Wilson-line geometries in amplitude and splitting function factorisation

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Outline

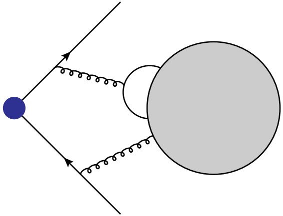
- Infrared singularities in form factors and splitting functions
- Factorisation
  - Properties of the soft functions
- Relations between Wilson-line geometries
- Conclusions

Gauge theory form factors

Easiest amplitude: two legs on shell with an off-shell current.
e.g. Massless quark form factor

$$\overline{u}(p_2)\gamma^\mu u(p_1)\, F_{\text{quark}}(q^2) = \int dx\, e^{-iq\cdot x}\, \langle p_2|\overline{\psi}(x)\gamma^\mu\psi(x)|p_1\rangle$$

- Single kinematic scale: $q^2 = (p_2 - p_1)^2$
- **IR sensitivity**: in $d = 4 - 2\epsilon$ **singularities for** $\epsilon \to 0$.



All-order representation

Evolution equation (Mueller 79; Collins 80; Collins, Soper 81; Sen 81)

$$Q^2 \frac{\partial}{\partial Q^2} \log F_i = \frac{1}{2}\left[\underbrace{K_i\left(\alpha_s(\mu^2,\epsilon)\right)}_{\text{Divergent}} + \underbrace{G_i\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2,\epsilon), \epsilon\right)}_{\text{Kinematics}}\right]$$

Renormalisation scale independence:

$$\mu \frac{d}{d\mu} G_i\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) = -\mu \frac{d}{d\mu} K_i\left(\alpha_s, \epsilon\right) \equiv \gamma_i^{\text{cusp}}\left(\alpha_s\right).$$

All-order representation (Magnea, Sterman 1990)

$$\log F_i(Q^2) = \int_0^{Q^2} \frac{d\lambda^2}{2\lambda^2}\left[G_i\left(1, \alpha_s(\lambda^2,\epsilon), \epsilon\right) - \gamma_i^{\text{cusp}}\left(\alpha_s(\lambda^2,\epsilon)\right) \log \frac{Q^2}{\lambda^2}\right]$$

Infrared singularities

All-order representation (Magnea, Sterman 1990)

$$\log F_i(Q^2) = \int_0^{Q^2} \frac{d\lambda^2}{2\lambda^2}\left[G_i\left(1, \alpha_s(\lambda^2, \epsilon), \epsilon\right) - \gamma_i^{\text{cusp}}\left(\alpha_s(\lambda^2, \epsilon)\right) \log \frac{Q^2}{\lambda^2}\right]$$

$\alpha_s(\mu^2, \epsilon)$ is the d-dimensonal coupling

$$\mu^2 \frac{d}{d\mu^2}\alpha_s(\mu^2, \epsilon) = -\epsilon\alpha_s(\mu^2, \epsilon) - b_0\alpha_s^2(\mu^2, \epsilon) - \ldots$$

Divergences from the $\lambda^2 \to 0$ boundary

$$-\frac{\alpha_s(Q^2, \epsilon)}{\epsilon} = \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2}\alpha_s(\lambda^2, \epsilon) + \mathcal{O}\left(\alpha_s^2\right)$$

$$-\frac{\alpha_s(Q^2, \epsilon)}{\epsilon^2} = \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2}\alpha_s(\lambda^2, \epsilon) \log\left(\frac{\lambda^2}{Q^2}\right) + \mathcal{O}\left(\alpha_s^2\right)$$

The anomalous dimensions γ^{cusp} and γ_G

The cusp anomalous dimension

γ_i^{cusp} gives the double poles of the form factor. Up to 3 loops

$$\frac{\gamma^{\text{cusp}}_{\text{quark}}}{C_F} = \frac{\gamma^{\text{cusp}}_{\text{gluon}}}{C_A} \qquad \textbf{Casimir scaling}$$

The *collinear* anomalous dimension

$G_i(1, \alpha_s, \epsilon)$ generates both **single poles** and **finite parts**.
$\boldsymbol{\gamma_{G_i}}$ is defined order-by-order in α_s to give **only poles**.

