ &\qquad\qquad\left. + \;\{i\leftrightarrow j\} \vphantom{\frac{1}{1}}\right\},\end{aligned} \tag{3.16}$$

In terms of the F(z) defined earlier,

$$D_1(z,\bar{z}) = u^{-2\epsilon} F(z,\bar{z}) + F\left(\frac{-z}{1-z}, \frac{-\bar{z}}{1-\bar{z}}\right)$$

$$D_2(z,\bar{z}) = u^{-2\epsilon} F(z,\bar{z}) - \left(\frac{u}{v}\right)^{-2\epsilon}\left[F\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) - F\left(\frac{1-z}{-z}, \frac{1-\bar{z}}{-\bar{z}}\right)\right]$$

Triple term S_{i,j,k} is manifestly invariant under z → (1/z , 1/(1-z) , z/(z-1))

In terms of SVHPLs:

$$D_2(z) = \frac{1}{\epsilon^2}\mathcal{L}_0\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_0(\mathcal{L}_1)^2 + \frac{2}{3}\mathcal{L}_0(\mathcal{L}_1)^3 + 6\,\zeta_2\left(\mathcal{L}_{0,1} - \mathcal{L}_{1,0}\right) + 2\left(\mathcal{L}_{0,0,0,1} - \mathcal{L}_{0,0,1,0} + \mathcal{L}_{0,1,0,0} + \mathcal{L}_{0,1,0,1} - \mathcal{L}_{1,0,0,0}\right).$$

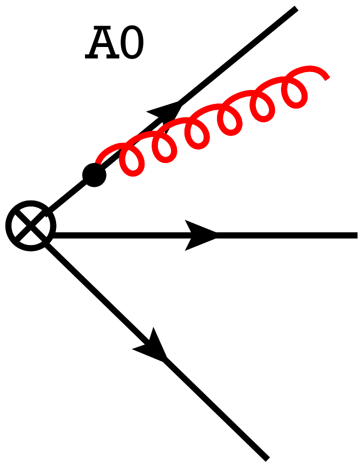
D_i (z) vanishes as z-> 0

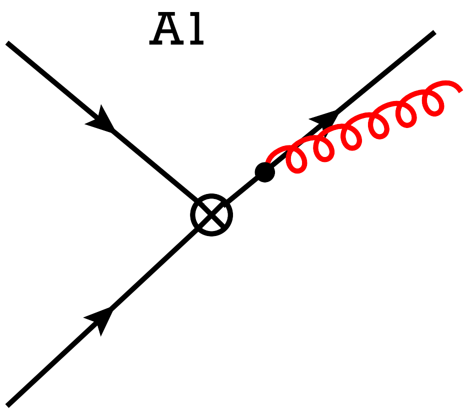
$$D_1(z) = -\frac{1}{\epsilon^2}(\mathcal{L}_1)^2 - \frac{1}{\epsilon}(\mathcal{L}_1)^3 - \frac{7}{12}(\mathcal{L}_1)^4 + 4\mathcal{L}_{1,0,1,0} + 2\mathcal{L}_{1,0,1,1} + 2\mathcal{L}_{1,1,1,0}$$

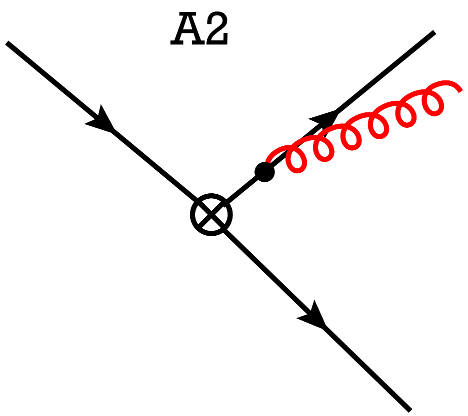
ANALYTIC CONTINUATION

$$\frac{s_{ik}s_{qj}}{s_{ij}s_{qk}} := u\,, \qquad \frac{s_{jk}s_{iq}}{s_{ij}s_{qk}} := v\,.$$

Region	Kinematics	analytic continuation	
A_0	all outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert$
A_1	j,k incoming, q,i outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert \mathrm{e}^{-2i\pi}$
A_2	i incoming, q,j,k outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert$







Analytic continuation of SVHPLs

In A1 region $D_i(z,\bar{z})|_{A_1} \;=\; D_i(z,\bar{z})|_{A_0} + \mathrm{disc}_{A_1} D_i(z,\bar{z})$ $\quad \mathrm{disc}_{A_1} D_i(z,\bar{z}) \;=\; \underset{z \to z\, e^{-2\pi \mathrm{i}}}{\mathrm{disc}} [D_i(z,\bar{z})]$

