

Implications of Conformal Symmetry for Scattering Amplitudes

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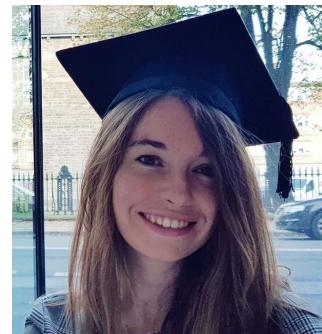
The conformal team



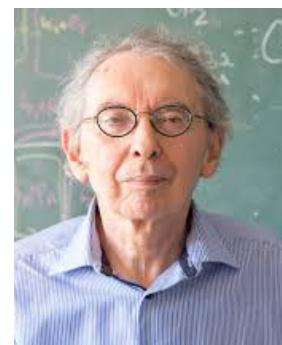
Dmitry Chicherin
MPI Munich



Johannes Henn
MPI Munich



Bláithín Power
LMU Munich



Emery Sokatchev
LAPTh Annecy



Edward Wang
École Polytechnique
Paris



JHEP 2002 (2020) 019, [arXiv:1911.12142](https://arxiv.org/abs/1911.12142)

PoS LL2018 (2018) 037, [arXiv:1807.06020](https://arxiv.org/abs/1807.06020)

Declaration of intent

Particle collisions at extremely high energies \Rightarrow the masses can sometimes be neglected

Symmetry enhancement: Poincaré \rightarrow Conformal group

Broad applications: gauge theories, Yukawa vertices, φ^4 ;
 φ^3 in D=6 dimensions

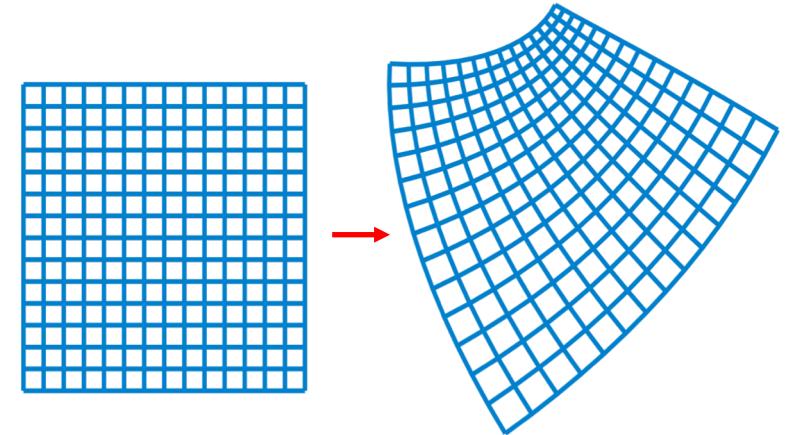
Most studies so far deal with correlation functions in position space
What are the consequences for on-shell scattering processes?

Plan of the talk

- Why is conformal symmetry difficult for scattering amplitudes?
- Why is it worth studying?
 - Calculation of certain loop integrals
 - Interplay with rational factors of loop amplitudes
 - Conformal symmetry of finite loop amplitudes
- What are the future research directions?

Conformal symmetry

In absence of dimensionful parameters,
the symmetry of the Lagrangian is enhanced



$$\text{Poincaré} + \begin{cases} \text{Dilatations } x^\mu \rightarrow \lambda x^\mu \\ \text{Conformal boosts } x^\mu \rightarrow \frac{x^\mu - b^\mu x^2}{1 - 2 b \cdot x + b^2 x^2} \end{cases} = \text{Conformal group}$$

Transformations which preserve causality

$$g^{\mu\nu} \rightarrow g'^{\mu\nu}(x') = \Omega^2(x) g^{\mu\nu}(x)$$

Conformal symmetry in position space

The generators are 1st order differential operators

$$\text{Conformal boosts} \quad K_\mu = x^2 \frac{\partial}{\partial x^\mu} - 2 x_\mu x^\nu \frac{\partial}{\partial x^\nu} - 2 \Delta x_\mu$$

Strong predictive power

E.g. 2- and 3-point correlation functions fixed up to a constant

$$\langle \varphi_1(x_1) \varphi_2(x_2) \varphi_3(x_3) \rangle = \frac{n_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} x_{31}^{\Delta_3 + \Delta_1 - \Delta_2}}$$

Conformal symmetry in position space

$$\langle \varphi_1(x_1) \dots \varphi_4(x_4) \rangle = f(u, v) \prod_{i < j}^4 x_{ij}^{\frac{\Delta}{3} - \Delta_i - \Delta_j} \quad \text{with } \Delta = \sum_{i=1}^4 \Delta_i$$

Cross ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2}$

The generator is 1st order

$$\begin{cases} K_\mu u = 0 \\ K_\mu v = 0 \end{cases} \xrightarrow{\text{red arrow}} K_\mu f(u, v) = 0$$

Conformal symmetry in momentum space

The generator of conformal boosts becomes **2nd order!**

Off-shell: $k_\mu = -p_\mu \frac{\partial}{\partial p^\nu} \frac{\partial}{\partial p_\nu} + 2 p^\nu \frac{\partial}{\partial p^\nu} \frac{\partial}{\partial p^\mu} + 2 (D - \Delta) \frac{\partial}{\partial p^\mu}$

On-shell: $k_{\alpha\dot{\alpha}} = \frac{\partial^2}{\partial \lambda^\alpha \partial \tilde{\lambda}^{\dot{\alpha}}} \quad [\text{Witten 2003}]$

Technical obstacle $k_\mu f(\{p_i\}) = 0 \rightarrow f = ???$

No notion of cross ratio

Importance to find ways to systematically construct conformal invariants

Quantum breaking of conformal symmetry

Tree-level amplitudes are conformally invariant $k_\mu M^{tree} = 0$

Two quantum sources of symmetry breaking:

- Ultraviolet effects $k_\mu M^{loop} \neq 0$
- Infrared (collinear/soft) effects

Goal: understand **how** conformal symmetry is broken

$$k_\mu M^{loop} = A_\mu$$

Let us start with finite loop amplitudes

Holomorphic anomaly

Even tree-level amplitudes are not “exactly” conformally invariant

$$M_{n;MHV}^{tree} = \frac{\langle ij \rangle^4 \delta^{(4)}(p_1 + \dots + p_n)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial^2}{\partial \lambda_i^\alpha \partial \tilde{\lambda}_i^{\dot{\alpha}}}$$

Holomorphic anomaly [Cachazo, Svrcek, Witten 2004]

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} \frac{1}{\langle \lambda \chi \rangle} = 2\pi \tilde{\chi}_{\dot{\alpha}} \delta(\langle \lambda \chi \rangle) \delta([\tilde{\lambda} \tilde{\chi}]) \quad \leftarrow \quad \frac{\partial}{\partial \bar{z}} \frac{1}{z} = \pi \delta^2(z)$$