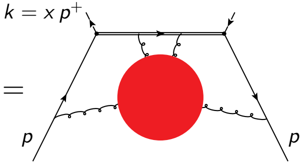
$$\int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} G_i(\alpha_s(\lambda^2, \epsilon), \epsilon) = \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \gamma_{G_i}(\alpha_s(\lambda^2, \epsilon), \epsilon) + \mathcal{O}\left(\epsilon^0\right)$$

$\gamma_{G_{\text{quark}}}$ and $\gamma_{G_{\text{gluon}}}$ are not related to each other.

Perturbative PDFs

Perturbative parton distribution functions (Collins, Soper 1981), *e.g.*

$$f_{q/p}(x) = \int \frac{dy}{4\pi}\, e^{-iy\, xp\cdot u} \langle p|\overline{\psi}_q(yu)\,(\gamma\cdot u)\, W_u(y,0)\psi_q(0)|p\rangle$$



- p external **parton** of momentum $p = (p^+, 0^-, \mathbf{0}_\perp)$.
- u lightcone direction $u = (0^+, 1^-, 0_\perp)$.
- x fraction of p^+ outgoing at the vertex with the **Wilson line**

$$W_u(y,0) = \mathbf{P}\exp\left(ig_s \int_0^y d\lambda\, u_\mu\, A^\mu\,(\lambda u_\mu)\right)$$

Splitting functions

$$f_{q/p}(x) = \int \frac{dy}{4\pi}\, e^{-iy\,xp\cdot u} \langle p|\overline{\psi}_q(yu)\,(\gamma\cdot u)\,W_u(y,0)\psi_q(0)|p\rangle$$

$$f_{g/p}(x) = \frac{1}{x\,p\cdot u}\int \frac{dy}{2\pi}\, e^{-iy\,xp\cdot u} \langle p|G^a_{\mu+}(yu)W_u(y,0)G^{a;+\mu}(0)|p\rangle$$

- Defined in **MS** scheme by their UV counterterms
- $f_{i/p}(x)$ have the **same UV** behaviour of the parton distributions at **hadron** level $f_{i/H}$

Renormalisation

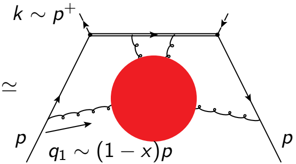
$f_{i/p}$ obey the DGLAP equation

$$\frac{df_{i/p}(x)}{d\log\mu} = 2\int_x^1 \frac{dz}{z} P_{ij}(z,\alpha_s) f_{j/p}\left(\frac{x}{z},\mu\right)$$

The splitting kernels become singular in the limit $x \to 1$.

- Singularity of infrared origin

$f_{q/p}(x \to 1) \quad \simeq$



- Leading behaviour captured by the diagonal terms (Korchemsky 1989, Berger 2002)

$$P_{ii} = \frac{\gamma_i^{\text{cusp}}}{(1-x)_+} + B_{\delta,i}\,\delta(1-x) + \mathcal{O}(\log(1-x))$$

- Divergences controlled by γ_i^{cusp} and $\boldsymbol{B}_{\delta;i}$.

Comparing the IR singularities

The coefficients γ_{G_i} and $B_{\delta,i}$ obey

$$\frac{\gamma_{G_q} - 2B_{\delta,q}}{C_F} = \frac{\gamma_{G_g} - 2B_{\delta,g}}{C_A}$$

Casimir scaling to 3 loops
(van Neerven, Ravindran, Smith 2004; Moch, Vermaseren, Vogt 2005).
Relations with **Wilson-line geometries**

$$\gamma_G - 2B_\delta = \frac{\Gamma_{\mathrm{DY}}}{2} \underset{\substack{at\\ 2\text{ loops}}}{=} \frac{\Gamma_\Box}{4}$$

- Γ_{DY} **soft anomalous dimension** in **Drell-Yan** (Belitsky 1998; Li, von Manteuffel, Schabinger, Zhu 2014).
- $\Gamma_\Box$ anomalous dimension of **parallelogram** Wilson loop (Korchemsky, Korchemskaya 1992)

Questions

- What is the origin of the simple relation between γ_G and B_δ?
  γ_{G_i} and $B_{\delta,i}$ depend on $i =$ quark, gluon, but their combination is **universal**.
- What is the connection between γ_G, B_δ and the Wilson loops?
- Are there simple relations between the anomalous dimensions of different Wilson loops?