Starting from weight 1, build the analytic continuation for
higher weight SVHPLs by requiring consistency with the differential equations.

$$d\; Disc_z\, L_w(z) = Disc_z\; d\; L_w\,(z)$$

Bottom up approach: compute the discontinuity of differential and integrate back

$$\begin{aligned}\mathrm{disc}_{A_1} D_1(z) = 2\mathrm{i}\pi \Bigg\{ & 8\left[\mathrm{Li}_3(z) + \mathrm{Li}_3\left(\frac{-z}{1-z}\right)\right] \\ & - \log(1-z)\left[4\Big(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\Big) + \log^2(1-z) - \log^2(1-\bar{z})\right]\Bigg\}\end{aligned}$$

D_1,2 are single-valued functions in A1 region

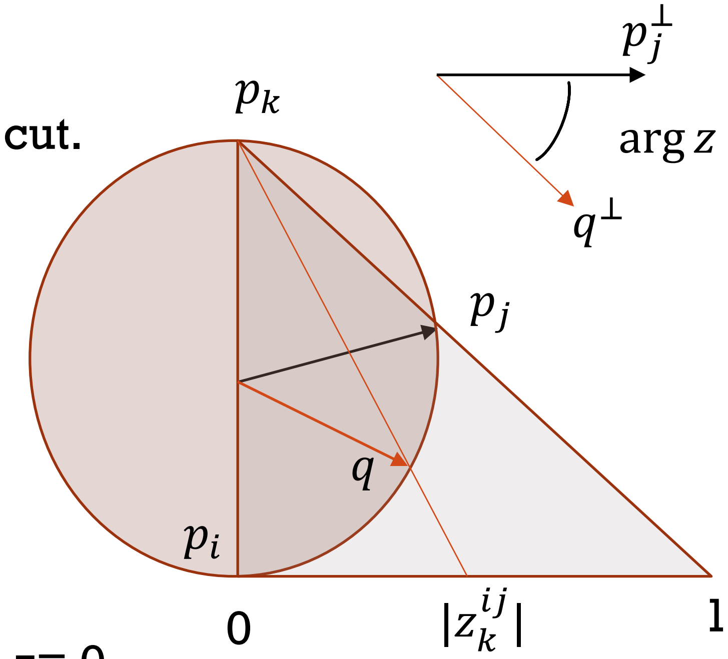
Disc_A1 D_1,2 no longer satisfies first entry condition,
they develop branch cut the real axis for |z|>1.

The argument of $\ln\frac{1-z}{1-\bar{z}}$ is ambiguous along the branch cut.

The boundary z= zb is kinematically accessible and does not correspond to physical singularity.
Ambiguity must cancel in the amplitude.

$$\ln\frac{1-z}{1-\bar{\bar{z}}} \quad (\ln\frac{1-z}{1-\bar{z}}+2\pi\, i\,)\ (\ln\frac{1-z}{1-\bar{z}}-2\pi\, i\,)$$

Disc_A1 D_1,2 are smooth function in the neighbourhood of Im z= 0 .



FACTORIZATION VIOLATION

Collinear factorization violation

In spacelike splitting, the picture of coherent soft emission breaks down.
The physical origin of the breakdown is related to the Feynman iε prescription, and therefore to the causality of the theory.

$$\Gamma_n^{\text{dip.}}(\{p_i\},\{\mathbf{T}_i\},\mu,\alpha_s) = -\frac{1}{2}\hat{\gamma}_K(\alpha_s)\sum_{i<j}\log\left(\frac{-s_{ij}-i0}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}(\alpha_s)$$

Strict collinear factorization breaks down in spacelike regime.
The splitting amplitude contains IR poles that depend on both the color and kinematics of non-collinear partons.

