Journeys beyond the soft approximation

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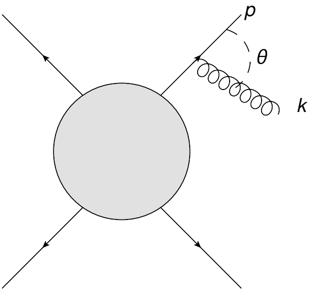
Overview

- Brief introduction to (next-to-) soft divergences.
- Applications in Collider Physics (mainly QCD).
- Applications in high energy scattering (mainly gravity).
- Outlook.

Infrared divergences

- In scattering amplitudes, get singularities due to soft or collinear gauge bosons:



$$\frac{1}{p \cdot k} = \frac{1}{|\boldsymbol{p}||\boldsymbol{k}|(1 - \cos\theta)}.$$

- Formal divergences cancel upon combining real and virtual graphs.

- Both soft and collinear radiation is *universal*.
- Physics: it has an infinite wavelength, so cannot resolve the underlying amplitude.

- Universality of soft / collinear radiation is expressed in *factorisation formulae*.
- Example: consider a tree-level amplitude $\mathcal{A}_{n+1}(\{p_i\}, k)$ where momentum k becomes soft. We then get the *soft theorems*

$$\lim_{k^\mu \to 0} \mathcal{A}_{n+1}(\{p_i\}, k) = \mathcal{S}^{(0)}(\{p_i\}, k)\mathcal{A}_n(\{p_i\}),$$

  where

$$\mathcal{S}^{(0)}_{\mathrm{QED}} = \sum_{i=1}^{n} \frac{\epsilon_\mu(k) p_i^\mu}{p_i \cdot k}, \quad \mathcal{S}^{(0)}_{\mathrm{grav.}} = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu}(k) p_i^\mu p_i^\nu}{p_i \cdot k}$$

  (Yennie, Frautschi, Suura; Weinberg).
- All dependence on the soft momentum k is in the overall factor $\mathcal{S}$.

- It is also possible to write such formulae at one order higher in the k expansion (Cachazo, Strominger; Casali):

$$\mathcal{A}_{n+1}(\{p_i\}, k) = \left[\mathcal{S}^{(0)} + \mathcal{S}^{(1)}\right] \mathcal{A}_n(\{p_i\}),$$

with

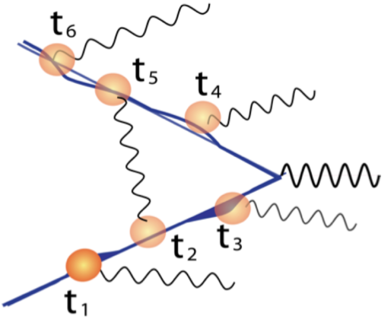
$$\mathcal{S}^{(1)}_{QED} = \sum_{i=1}^{n} \frac{\epsilon_\mu k_\rho J^{(i)\mu\rho}}{p_i \cdot k}, \quad \mathcal{S}^{(1)}_{grav.} = \sum_{i=1}^{n} \frac{\epsilon_\mu k_\rho J^{(i)\mu\rho}}{p_i \cdot k},$$

where $J^{(i)}_{\mu\nu}$ is the total angular momentum of (hard) particle i.

- `hep-th` calls these the *next-to-soft theorems*. Intense activity since 2014.
- However, there is a surprisingly long (pre)-history!

- Next-to-soft effects were first studied in gauge theory (QED) by Low (1958).
- He considered external scalars; generalised to fermions by Burnett and Kroll (1968).
- Both groups only considered massive particles (no collinear effects).
- Similar work in gravity by Gross, Jackiw (1968).
- Del Duca (1990) generalised the Low-Burnett-Kroll result to include collinear effects.

- Next-to-soft effects for massive particles considered using worldline methods by Laenen, Stavenga, White (2008).



- Can replace propagators for external legs by quantum mechanics path integrals.
- Leading term in perturbative expansion is classical trajectory (soft limit).
- First-order wobbles give next-to-soft behaviour.

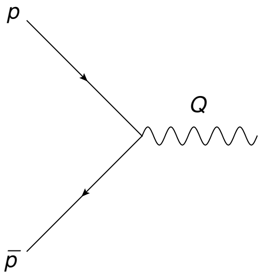
- Also works for gravity (White, 2011).