Anomaly of tree amplitudes localised on collinear configurations

[Beisert et al. 2009]

Studied at the level of cuts of loop amplitudes

[Korchemsky, Sokatchev 2009] [Bargheer, Beisert, Galleas, Loebert, McLoughlin 2009]
[Cachazo 2004] [Cachazo, Svrcek, Witten 2004]

Conformal anomaly of finite loop integrals

[Chicherin, Sokatchev 2018]

Consider φ^3 theory in $D = 6$

$$\langle \varphi(q)\varphi(-p-q)|\varphi(p)\rangle_{tree} = \begin{array}{c} (p+q)^2 \neq 0 \\ p^2 = 0 \\ q^2 \neq 0 \end{array}$$

$$k_\mu \frac{1}{(q^2 + i\epsilon)((p+q)^2 + i\epsilon)} = 0 \quad ?$$

Conformal anomaly of finite loop integrals

[Chicherin, Sokatchev 2018]

Consider φ^3 theory in $D = 6$

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$$k_\mu \frac{1}{(q^2 + i\epsilon)((p+q)^2 + i\epsilon)} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1-\xi) \delta^{(6)}(q + \xi p)$$

Contact-type anomaly localized on collinear configuration $q \sim p$

Conformal symmetry seems to be broken beyond repair at loop level

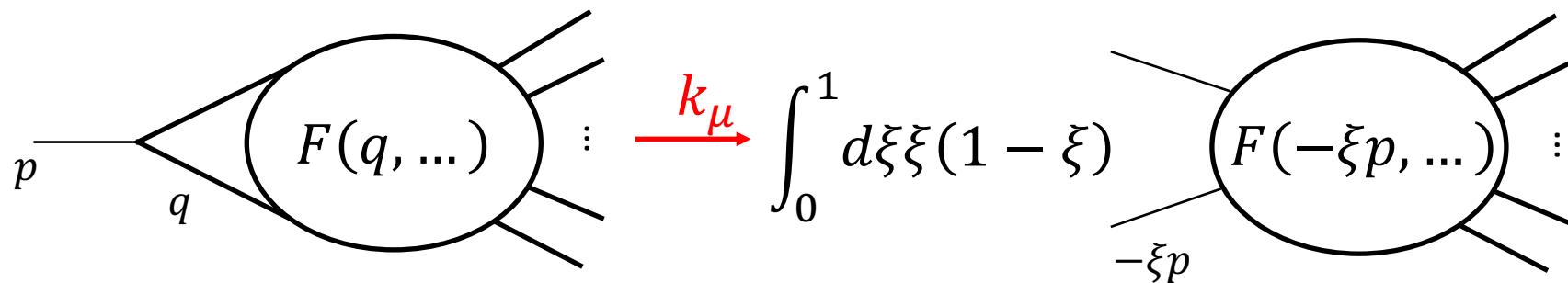
Even finite loop amplitudes/integrals are in general not conformal!



Anomalous conformal Ward identities

[Chicherin, Sokatchev 2018]

The contact term localizes the loop integration



System of inhomogeneous 2nd order PDE

$$k^\mu I^{(\ell,n)} = \sum_i p_i^\mu \int_0^1 d\xi I_i^{(\ell-1,n+1)}(\xi)$$

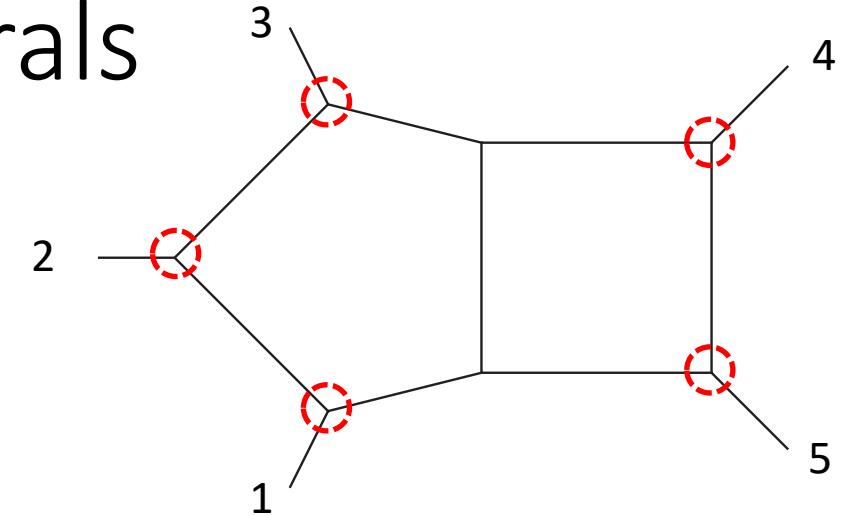
Predictive power!

Bootstrap of multi-loop integrals

Example: 6D scalar penta-box

Five-particle massless scattering: 31-letter
alphabet of pentagon functions

[Gehrmann, Henn, Lo Presti 2015][Chicherin, Henn, Mitev 2018]



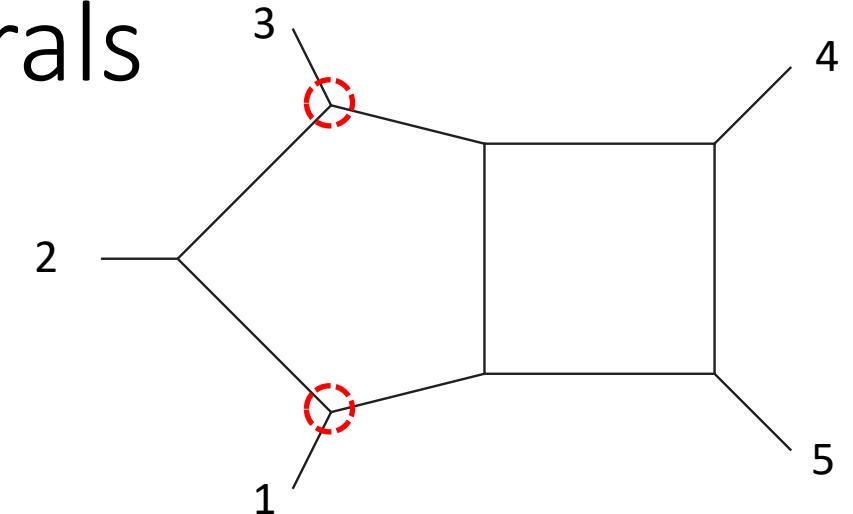
Bootstrap of multi-loop integrals

Example: 6D scalar penta-box

Five-particle massless scattering: 31-letter
alphabet of pentagon functions

[Gehrmann, Henn, Lo Presti 2015][Chicherin, Henn, Mitev 2018]

$$(n \cdot k) \mathcal{S}[I_{PB}] = \underbrace{(n \cdot p_1)A_1 + (n \cdot p_3)A_3}_{\text{Weight-3}}$$



$$n \cdot p_i = 0 \forall i = 2,4,5$$

[SZ, Proceedings of
Loops and Legs 2018]