$$\Gamma_{\square} \stackrel{?}{=} 2\,\Gamma_{\mathrm{DY}}$$

Infrared factorisation

Infrared contributions to the form factors

Soft and collinear singularities **decouple** from the hard scattering (Collins 1980, Sen 1981)

$$F(q^2) = H\left(\frac{q^2}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2)\right) \prod_{i=1}^{2} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$
$$\times \left(\frac{\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon)}{\prod_{i=1}^{2} \mathcal{J}_i\left(\frac{(2\beta_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)}\right)$$

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- $J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$: emissions **collinear** to p_i.
  $\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon)$: **soft** particle exchanges.
  $\mathcal{J}_i\left(\frac{(2\beta_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$: **soft and collinear** emissions.
- Use these as building blocks of the K and G functions.

Building blocks of the factorisation

$\mathcal{S}$ and $\mathcal{J}$ are **Wilson-line correlators** (Dixon, Magnea, Sterman 2008)

Soft and **eikonal jet** functions

$$\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon) = \langle 0 | \, T \left[W_{\beta_1}(\infty, 0) W_{\beta_2}(0, \infty) \right] | 0 \rangle$$

$$\mathcal{J}_i \left(\frac{(2\beta_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) = \langle 0 | \, T \left[W_{n_i}(\infty, 0) W_{\beta_i}(0, \infty) \right] | 0 \rangle$$

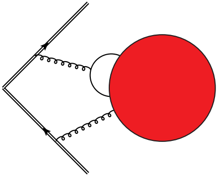
- β_i velocity of the particle with momentum p_i.
- n_i auxiliary vectors with $n_i^2 \neq 0$ (avoid spurious singularities).

- Dependence on the **colour representation** of external particles $\rightarrow$ **Casimir scaling**
- Independence on mass scales $\rightarrow$ **vanishing bare results**

The (renormalised) soft function

A decomposition of the $K + G$ type applies, as in the **form factor**

$$\langle 0 | T \left[W_{\beta_1}(\infty, 0) W_{\beta_2}(0, \infty) \right] | 0 \rangle =$$



All-order representation (Dixon, Magnea, Sterman 2008)

$$\log \mathcal{S} = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[\Gamma_{\wedge} + \gamma^{\text{cusp}} \log \left(\frac{\beta_1 \cdot \beta_2 \mu^2}{\lambda^2} \right) \right]$$

- $\gamma^{\text{cusp}} \to$ **double IR poles** and **kinematic logs**.
- $\Gamma_{\wedge} \to$ **single IR poles**.

Computing $\mathcal{S}$

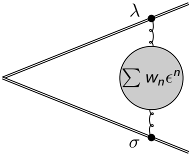
Problem: **disentangle UV and IR poles** in a **scaleless** quantity.

Non-abelian exponentiation

$\log \mathcal{S}$ has a **single IR** and **collinear** pole. (Sterman 1981; Gatheral, Frenkel, Taylor 1984; Berger 2002; Erdoğan, Sterman 2014)

Coordinate-space representation (Erdoğan, Sterman 2015)

$$\log \mathcal{S}^{\text{bare}} = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\lambda}{\lambda} \int_0^{\infty} \frac{d\sigma}{\sigma}\, \epsilon^n \underbrace{w_n\left(\alpha_s\left(\frac{1}{\lambda\sigma}, \epsilon\right)\right)}_{\text{finite}}$$

$=$ 

$$\begin{cases} \text{overall IR: } \lambda, \sigma \to \infty \\ \text{cusp: } \lambda, \sigma \to 0 \\ \text{collinear } \sigma\text{: } \lambda \ll \sigma \end{cases}$$

Finding γ^{cusp} and $\Gamma_\wedge$

Divergences are **localised** around $\lambda, \sigma \to 0$.
Renormalisation: subtraction of the **short-distance** contribution (Erdoğan, Sterman 2015)

$$\log \mathcal{S} = \sum_{n=0}^{\infty} \int_{\frac{1}{\mu}}^{\infty} \frac{d\lambda}{\lambda} \int_{\frac{1}{\mu}}^{\infty} \frac{d\sigma}{\sigma}\, \epsilon^n\, w_n\left(\alpha_s\left(\frac{1}{\lambda\sigma}, \epsilon\right)\right)$$
$$= -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\Gamma_\wedge + \gamma^{\text{cusp}} \log\left(\frac{\beta_1 \cdot \beta_2 \mu^2}{\lambda^2}\right)\right]$$