Applications

- The history of next-to-soft physics suggests that there are many applications.
- Indeed, these have been reinvigorated by the recent work on next-to-soft theorems.
- The aim of this talk is to review some of these applications.

Key message: next-to-soft physics connects `hep-th`, `hep-ph`, `hep-ex` and `gr-qc`!

- A major application of (next-to) soft physics is to collider physics.
- We saw earlier that IR singularities cancel when real and virtual diagrams are combined.
- However, the cancellation can leave behind large contributions to perturbative quantities.
- Consider e.g. the production of a vector boson at a collider ("Drell-Yan production"):



- Let $z = Q^2/s$ be the fraction of (squared) energy s carried by the vector boson.
- At LO, $z = 1$, and thus the cross-section is

$$\frac{d\sigma^{(0)}}{dz} \propto \delta(1 - z).$$

- At next-to-leading order (NLO), radiation can carry energy, so that
$$0 \leq z \leq 1.$$
- The NLO cross-section then turns out to be
$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dz} \sim \frac{\alpha_s}{2\pi}\left[4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln(z) + \delta(1-z)\left(\frac{2\pi^2}{3}-8\right)\right].$$
- It contains highly divergent terms as $z \to 1$.
- Looks like perturbation theory is in trouble!
- Let's go one order higher and see what happens...

- At NNLO the problem is even worse! One has

$$\frac{d\sigma^{(2)}_{q\bar{q}}}{dz} \sim C_F^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \left[128 \left(\frac{\ln^3(1-z)}{1-z}\right)_+ - 256 \left(\frac{\ln(1-z)}{1-z}\right)_+ + \ldots\right],$$

  where $\ldots$ denotes terms suppressed by $(1-z)$.
- Logs get higher at higher orders in perturbation theory...
- ... which indeed breaks down as $z \to 1$.
- Precisely the regime where the vector boson is produced near threshold, so that extra radiation is soft / collinear!
- The problem terms are echoes of IR singularities having been present.
- Thus, this problem affects many different scattering processes...

- For heavy particles produced near threshold, we can define a ξ, where $\xi \to 0$ at threshold (e.g. $\xi = (1-z)$).
- Then the general structure of any such cross-section is:

$$\frac{d\sigma}{d\xi} = \sum_{n,m} \alpha^n \left[c_{nm}^{(0)} \left(\frac{\ln^m \xi}{\xi} \right)_+ + c_{nm}^{(1)} \ln^m \xi + \ldots \right].$$

- First set of terms correspond to (leading) threshold logs: pure soft and / or collinear.
- Second set of terms is next-to-leading power (NLP) threshold logs: next-to-soft and / or collinear.
- For $\xi \to 0$, we need to rethink perturbation theory.

- The solution to this problem is to somehow work out what the large logs are to all orders in α_s.
- Then we can sum them up to get a function of α_s that is better behaved than any fixed order perturbation expansion.
- Toy example: consider the function

$$e^{-\alpha_s x} = \sum_{n=0}^{\infty} \frac{\alpha_s^n (-x)^n}{n!}.$$

- Each term diverges as $x \to \infty$, but the all-order result is well-behaved.

- Many approaches exist for resumming leading threshold logs.
- There are many (hundreds?) of observables at e.g. the LHC for which this is relevant.
- Original diagrammatic approaches by e.g. Sterman; Catani, Trentadue.
- Can also use Wilson lines (Korchemsky, Marchesini), or the renormalisation group (Forte, Ridolfi).
- A widely used approach is to treat soft and collinear gluons as separate fields in an effective theory: soft-collinear effective theory (SCET) (Becher, Neubert; Schwartz; Stewart).
- All approaches have the *factorisation* of soft / collinear physics at their heart.

- The general structure of an n-point amplitude is

$$\mathcal{A}_n = \mathcal{H}_n \times \mathcal{S} \times \frac{\prod_i J_i}{\prod_i \mathcal{J}_i}.$$

- This is the virtual generalisation of the soft theorem.
- Here $\mathcal{H}_n$ is the *hard function*, and is IR finite.
- The *soft* and *jet functions* $\mathcal{S}$ and J_i collect soft / collinear singularities respectively.
- The *eikonal jets* $\mathcal{J}_i$ remove any double counting.
- The soft and jet functions have universal definitions in terms of Wilson line operators.