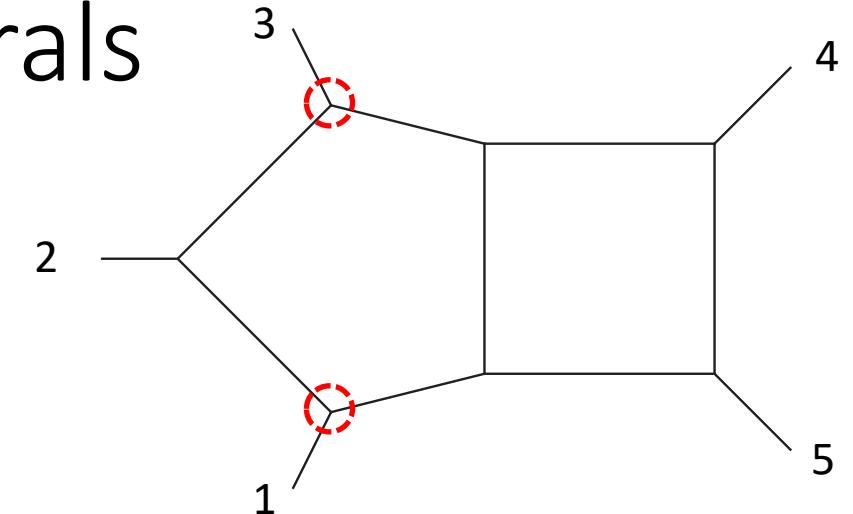
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Ansatz in terms of weight-5 integrable symbols

$$\mathcal{S}[I_{PB}] = \frac{1}{\sqrt{\Delta}} \sum_{i_1, \dots, i_5} c_{i_1 \dots i_5} (W_{i_1} \otimes \dots \otimes W_{i_5})$$

Entirely fixed by just one
projection $n \cdot k!$

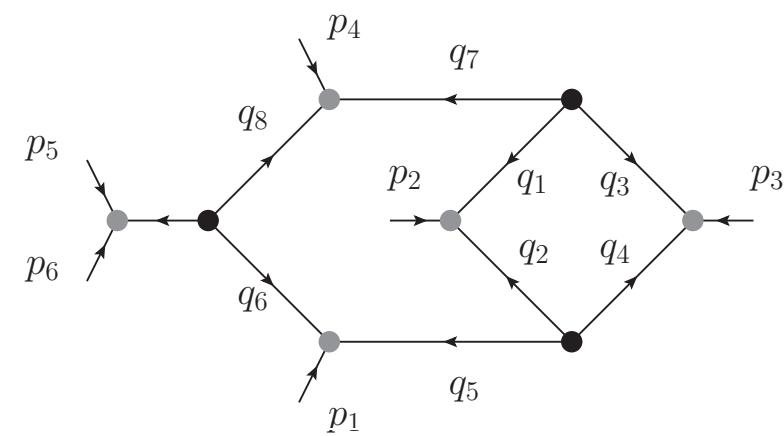
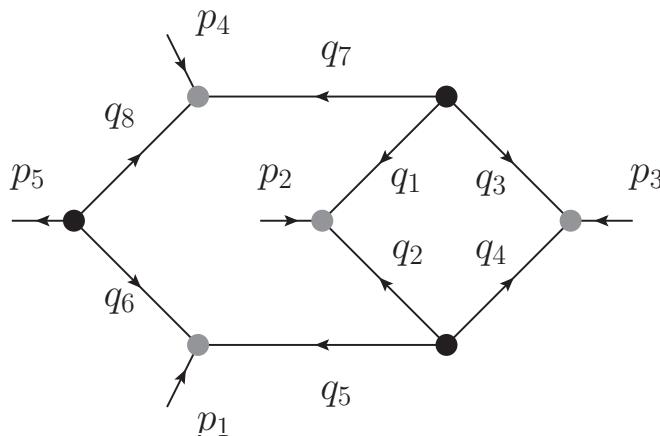
A supersymmetric detour

[Chicherin, Henn, Sokatchev 2018 x2]

4D Wess-Zumino model of massless $\mathcal{N} = 1$ supersymmetric matter

Superconformal Ward identities \Rightarrow 1st order PDE

They can be solved directly, no assumptions needed



Conformal anomaly for finite amplitudes

φ^4 and Yukawa in D=4 dimensions, φ^3 in D=6

[Chicherin, Sokatchev 2018]

Extension to **gauge theories** is complicated

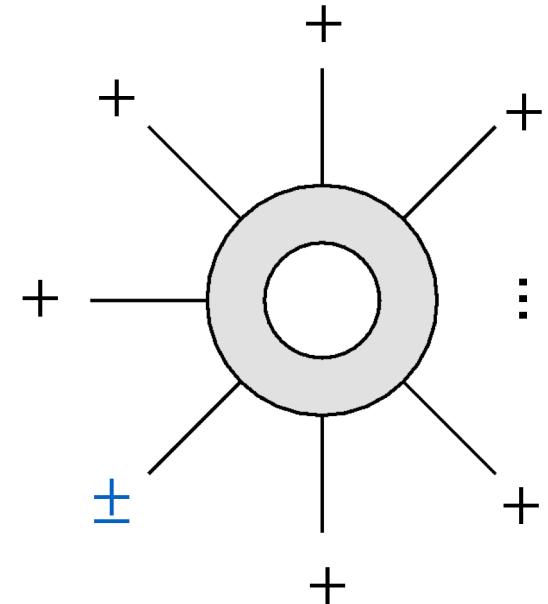
- Gauge invariance forces to work with sums of diagrams
- Most of the diagrams with external gluons suffer from infrared divergences