Catani, de Fllorian, Rodrigue 2012

Collinear limit of two-loop soft factor in A1 region

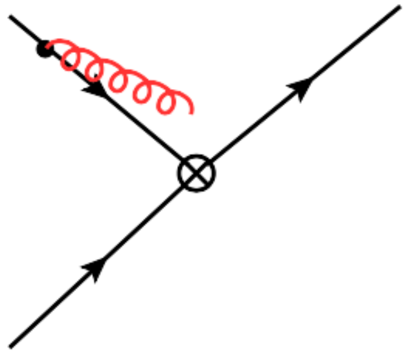
Consider spacelike splitting at two loops,
where particle 1 is an incoming parton with momentum −p1

$V_{ij}^{q} := \left[\frac{\mu^2\,(-s_{ij})}{(-s_{iq})(-s_{qj})}\right]^{\epsilon}$ develop a phase when i,j are both incoming.

$$\mathbf{Sp}^{(2)}\Big|_{\text{dipole}} \overset{q-\text{soft}}{\simeq} -\left(\frac{\mu^2}{x_q s_{1q}}\right)^{2\epsilon} C_2(\epsilon) \sum_{k\neq 1} \boldsymbol{T}_q \cdot \boldsymbol{T}_k \exp\left[(-1)^{\lambda_{kq}+1} 2\mathrm{i}\pi\epsilon\right] \mathbf{Sp}^{(0)}$$

Factorization breaking term is purely imaginary (anti-hermitian),
cancels in the squared amplitudes

Without loss of generality, consider the analytic continuation of the tripole term to the A1 region where {1, k} are incoming and {i, q} are outgoing.



$$\lim_{z,\bar{z}\to 1}\left[\boldsymbol{S}^{+,(2)}_{a,\{i,1,k\}}\Big|_{A_1}\right] = \lim_{z,\bar{z}\to 1}\,\mathrm{disc}_{A_1}\,\boldsymbol{S}^{+,(2)}_{a,\{i,1,k\}}$$

$$= \boldsymbol{T}_1^{a_1}\frac{1}{\sqrt{-x_q}\,\langle 1q\rangle}\left(\frac{\mu^2}{x_q s_{1q}}\right)^{2\epsilon}\exp\left[-2i\pi\epsilon\right]$$

$$\times\; 2\,\boldsymbol{T}_i^{a_i}\boldsymbol{T}_k^{a_k}\lim_{z,\bar{z}\to 1}\Big[f^{aa_ib}f^{ba_1a_k}\,\mathrm{disc}_{A_1}D_1(1-z,1-\bar{z})$$

$$+ f^{aa_1b}f^{ba_ka_i}\,\mathrm{disc}_{A_1}D_2(1-z,1-\bar{z})\Big]$$

$$
\begin{aligned}
\mathbf{Sp}^{(2)}\Big|_{\text{tripole}} &\overset{q-\text{soft}}{\simeq} -\frac{1}{4}\sum_{\substack{\text{tripoles}\\ \{i,1,k\}}} \boldsymbol{S}^{+,(2)}_{a,\{i,1,k\}}\Big|_{q\|p_1} \\
&= \left(\frac{\mu^2}{x_q s_{1q}}\right)^{2\epsilon} \sum_{i\neq k\neq 1} \delta_{0,\lambda_{ik}}\delta_{1,\lambda_{1k}} \Bigg\{ f^{b a_k a_i} \boldsymbol{T}^b_q \boldsymbol{T}^{a_k}_k \boldsymbol{T}^{a_i}_i \times \Bigg[\\
&\frac{1}{\epsilon^2}\left(\mathrm{i}\pi \log v_k^{1i} - \pi^2\right) - \frac{\mathrm{i}\pi^3}{3}\log v_k^{1i} + 4\mathrm{i}\pi\zeta_3 + 30\zeta_4 + \frac{8\pi}{3}\left(\arg(z_k^{1i})^3 - \pi^2 \arg(z_k^{1i})\right)\Bigg] \\
&+ \Big[(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)(\boldsymbol{T}_q\cdot\boldsymbol{T}_k) + (\boldsymbol{T}_q\cdot\boldsymbol{T}_k)(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)\Big]\left(\frac{\pi^2}{\epsilon^2} - 30\zeta_4\right)\Bigg\}\mathbf{Sp}^{(0)}, \qquad (4.27)
\end{aligned}
$$

commutator between two Hermitian operator $[(T_q \cdot T_i), (T_q \cdot T_k)]$, when sandwiched between tree amplitudes $\langle M(0) | \cdots | M(0) \rangle$ the color sum vanishes.