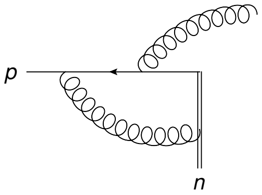
- The soft-collinear factorisation formula leads directly to resummation of threshold effects.
- Related ideas in other approaches (e.g. SCET).
- Summing successive towers of threshold logs requires calculating the soft and jet functions to a given order in perturbation theory.
- State of the art is two loops (Sterman, Aybat, Dixon, Kidonakis, Mitov, Sung, Becher, Neubert, Beneke, Falgari, Schwinn, Ferroglia, Pecjak, Yang).
- Progress towards three-loops and beyond (Gardi, Laenen, Stavenga, Smillie, White, Almelid, Duhr, Korchemsky, Henn, Huber, Grozin, Marquard, Correa, Maldacena, Sever).

- To date, much less has been known about NLP effects.
- Known for a while to be numerically significant e.g. in Higgs production (Kramer Laenen, Spira; Harlander, Kilgore; Catani, de Florian, Grazzini, Nason).
- This has been confirmed by recent N^3LO Higgs results (Anastasiou, Duhr, Dulat, Herzog, Mistlberger).
- There are three good reasons to study NLP logs:
  1. Resummation of them will improve precision.
  2. Even without resummation, NLP logs may provide good approximate N^nLO cross-sections.
  3. Can improve the stability of numerical codes.

- Next-to-soft effects in particular scattering processes classified to all orders by (Almasy, Moch, Presti, Soar, Vermaseren, Vogt).
- Can also be classified using the *method of regions* (Beneke, Smirnov, Pak, Jantzen). See e.g. Bonocore, Laenen, Magnea, Vernazza, White.
- None of the previous approaches is fully general - but strong hints of an underlying structure.
- Can we predict NLP logs in an arbitrary process?
- Can they be written in terms of universal functions (like LP effects)?
- Encouraging recent progress...

- It is well-known that LP effects can be described using *Soft-Collinear Effective Theory* SCET (Stewart, Schwartz, Bauer, Fleming; Becher, Neubert).
- The same language can be extended to NLP level.
- Originally explored in B physics (Beneke, Campanario, Mannel, Pecjak).
- Recent study for scattering amplitudes (Larkoski, Neill, Stewart).
- Phenomenology explored by Feige, Kolodrubetz, Moult, Stewart, Rothen, Tackmann, Zhu; Boughezal, Liu, Petriello.
- Recent resummation of leading NLP log for some observables (Moult, Stewart, Vita, Zhu; Beneke, Broggio, Jaskiewicz, Vernazza).

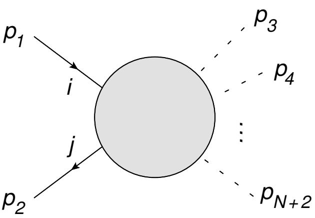
- The soft-collinear factorisation formula can be generalised to next-to-leading power level (Bonocore, Laenen, Magnea, Melville, Vernazza, White).
- A new quantity appears at next-to-soft level: the *jet emission function*.
- Has been calculated at one-loop level for quarks.



- Calculation for gluons in progress.
- Further such functions are needed for general processes (Gervais)...
- ...which have counterparts in the SCET approach.

- For leading logs, the jet emission functions do not contribute.
- One may then show that the LL NLP logs indeed exponentiate, and can be resummed (Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, Vernazza. White).
- Results agree with SCET approach (Beneke, Broggio, Jaskiewicz, Vernazza)...
- ...and previous conjectures (Moch, Vogt).
- Furthermore, the argument works for arbitrary colour-singlet production processes (e.g. (multi-) Higgs production).
- Further work will involve:
  1. Inclusion of other partonic channels.
  2. Extension to arbitrary processes.
  3. Numerical studies and implementations.
  4. Extension to NLL and beyond (difficult!).

- Even at fixed order, next-to-soft corrections can be useful.



- Consider emission of an additional gluon of momentum k, up to NLP level.
- Next-to-soft theorems imply the general NLP amplitude (Del Duca, Laenen, Magnea, Vernazza, White):

$$|\mathcal{A}_{\mathrm{NLP}}|^2 \sim \frac{p_1 \cdot p_2}{p_1 \cdot k\, p_2 \cdot k} |\mathcal{A}_{\mathrm{LO}}(p_1 + \delta p_1, p_2 + \delta p_2)|^2,$$

where

$$\delta p_{1,2}^{\alpha} = -\frac{1}{2}\left(\frac{p_{2,1} \cdot k}{p_1 \cdot p_2} - \frac{p_{1,2} \cdot k}{p_1 \cdot p_2} p_2^{\alpha} + k^{\alpha}\right).$$