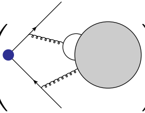
Anomalous dimension found comparing the **integrands**

$$\frac{\Gamma_\wedge}{C_i} = \left(\frac{\alpha_s}{\pi}\right)^2 \left[C_A\left(\frac{101}{54} - \frac{11}{6}\zeta(2) - \frac{\zeta(3)}{4}\right) + n_f\, T_f\left(\frac{2\zeta(2)}{3} - \frac{14}{27}\right)\right]$$

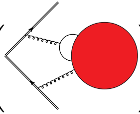
Hard-collinear singularities of the form factor

Subtracting the soft function from the form factor one **isolates purely collinear poles**

$$\log\left(\frac{J_i|_{\text{pole}}}{\mathcal{J}_i}\right) = \log\Bigg($$



$$\Bigg) - \log\Bigg($$



$$\Bigg)$$

$$= \frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\gamma_{J_i/\mathcal{J}_i} - \frac{\gamma_i^{\text{cusp}}}{2}\log\left(\frac{2(p_i\cdot n_i)^2}{(\beta_i\cdot n_i)^2\mu^2}\right)\right]$$

where $\gamma_{J_i/\mathcal{J}_i}$ is **defined** by the difference

$$\boxed{2\gamma_{J_i/\mathcal{J}_i} = \gamma_{G_i} + \Gamma_\wedge}$$

Summary: form factor

Factorisation decomposes the *collinear* anomalous dimension γ_{G_i}

- **hard-collinear** part $\gamma_{J/\mathcal{J}}$
- **soft** contribution $\Gamma_\wedge$

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_\wedge$$

- $\gamma_{J_i/\mathcal{J}_i}$ is **independent** on **process kinematics** $\longrightarrow$ **universal**.
- $\Gamma_\wedge$ is associated to the **Wilson loop** that captures the soft virtual corrections.

Next: factorising singularity $B_\delta\,\delta(1-x)$ in the splitting functions

PDFs at large-x

Soft and **collinear** contributions factorise in PDFs at $x \to 1$

(Korchemsky 1989; Korchemsky, Marchesini 1992; Berger 2002)

Factorisation formula

In Mellin space $\widetilde{f}_i(N) = \int_0^1 dx\, x^{N-1} f_i(x)$

$$\widetilde{f}_i(N,\mu) = \left(\prod_{i=1}^{2} \frac{J_i\left(\frac{2(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s, \epsilon\right)\Big|_{\text{pole}}}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s, \epsilon\right)} \right) \widetilde{\mathcal{S}}_{\sqcap}\left(N, \frac{\beta\cdot u\mu}{p\cdot u}, \alpha_s, \epsilon\right)$$

Same collinear singularities as the **form factors**.

Splitting functions

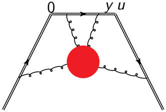
$$P_{ii} = 2\gamma_{J_i/\mathcal{J}_i} - 2\gamma_i^{\text{cusp}} \log\left(\frac{\sqrt{2}p\cdot n}{\beta\cdot n}\right) + \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma^{\text{cusp}} + \frac{d\log\widetilde{\mathcal{S}}_{\sqcap}}{d\log\mu}$$

The PDF soft function

$\mathcal{S}_\sqcap$ is a Wilson-line correlator (Korchemsky, Marchesini 1992)

$$\mathcal{S}_\sqcap = (p \cdot u) \int \frac{dy}{2\pi} e^{iy(1-x)\, p\cdot u} W_\sqcap(y)$$

$$W_\sqcap(y) = \langle 0 | \, T \Big[W_\beta(\infty, y) W_u(u, 0) W_\beta(0, -\infty) \Big] |0\rangle =$$



Conditions on the analytic structure of $W_\sqcap$

- **Support condition**: $\mathcal{S}_\sqcap(x > 1) = 0$
- **Reality condition**: $\mathcal{S}_\sqcap$ real function of x

Both conditions automatically satisfied writing $W_\sqcap$ as a function of

$$\rho(y) = (i(y \cdot u - i0)) = (\rho(-y))^*$$

Computing $W_\sqcap$

Non-abelian exponentiation: **singularities** of $\log W_\sqcap$ from the **cusp** and **collinear limits**

$$\log W_\sqcap^{\text{bare}} = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\lambda}{\lambda} \int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma}\, \epsilon^n \underbrace{w_n^\sqcap\left(\frac{1}{\lambda\sigma}\right)}_{\text{finite}}$$