Squared splitting amplitude

$$\mathbf{Sp}^{\dagger}\mathbf{Sp}\Big|_{\text{non-fac.}} \overset{q-\text{soft}}{\simeq} \bar{a}^2 g_s^2 \sum_{i\neq k\neq 1} \delta_{0,\lambda_{ik}} \mathbf{Sp}^{(0)\dagger} \Bigg\{ \Big[(\boldsymbol{T}_q \cdot \boldsymbol{T}_i)(\boldsymbol{T}_q \cdot \boldsymbol{T}_k) + (\boldsymbol{T}_q \cdot \boldsymbol{T}_k)(\boldsymbol{T}_q \cdot \boldsymbol{T}_i) \Big] (-15\,\zeta_4)$$
$$+ 2\pi \mathrm{i}\, \delta_{1,\lambda_{1k}}\, f^{b a_k a_i} \boldsymbol{T}_q^b \boldsymbol{T}_k^{a_k} \boldsymbol{T}_i^{a_i} \left(\frac{\mu^2}{x_q s_{1q}} \right)^{2\epsilon} \left[\left(\frac{1}{\epsilon^2} - 2\zeta_2 \right) \log v_k^{1i} + 4\zeta_3 \right] \Bigg\} \mathbf{Sp}^{(0)} + \mathcal{O}(\bar{a}^4)\,.$$

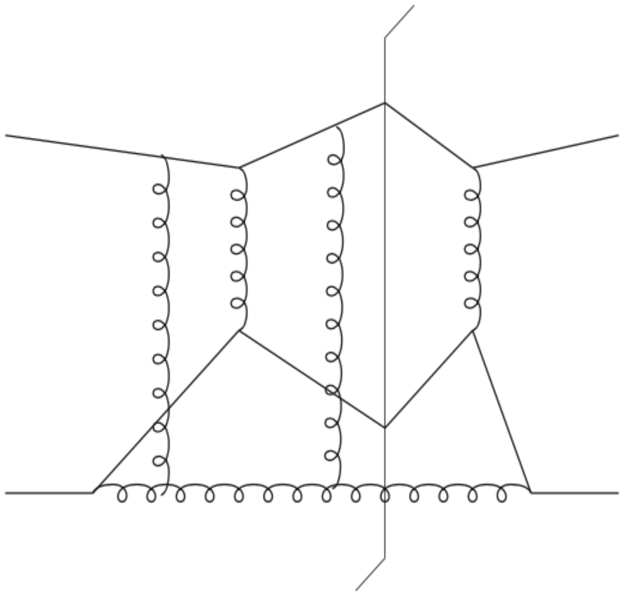
$$v_k^{1i} = \frac{s_{ik} s_{1q}}{s_{1i} s_{kq}}\,, \qquad z_k^{1i} = \frac{\langle ki\rangle\langle 1q\rangle}{\langle 1i\rangle\langle kq\rangle}$$

Phase-space integrals of the splitting function might generate collinear divergences that cannot be removed by pdf counterterm

$$E_3 E_4 \frac{d\sigma}{d^3\boldsymbol{p}_3 d^3\boldsymbol{p}_4} = \sum \int d\hat{\sigma}_{i+j\to k+l+X}\, f_{i/1}\, f_{j/2}\, d_{3/k}\, d_{4/l}$$
$$+ \text{power-suppressed correction.}$$

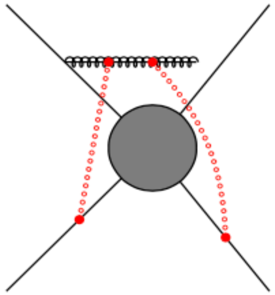
hard-scattering / pdf factorization is endangered in the production of high-pT hadrons in hadron-hadron collisions at N^3LO?

Collins,Qiu, 0705.2141



An counterexample for TMD factorization was construct is for the single-spin asymmetry
with one beam transversely polarized. (in a greatly simplified model theory)

SCET effective operator Glauber mode exchanged between hard partons. The double Glauber ladder diagram produce the same two-loop constant as we find the soft emission factor.



$$= -(\mathbf{T}_2 \cdot \mathbf{T}_j)(\mathbf{T}_2 \cdot \mathbf{T}_3)\,\mathbf{Sp}^0\,\overline{\mathcal{M}}^0 \times \left(\frac{\alpha_s}{2\pi}\right)^2 (i\pi)^2 \left(\frac{4\pi\mu^2}{\vec{p}_{2,\perp}^{\,2}}\right)^{2\epsilon} [\Gamma(-\epsilon)]^2 \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}$$

Calculations in these studies was done without assuming soft limit

Schwartz, Yan, Zhu 1703.08572

Summary

We provide results for two-loop soft emission factor which involves three parton correlation. It may serve as a building block for IR subtraction for N3LO phase-space integral both in e+e- and hadron colliders.

The intricate analytic property of tripole terms poses a strong constraint which may be useful for obtaining higher-loop results of full amplitudes by their analytic properties

Collinear factorization breaks down at NNLO in the scattering amplitude. This observation could potentially endanger factorization for inclusive cross section in dijet production at high-pT .

Thank you for your attention .