- The formula works fully differentially.
- Also leads to a universal form of the cross-section (same in quark or gluon channel):

$$\frac{1}{\hat{\sigma}_{\rm LO}(zs)}\frac{d\hat{\sigma}_{\rm NLP}}{dz}=\frac{\alpha_s}{\pi}\left(\frac{\bar{\mu}^2}{s}\right)^{\epsilon}\left[\frac{2-\mathcal{D}_0}{\epsilon}+4\mathcal{D}_1(z)-4\log(1-z)\right],$$

  where $z\to 1$ at threshold.
- Formula works if LO process is tree-level or loop induced.
- Recent extension to prompt photon production (van Beekveld, Beenakker, Laenen, White).
- These and similar ideas can be used to:
  1. Provide approximate higher-order cross-sections.
  2. Constrain future analytic calculations.
  3. Improve stability of NLO subtraction schemes (see e.g. Boughezal, Isgrò, Petriello).

- Next-to-soft physics has a large number of applications in collider physics.
- Typically this involves summing up large terms in perturbative cross-sections...
- ... or finding approximate forms for fixed-order cross-sections.
- Such calculations improve the precision of theory predictions at the LHC.
- Current data demands this precision!

- So far we have focused on next-to-soft corrections in QCD.
- However, they have a different role to play in understanding the conceptual structure of quantum gravity...
- ...and may even have phenomenological consequences!
- More specifically, they are relevant to high energy scattering.
- Many papers from the 1990s onwards (Amati, Ciafaloni, Veneziano, Colferai, Falcioni; 't Hooft; Verlinde^2; Jackiw, Kabat, Ortiz).

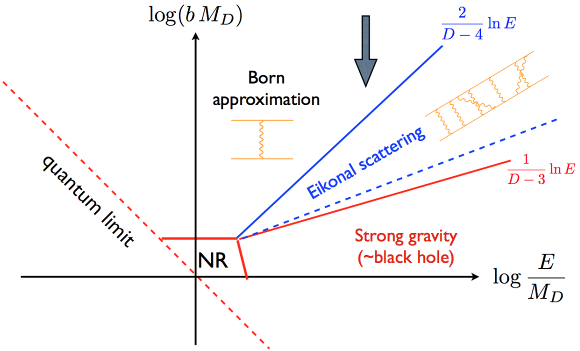
- More specifically, we will focus on $2 \to 2$ scattering in the high energy or *Regge* limit

$$s \gg |t|,$$

where s is the squared centre of mass energy, and $|t|$ the momentum transfer.

- Corresponds to scattering above the Planck scale in gravity.
- Naïvely, we might think that non-renormalisability is a problem.
- However, in this limit infinite numbers of *soft* gravitons are exchanged, and the results are well-behaved!

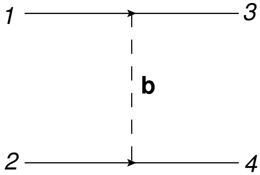
- Can consider different regions in impact parameter b (conjugate to $|t|$), and energy $E \sim \sqrt{s}$:



(see e.g. Giddings, Schmidt-Sommerfeld, Andersen).

- Next-to-soft corrections probe unknown parts of this diagram.

- It is possible to understand the gravity behaviour using QCD-like methods.
- Starting with QCD, there is a nice way to understand the Regge limit in terms of two Wilson lines separated by a transverse distance (Korchemsky, Korchemskaya).
- See also Balitsky; Caron-Huot.



- Take particles of mass m, such that

$$s \gg -t \gg m^2.$$

- $\boldsymbol{b}$ is the (2-d) impact parameter (distance of closest approach).

- Using known exponentiation properties of Wilson lines, one finds

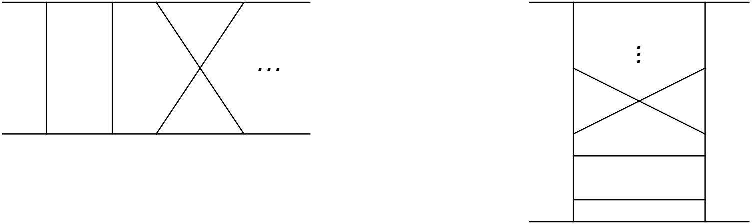
$$\mathcal{A} = \exp\left\{K\left[i\pi\mathbf{T}_s^2 + \mathbf{T}_t^2\log\left(\frac{s}{-t}\right)\right] + \ldots\right\}, \quad K = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\boldsymbol{b}^2)^\epsilon}{2\epsilon}$$