Subtraction of the **ultraviolet** contribution

$$\log W_\sqcap^{\text{ren}} = -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[2\gamma^{\text{cusp}} \log\left(\frac{\rho\mu}{\sqrt{2}}\right) + \Gamma_\sqcap\right]$$

Two-loop result for $\Gamma_\sqcap$

$$\frac{\Gamma_\sqcap}{C_i} = \left(\frac{\alpha_s}{\pi}\right)^2 \left[C_A\left(\frac{101}{27} - 2\zeta(3) - \frac{11\zeta(2)}{3}\right) - n_f T_f\left(\frac{28}{27} - \frac{4\zeta(2)}{3}\right)\right]$$

Factorisation of B_δ

Fourier and Mellin transform of $W_\sqcap$ leads to the **soft function**

$$\log \widetilde{\mathcal{S}}_\sqcap = -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[2\gamma^{\text{cusp}}\log\left(\frac{N\mu\beta\cdot u}{\sqrt{2}p\cdot u}\right)+\Gamma_\sqcap\right]$$

Differentiation with respect to $\log\mu$

$$\widetilde{P}_{ii}(N) = -\gamma^{\text{cusp}}\log(N) + \gamma_{J/\mathcal{J}} - \frac{\Gamma_\sqcap}{2}$$

Singularity $\delta(1-x)$

Read off B_δ from the subleading-N piece

$$\boxed{2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_\sqcap}$$

Relations between Wilson-line geometries

Eikonal relation between γ_G and B_δ

- Separation of **soft** and **purely collinear** contributions in B_δ

$$2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_{\sqcap}$$

- Comparison with the form factor singularities

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_{\wedge}$$

Eikonal relation between γ_G and B_δ

- Separation of **soft** and **purely collinear** contributions in B_δ

$$2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_\sqcap$$

- Comparison with the form factor singularities

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_\wedge$$

Conclusion

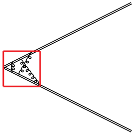
Relation between form factor singularities and splitting functions

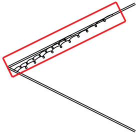
$$\boxed{\gamma_G - 2B_\delta = \Gamma_\sqcap - \Gamma_\wedge}$$

Checked to two loops via direct computation of $\Gamma_\sqcap$ and $\Gamma_\wedge$.

Effects of the Wilson-line geometry

Coordinate-space origin of the singularities (Erdoğan, Sterman 2015)





Cusp configuration $\rightarrow \gamma^{\text{cusp}}$ Lightlike collinear $\rightarrow \Gamma$

- **Independent** on the **global geometry**.
- Comparison of $\Gamma_{\sqcap}$ and $\Gamma_{\wedge}$: **difference** in **finite/infinite** lightlike lines

$$\boxed{\Gamma_{\sqcap} - \Gamma_{\wedge} \equiv \Gamma_{\text{finite}}}$$

Consistency check: two-loop **parallelogram Wilson loop** (Korchemskaya, Korchemsky 1992)

$$\frac{\Gamma_{\square}}{4} = \Gamma_{\text{finite}}.$$

Conclusions

Factorisation shows

- γ_{G_i} and $B_{\delta,i}$ share the **same hard-collinear** contributions.
- $\gamma_{G_i} - 2B_{\delta,i}$ identifies a **difference** of **Wilson loops**
  - **simple relations** between **quark** and **gluon** quantities.

Coordinate-space analysis of the Wilson loops implies

- The **singularities** of the Wilson loops are sensitive to **cusp** and **collinear** configurations
  - **Anomalous dimensions** constructed by *counting* cusps, finite and infinite lightlike lines *e.g.*

$$\Gamma_{\sqcap} - \Gamma_{\wedge} = \frac{\Gamma_{\square}}{4}$$

Outlook

- Test the relations between finite and infinite line anomalous dimensions on different contours.
- Test the agreement between $\Gamma_{\square}$ and Γ_{DY}, using the known 3-loop results for the latter.
- Can we extend the relations beyond the singularities of the Wilson loops?
- $\Gamma_{\wedge}$ gives the finite parts of the **gluon Regge trajectory**. Can we explain this agreement?

Thank you