First step: study finite amplitudes in Yang-Mills theory/QCD

Finite loop amplitudes in QCD

The **all-plus** and **single-minus** amplitudes in QCD

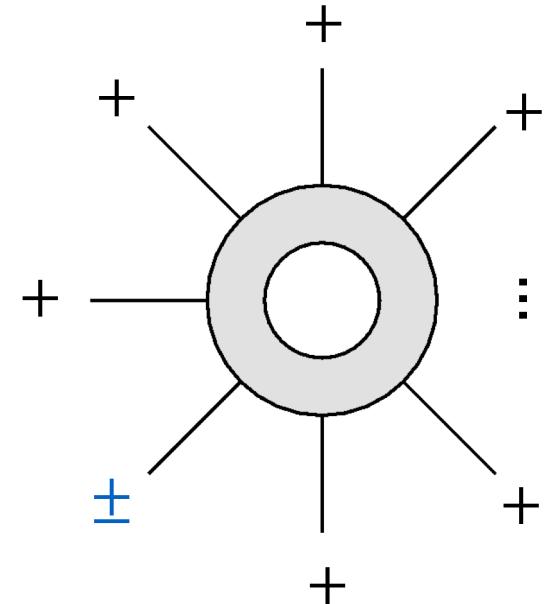
- vanish @ tree-level
- are finite and rational @ 1-loop



Finite loop amplitudes in QCD

The **all-plus** and **single-minus** amplitudes in QCD

- vanish @ tree-level
- are finite and rational @ 1-loop



The one-loop **all-plus** amplitudes are **conformally invariant**

The one-loop **single-minus** amplitudes are **not conformally invariant**

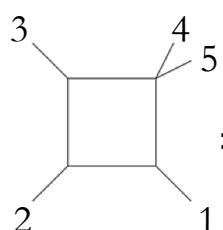
Five-gluon all-plus amplitude @ 2-loop

Permutations
of external legs

Color $SU(N_c)$

$$\kappa = \frac{d_s - 2}{6} \text{ with } d_s = g^\mu_\mu \text{ gluon spin dimension}$$

$$\begin{aligned} \mathcal{H}_{\text{double trace}}^{(2)} &= \sum_{S_5/\Sigma} \text{Tr}(12)[\text{Tr}(345) - \text{Tr}(543)] \sum_{\Sigma} \left\{ 6 \kappa^2 \left[\frac{\langle 24 \rangle [14][23]}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} + 9 \frac{\langle 24 \rangle [12][23]}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle^2} \right] \right. \\ &\quad \left. + \kappa \frac{[15]^2}{\langle 23 \rangle \langle 34 \rangle \langle 42 \rangle} \left[\begin{array}{c} 4 \\ & 1 \\ & 5 \\ 3 & & 2 \\ & & 4 \end{array} \right] + \left[\begin{array}{c} 4 \\ & 1 \\ & 5 \\ 3 & & 2 \\ & & 2 \end{array} \right] - \left[\begin{array}{c} 5 \\ & 1 \\ & 5 \\ 3 & & 4 \\ & & 2 \end{array} \right] - \left[\begin{array}{c} 5 \\ & 1 \\ & 5 \\ 3 & & 4 \\ & & 3 \end{array} \right] - \left[\begin{array}{c} 4 \\ & 1 \\ & 2 \\ 3 & & 5 \\ & & 4 \end{array} \right] - \left[\begin{array}{c} 4 \\ & 1 \\ & 2 \\ 3 & & 3 \\ & & 4 \end{array} \right] \right] \right\} \end{aligned}$$



$$= \text{Li}_2 \left(1 - \frac{s_{12}}{s_{45}} \right) + \text{Li}_2 \left(1 - \frac{s_{23}}{s_{45}} \right) + \log^2 \left(\frac{s_{12}}{s_{23}} \right) + \frac{\pi^2}{6}$$

[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, SZ 2019]

[Dunbar, Godwin, Perkins, Strong 2019]

Hints of conformal symmetry in the polylogarithmic part of the amplitude

$$+ \kappa \frac{[15]^2}{\langle 23 \rangle \langle 34 \rangle \langle 42 \rangle} \left[\begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ - \\ \text{Diagram 3} \\ - 4 \\ \text{Diagram 4} \\ - 4 \\ \text{Diagram 5} \\ - 4 \\ \text{Diagram 6} \end{array} \right]$$

The diagrams are six 4-point Feynman-like graphs with indices 1 through 5 around them. Diagram 1 has indices 4, 1, 5, 3 clockwise; Diagram 2 has 1, 5, 4, 2; Diagram 3 has 1, 5, 5, 2; Diagram 4 has 1, 2, 4, 3; Diagram 5 has 1, 2, 5, 3; Diagram 6 has 1, 2, 2, 4.

Manifestly conformally invariant rational factors

$$k_{\alpha\dot{\alpha}} \frac{[45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} = 0$$

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^5 \frac{\partial^2}{\partial \lambda_i^\alpha \partial \tilde{\lambda}_i^{\dot{\alpha}}}$$

These rational factors can be obtained via 4D unitarity

An alternative way to the hard function

[Dunbar, Godwin, Perkins, Strong 2019]

$$\mathcal{H}_n^{(2;\lambda)} = P_n^{(2;\lambda)} + R_n^{(2;\lambda)}$$

```
graph TD; A["4D unitarity cuts"] --> B["Polylogarithmic"]; C["Recursion relations"] --> D["Rational"]; B --> E["H_n^(2;lambda) = P_n^(2;lambda) + R_n^(2;lambda)"]; D --> E;
```

The diagram illustrates the decomposition of the hard function $\mathcal{H}_n^{(2;\lambda)}$ into its components. The equation $\mathcal{H}_n^{(2;\lambda)} = P_n^{(2;\lambda)} + R_n^{(2;\lambda)}$ is at the top. Below it, two red arrows point upwards from the labels "Polylogarithmic" and "Rational". The label "Polylogarithmic" is connected by a red arrow from the label "4D unitarity cuts" located below it. The label "Rational" is connected by a red arrow from the label "Recursion relations" located below it.

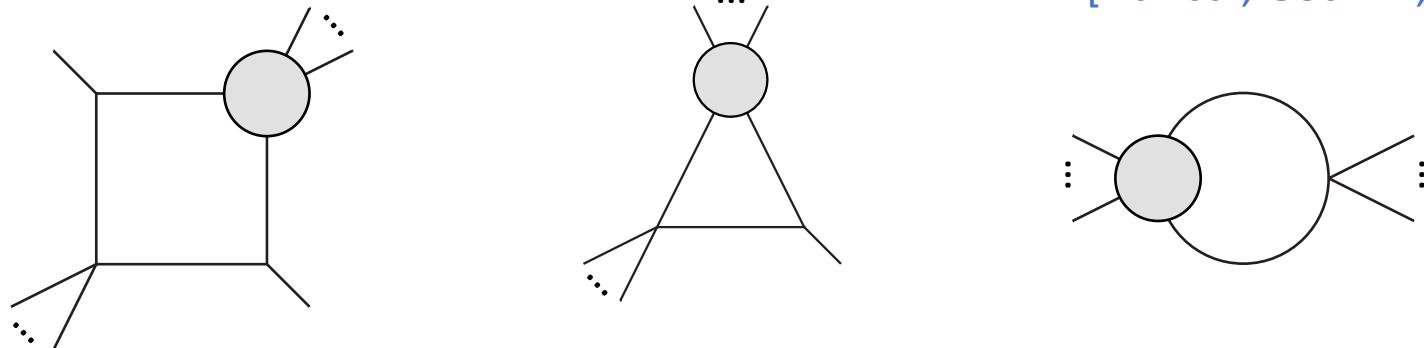
Four-dimensional unitarity – The all-plus case