- There are two terms with non-trivial colour dependence:
  (I) A t-channel term: $\propto \mathbf{T}_t^2\log(\frac{s}{-t})$.
  (II) A pure *eikonal phase*: $\propto i\pi\mathbf{T}_s^2$.
- The former is responsible for *Reggeisation* of t-channel exchanges:

$$-\frac{i\eta_{\mu\nu}}{q^2} - > -\frac{i\eta_{\mu\nu}}{q^2}\left(\frac{s}{-t}\right)^\alpha$$

- The latter describes a spectrum of bound states (e.g. positronium).

- The eikonal phase comes from horizontal (crossed) ladder diagrams, whereas the Regge trajectory comes from vertical ladders.



(a) (b)

- In QCD, the vertical ladders dominate.
- It is known that horizontal ladders dominate in gravity: the eikonal phase is enhanced by a factor $s/(-t)$ w.r.t. the Reggeisation term.
- The Wilson line approach gives an elegant view on this.

- First, we need to find appropriate Wilson lines for gravity.
- Here, we mean specifically the operator describing soft graviton emission.
- The relevant quantity has appeared in various places (Brandhuber, Heslop, Spence, Travaglini; Naculich, Schnitzer; White):

$$\exp\left[\frac{i\kappa}{2}\int_{\mathcal{C}} ds\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,h_{\mu\nu}(x)\right].$$

- For straight line contours $x^{\mu} = x_0^{\mu} + p^{\mu}s$, this becomes

$$\exp\left[\frac{i\kappa}{2}\,p^{\mu}\,p^{\nu}\int_{\mathcal{C}} ds h_{\mu\nu}(x)\right].$$

- Closely related to its QCD counterpart!

- Carrying out the Wilson line Regge limit calculation in gravity gives (Melville, Naculich, Schnitzer, White)

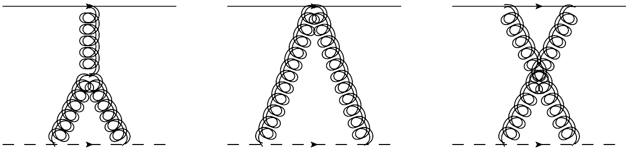
$$\mathcal{M} = \exp\left\{-K_g(\mu^2\boldsymbol{b}^2)^\epsilon\left[i\pi s + t\log\left(\frac{s}{-t}\right)\right] + \mathcal{O}(\epsilon^0)\right\},$$
$$K_g = \left(\frac{\kappa}{2}\right)^2\frac{\Gamma(1-\epsilon)}{8\pi^{2-\epsilon}}.$$

- The eikonal phase wins as $\frac{s}{-t}\to\infty$, in contrast to QCD.
- However, the structure of the result is basically the same, and can be obtained by the procedure

$$g_s \to \frac{\kappa}{2}; \quad \mathbf{T}^2_{s,t} \to s, t.$$

- This is the BCJ double copy! (see also Akhoury, Saotome; Sabio Vera, Campillo, Vazquez-Mozo, Johansson).

- Diagrammatic study of Regge limit by Akhoury, Saotome, Sterman.



- Considered a light particle scattering on a black hole.
- Next-to-soft corrections lead to a modifed eikonal phase:

$$\chi \to \chi_{\mathrm{E}} + \chi_{\mathrm{NE}}.$$

- Similar results from the Wilson line picture (Luna, Melville, Naculich, White).
- Correction corresponds to classical deflection angle of light particle (see also D'Appollonio, Di Vecchia, Russo, Veneziano; Bjerrum-Bohr, Donoghue, Holstein, Plante, Vanhove; Chi).

- More generally, how to get classical GR results from scattering amplitudes is an open problem.
- Huge amounts of recent attention, due to LIGO!
- The (next-to)soft expansion appears to be at least partially related to the $\hbar$ expansion (Kosower, Maybee, O'Connell).
- In any case, we would like to be able to classify the general structure of next-to-soft effects, including their universality (c.f. QCD).
- One way to do this is to use explicit (super-)gravity results at higher loop orders (Henn, Mistlberger; Bern, Ita, Parra-Martinez, Ruf; Abreu, Febres Cordero, Ita, Jaquier, Page, Ruf, Sotnikov).