The all-plus amplitude vanishes @ tree-level

- ⇒ It is finite and rational @ 1-loop in $D = 4$ dimensions
- Effectively it becomes an extra on-shell vertex

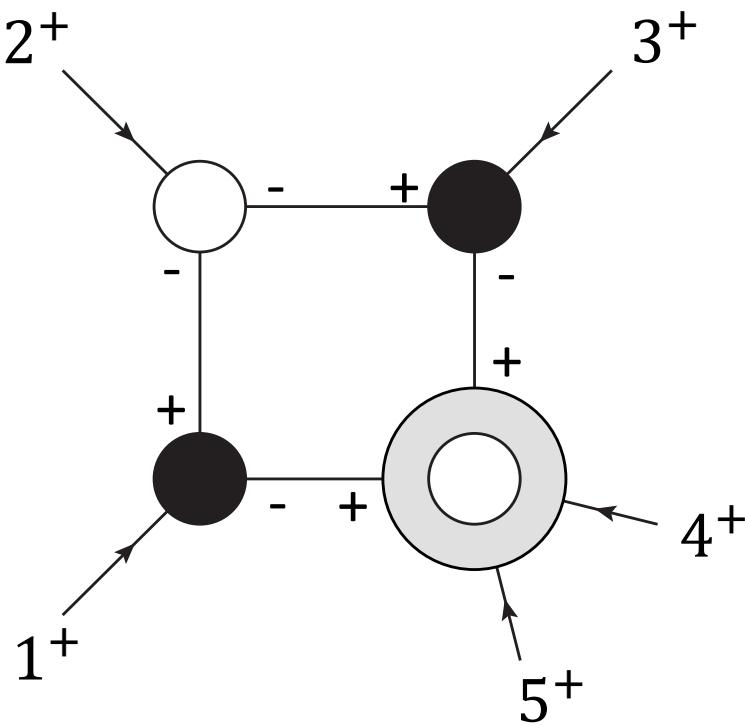
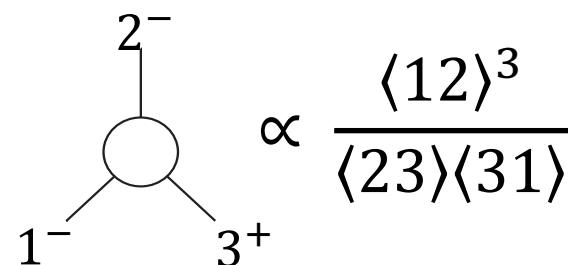
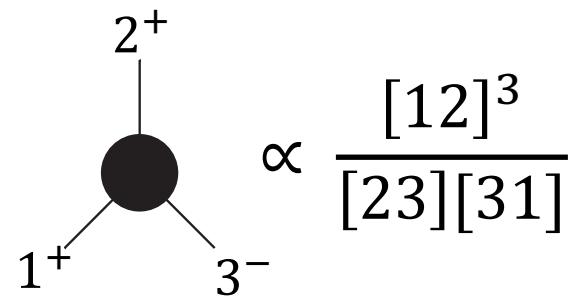
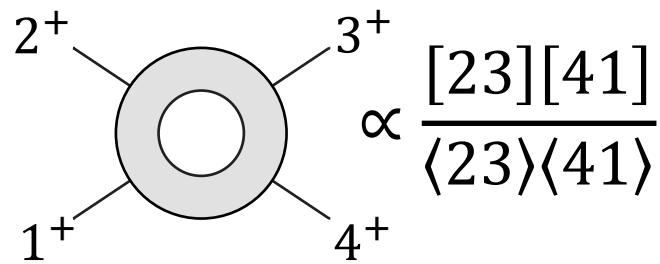
Huge simplification: the 2-loop cuts become 1-loop cuts with a single insertion of this vertex

[Dunbar, Perkins 2016]
[Dunbar, Godwin, Perkins, Strong 2019]



Only the **quadruple cuts** contribute to the polylogarithmic part

The quadruple cut with on-shell diagrams



The quadruple cut with on-shell diagrams

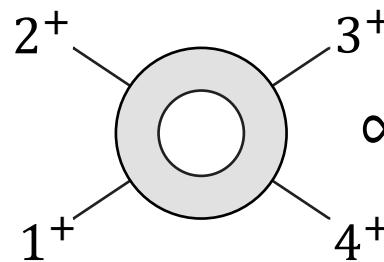


Diagram showing a shaded loop with four external lines labeled 1^+ , 2^+ , 3^+ , and 4^+ .

$$\propto \frac{[23][41]}{\langle 23 \rangle \langle 41 \rangle} = \frac{[23]^2}{\langle 41 \rangle^2}$$

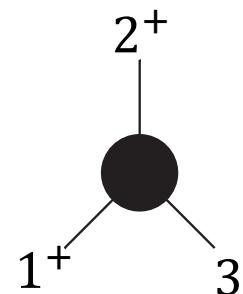


Diagram showing a black loop with three external lines labeled 1^+ , 2^+ , and 3^- .

$$\propto \frac{[12]^3}{[23][31]}$$

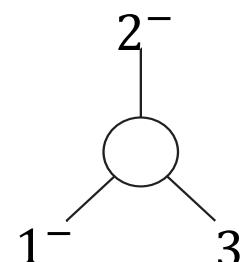
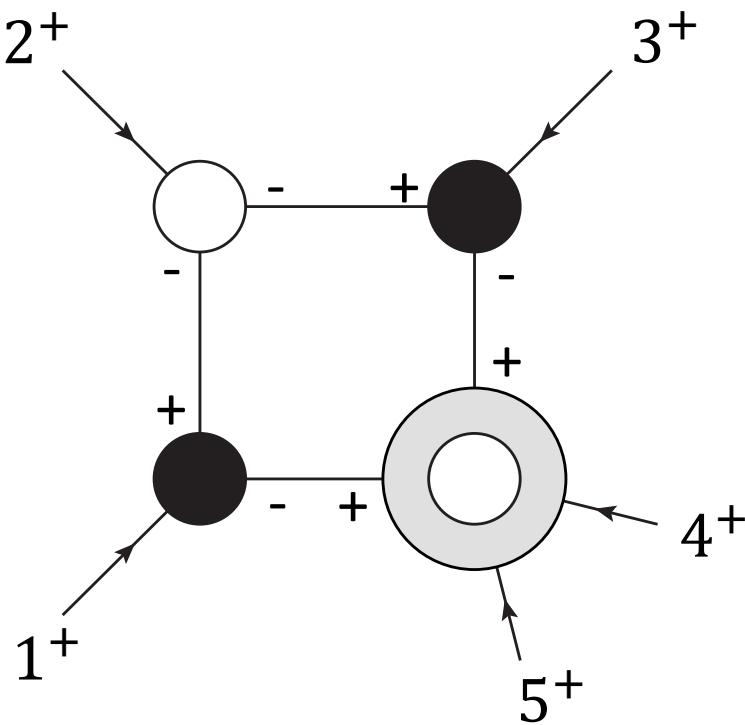
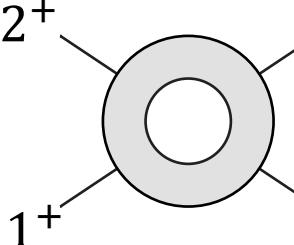


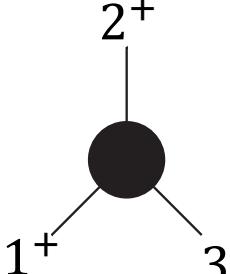
Diagram showing an empty loop with three external lines labeled 1^- , 2^- , and 3^+ .

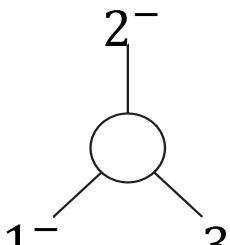
$$\propto \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$



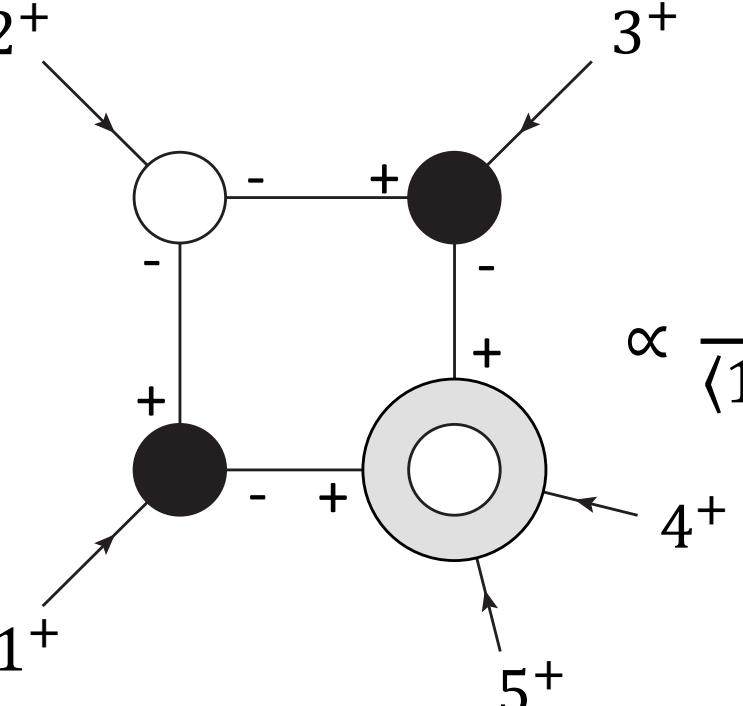
The quadruple cut with on-shell diagrams


$$\propto \frac{[23][41]}{\langle 23 \rangle \langle 41 \rangle} = \frac{[23]^2}{\langle 41 \rangle^2}$$


$$\propto \frac{[12]^3}{[23][31]}$$

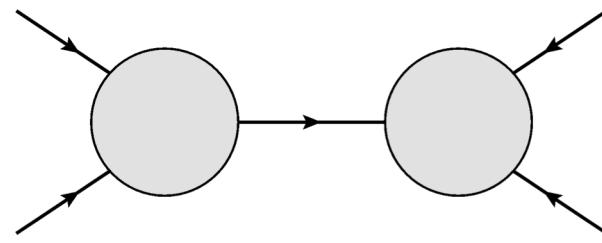

$$\propto \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

All the vertices are manifestly conformally invariant, and so is the quadruple cut!

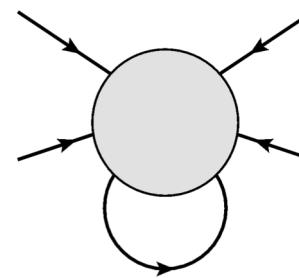

$$\propto \frac{[45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

Conformal invariance of on-shell diagrams

All on-shell diagrams can be obtained by iteration of two operations



Factorisation channel



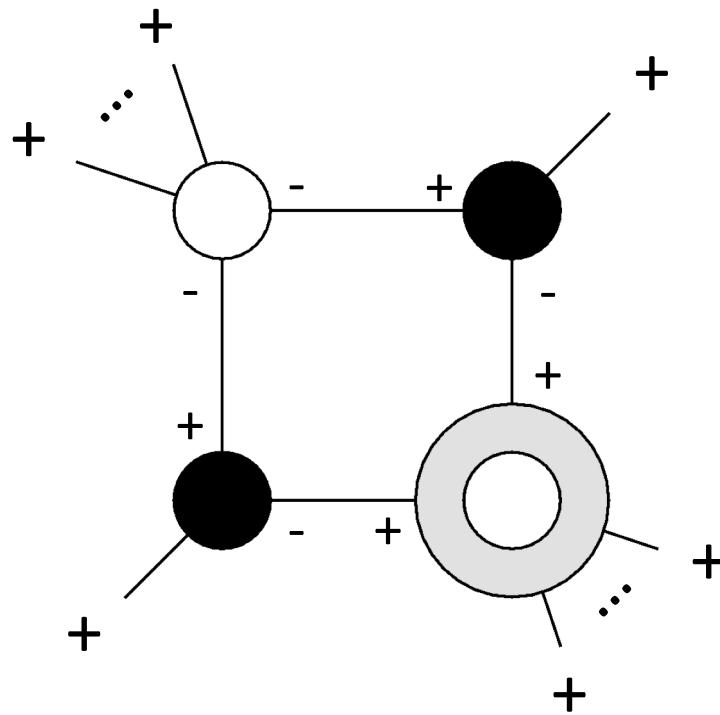
Forward limit

They preserve conformal symmetry!

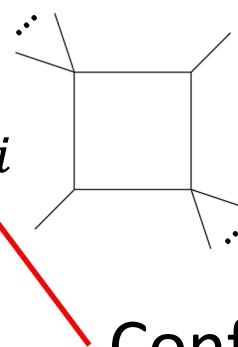
Systematic way of constructing higher-multiplicity conformal invariants

Rational factors for any n @ 2-loop

If the one-loop all-plus amplitudes are conformally invariant for any number of gluons n ,



$$P_n^{(2;\lambda)} \sim \sum_i r_i$$



Conformally invariant!