- Recently, $2 \to 2$ graviton scattering was calculated at 3-loop order in $\mathcal{N} = 8$ SUGRA (Henn, Mistlberger).
- In SUGRA theories, the tree-level amplitude can be factored out.
- Given the known exponentiation of IR divergences, it is then conventional to write the amplitude as

$$A = A^{(0)} e^{\alpha_G \mathcal{A}^{(1)}} e^{F}, \quad \alpha_G = \frac{G_N}{\pi \hbar} (4\pi\hbar^2)^{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)},$$

  where the *remainder function* F starts at 2-loop order, and sbould be IR finite.
- This exponentiation takes place in momentum space.

- The remainder function calculated by Henn & Mistlberger contains terms of the form

$$F^{(2)} = 3\pi^2 s^2 \epsilon \zeta_3 + \mathcal{O}(\epsilon^2, s), \quad F^{(3)} = -\frac{2i}{3}\pi^3 s^3 \zeta_3 + \mathcal{O}(\epsilon, s^2)$$

  which are leading in the Regge limit.
- This was initially puzzling, given that all leading Regge contributions should be captured by the exponentiation of the one-loop result...
- ...which leaves no room for them in the remainder function!

A RESOLUTION

- To see what has gone wrong, note that the exponentiation of the eikonal phase happens in *position space*.
- This is different to exponentiating in momentum space, at subleading orders in ϵ.
- One may indeed show that exponentiating in position space before Fourier transforming correctly predicts the high energy behaviour of the remainder function (Di Vecchia, Luna, Naculich, Russo, Veneziano, White).
- Problem solved, but can we go further?
- E.g. can we test possible ansätze for corrections to the Regge limit, using the known remainder functions?

- A possible guess for the (momentum-space) amplitude up to first subleading power in $s/(-t)$ is (Di Vecchia, Naculich, Russo, Veneziano, White)

$$\frac{iA}{2s} = A^{(0)} \int d^{D-2}b e^{-ibq/\hbar} \left[(1 + 2i\Delta(s,b))\, e^{2i\delta(s,b)} - 1 \right].$$

- Here q is the momentum transfer, which is conjugate to the impact parameter b.
- The exponent $\delta(s,b)$ is a generalised eikonal phase, whose higher orders lead to corrections to the classical deflection angle.
- The function $\Delta(s,b)$ encodes quantum corrections, and does not have to be formally exponentiated.

- One may expand the previous formula perturbatively, and fix higher-order contributions to the functions δ and Δ.
- Once they have been fixed, one can then see if all terms in the remainder function of Henn & Mistlberger are correctly predicted, up to first subleading power in s/t.
- This does not quite work (Di Vecchia, Naculich, Russo, Veneziano, White)!
- Mismatch occurs at sub-sub-sub-leading level in ϵ.
- May be a subtle dimensional regularisation issue...
- ...so I will not speculate more for now.
- But the agreement is otherwise very impressive!

Results in other theories

- Recently, two-loop results for the $2 \to 2$ graviton amplitude have been presented in pure GR, and $\mathcal{N} \geq 4$ SUGRA (Bern, Ita, Parra-Martinez, Ruf).
- The correction to the classical deflection angle (i.e. subleading correction to δ) is found to be universal.
- There is no obvious explanation for this at present...
- ...so finding one would be interesting!
- It would also be nice to see how much structure of the results one can get right using the generalised eikonal ansatz.

- Next-to-soft corrections are relevant to transplanckian scattering in gravity...
- ... and scattering black holes.
- Corrections to the Regge limit allow us to systematically obtain the classical deflection angle, as well as predict lots of other structure in higher-loop amplitudes.
- How to go beyond the leading Regge limit is very much an open question in a variety of theories.

- (Next-to)-soft physics has a large number of applications, in different areas of physics.
- For hep-ph, hep-ex: increased precision for collider observables.
- For hep-th, gr-qc: transplanckian scattering in gravity, black hole scattering.
- Common languages for QCD and gravity (e.g. Wilson lines) make underlying structures / common behaviour clearer.

Open questions

- Can we resum next-to-leading power (NLP) threshold logs?
- Other applications in precision physics?
- How do we use next-to-soft corrections to get classical GR results...
- ...or other interesting bits of amplitudes?
- How exactly is this related to the $\hbar$ expansion?
- What else are gravitational Wilson lines useful for?

Thanks for listening!