Confirmed by

[Dunbar, Perkins, Strong 2020]

[Dalgleish, Dunbar, Perkins, Strong 2020]

All- n formula for the 1-loop all-plus amplitude

$$A_n^{(1,1)} = -\frac{i}{3} \left(1 - \frac{n_f}{N_c}\right) \frac{\sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \langle i_1 i_2 \rangle [i_2 i_3] \langle i_3 i_4 \rangle [i_4 i_1]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

[Bern, Dixon, Kosower 1993] [Mahlon 1994] [Bern, Chalmers, Dixon, Kosower 1994]

$$A_5^{(1,1)} = \frac{i}{6} \left(1 - \frac{n_f}{N_c}\right) \frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + \text{Tr}(\gamma_5 p_1 p_2 p_3 p_4)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

[Bern, Dixon, Kosower 1993]

Not manifestly conformally invariant

Manifestly conformally invariant form of the all-plus amplitudes @ 1-loop

[Henn, Power, SZ 2020]

$$A_n^{(1,1)}(1^+, \dots, n^+) = -\frac{i}{3} \left(1 - \frac{n_f}{N_c}\right) \sum_{k=3}^{n-1} \sum_{m=k+1}^n c_{kmn}$$

$$c_{kmn} = - \frac{\langle 1 | x_{1k} x_{km} | 1 \rangle^2}{PT(1,2,\dots,k-1)PT(1,k,k+1,\dots,m-1)PT(1,m,m+1,\dots,n)}$$

$$PT(i_1, i_2, \dots, i_n) = \langle i_1 i_2 \rangle \langle i_2 i_3 \rangle \dots \langle i_{n-1} i_n \rangle \langle i_n i_1 \rangle$$

$$x_{ab} = p_a + p_{a+1} + \dots + p_{b-1}$$

A similar formula appeared previously
in the context of string amplitudes

[Mafra, Schlotterer 2014]

[He, Monteiro, Schlotterer 2016]

Conformal invariance term by term

$$A_n^{(1,1)}(1^+, \dots, n^+) = -\frac{i}{3} \left(1 - \frac{n_f}{N_c}\right) \sum_{k=3}^{n-1} \sum_{m=k+1}^n C_{kmn} \quad \xrightarrow{\text{red arrow}} \boxed{k_{\alpha\dot{\alpha}} C_{kmn} = 0}$$

(up to contact terms)

$$C_{kmn} = - \frac{\langle 1 | x_{1k} x_{km} | 1 \rangle^2}{PT(1,2,\dots,k-1)PT(1,k,k+1,\dots,m-1)PT(1,m,m+1,\dots,n)}$$

The price to pay:

- Not manifestly permutation invariant
- Spurious poles $\langle ij \rangle = 0$

A few explicit examples

$$A_4^{(1,1)} \propto \frac{[23]^2}{\langle 14 \rangle^2}$$

$$A_5^{(1,1)} \propto \frac{[45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} + \frac{[23]^2}{\langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} + \frac{[52]^2}{\langle 41 \rangle \langle 13 \rangle \langle 34 \rangle}$$

$$\begin{aligned} A_6^{(1,1)} &\propto \frac{[23]^2}{\langle 14 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} + \frac{[26]^2}{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} + \frac{[56]^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\ &- \frac{\langle 1|p_3 + p_4|2]^2}{\langle 13 \rangle \langle 34 \rangle \langle 41 \rangle \langle 15 \rangle \langle 56 \rangle \langle 61 \rangle} - \frac{\langle 1|p_5 + p_6|4]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 15 \rangle \langle 56 \rangle \langle 61 \rangle} - \frac{\langle 1|p_2 + p_3|6]^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 14 \rangle \langle 45 \rangle \langle 51 \rangle} \end{aligned}$$

More and more complicated objects appear at higher multiplicity

[Preliminary study in Edward Wang's BSc. Thesis]

BCFW recursion

[Britto, Cachazo, Feng, Witten 2005]

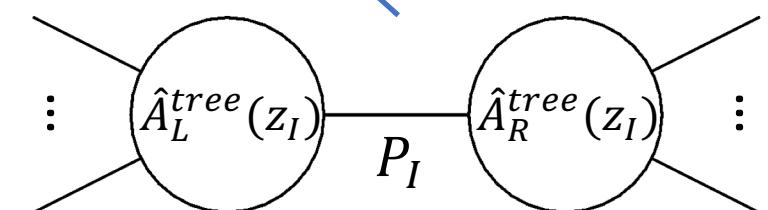
Complex shift of the momenta:

$$\begin{aligned}\lambda_i &\rightarrow \lambda_i + z \lambda_j \\ \tilde{\lambda}_j &\rightarrow \tilde{\lambda}_j - z \tilde{\lambda}_i\end{aligned}$$

Use analytic properties of the tree amplitude as a rational function of the shift parameter:

$$A_n^{tree} = \hat{A}_n^{tree}(0) = \oint_{C_0} dz \frac{\hat{A}_n^{tree}(z)}{z} = C_n^\infty - \sum_I \text{Res} \left[\frac{\hat{A}_n^{tree}(z)}{z}, z_I \right]$$

Residues factorise into lower-point tree-level amplitudes \Rightarrow **Recursion**



BCFW recursion at loop level

[Bern, Dixon, Kosower 2005]

- Loop amplitudes are in general not rational
 \Rightarrow The one-loop all-plus amplitudes are rational
- The one-loop all-plus and all-minus splitting function introduce double poles
 \Rightarrow They are removed by vanishing tree amplitudes
- There is in general no BCFW shift that removes the pole at infinity

$$A_n^{(1)} = \mathcal{C}_n^\infty + \sum_{\ell=0,1} \sum_{j,\sigma} A_{n_j}^{(1-\ell)\sigma}(z=z_j) \frac{i}{P_j^2} A_{n+2-n_j}^{(\ell)-\sigma}(z=z_j)$$

BCFW shift and conformal symmetry

It is possible to find more refined shifts s.t. $\lim_{z \rightarrow \infty} \hat{A}(z) = 0$

$$\text{E.g. } \tilde{\lambda}_j \rightarrow \tilde{\lambda}_j - z \tilde{\lambda}_l - z \frac{\langle nj \rangle}{\langle lj \rangle} \tilde{\lambda}_n, \quad \lambda_l \rightarrow \lambda_l + z \lambda_j, \quad \lambda_n \rightarrow \lambda_n + z \frac{\langle nj \rangle}{\langle lj \rangle} \lambda_j$$

[Bern, Dixon, Kosower 2005]

Such a shift breaks the manifest conformal invariance of the separate terms in the recursion

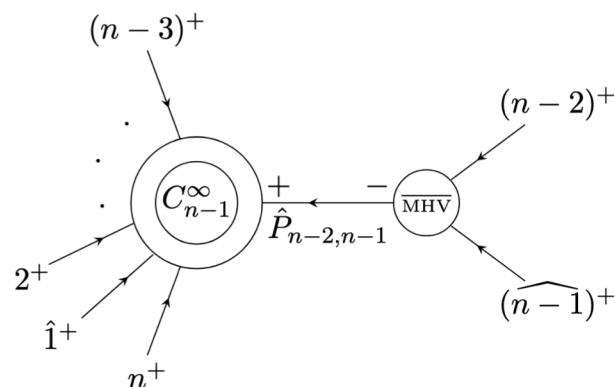
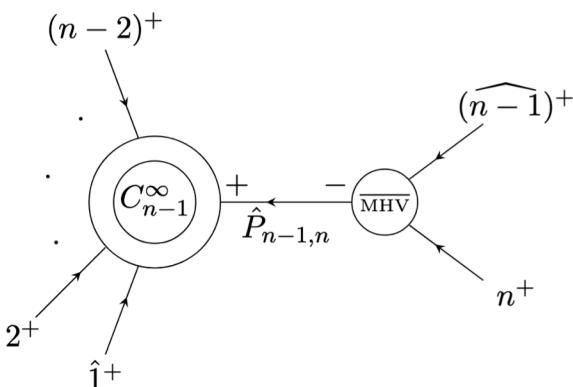
The ordinary BCFW shift preserves conformal symmetry

$$\lambda_i \rightarrow \lambda_i + z \lambda_j, \quad \tilde{\lambda}_j \rightarrow \tilde{\lambda}_j - z \tilde{\lambda}_i \quad [\text{Mason, Skinner 2010}]$$

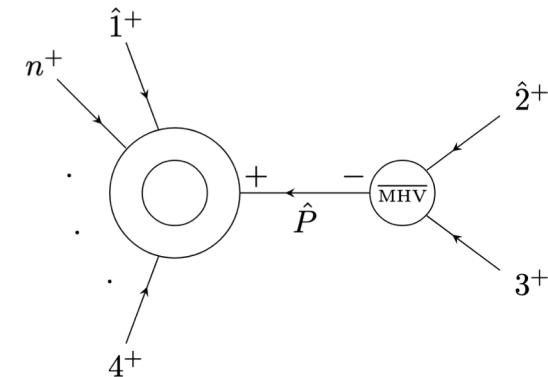
Two BCFW shifts to capture the surface term

$$A_n^{(1)} = \mathcal{C}_n^\infty + \sum_{\ell=0,1} \sum_{j,\sigma} A_{n_j}^{(1-\ell)\sigma}(z = z_j) \frac{i}{P_j^2} A_{n+2-n_j}^{(\ell)-\sigma}(z = z_j)$$

$$\begin{aligned}\lambda_{n-1} &\rightarrow \lambda_{n-1} + w\lambda_1 \\ \tilde{\lambda}_1 &\rightarrow \tilde{\lambda}_1 - w\tilde{\lambda}_{n-1}\end{aligned}$$



$$\begin{aligned}\lambda_2 &\rightarrow \lambda_2 + z\lambda_1 \\ \tilde{\lambda}_1 &\rightarrow \tilde{\lambda}_1 - z\tilde{\lambda}_2\end{aligned}$$



All terms in the recursion are manifestly conformally invariant!

Dual conformal symmetry

Similarity with formulas for NMHV super-amplitudes in $N = 4$ sYM

The latter enjoy **dual conformal symmetry**

[Drummond, Henn, Korchemsky, Sokatchev 2008] [Drummond, Henn, Plefka 2009]

Dual CS = ordinary CS in dual variables $p_i = x_i - x_{i+1}$

Infinitesimal generator $K^{\alpha\dot{\alpha}} = \sum_{i=1}^n \left(x_i^{\alpha\dot{\beta}} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^\beta} \right)$

Infinitesimal variation $\delta_b = b \cdot K$

1st order

The 1-loop 4-gluon all-plus amplitude is dual conformally invariant

$$A_4^{(1,1)} \propto \frac{[23]^2}{\langle 14 \rangle^2}$$

$$\longrightarrow \delta_b A_4^{(1,1)} = 0$$

$$\delta_b \langle i \ i + 1 \rangle = 2 (b \cdot x_i) \langle i \ i + 1 \rangle$$

$$\delta_b [i \ i + 1] = 2 (b \cdot x_{i+2}) [i \ i + 1]$$

Does dual conformal symmetry propagate to $n > 4$?

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Does dual conformal symmetry propagate to $n > 4$? Sort of.

Dual conformal symmetry is lost for $n > 4$

$$C_{kmn} = - \frac{\langle 1|x_{1k}x_{km}|1\rangle^2}{PT(1,2,\dots,k-1)PT(1,k,k+1,\dots,m-1)PT(1,m,m+1,\dots,n)}$$

Projecting by $|1\rangle$ breaks the symmetry:

$$\delta_b \langle 1|x_{1k}x_{km}|1\rangle = \underbrace{2 b \cdot (x_1 + x_k + x_m) \langle 1|x_{1k}x_{km}|1\rangle}_{\text{Covariant}} - \underbrace{\langle 1|x_{1k}x_{km}x_{m1}b|1\rangle}_{\text{Non-covariant}}$$

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Vanishes if $b \propto p_1$

Directional dual conformal covariance!

Directional dual conformal symmetry

[Bern, Enciso, Ita, Zeng 2017] [Bern, Enciso, Shen, Zeng 2018] [Chicherin, Henn, Sokatchev 2018]

$$\delta_{\mathbf{p}_1} C_{kmn} = 2 p_1 \cdot \left(2x_1 + x_k + x_m - \sum_{i=1}^n x_i \right) C_{kmn}$$

Directional dual conformal symmetry

[Bern, Enciso, Ita, Zeng 2017] [Bern, Enciso, Shen, Zeng 2018] [Chicherin, Henn, Sokatchev 2018]

$$\delta_{\mathbf{p}_1} C_{kmn} = 2 p_1 \cdot \left(2x_1 + x_k + x_m - \sum_{i=1}^n x_i \right) C_{kmn}$$

Why is p_1 special? The amplitude is **permutation invariant!**

Directional dual conformal symmetry

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$$\delta_{\mathbf{p}_1} C_{kmn} = 2 p_1 \cdot \left(2x_1 + x_k + x_m - \sum_{i=1}^n x_i \right) C_{kmn}$$

Why is p_1 special? The amplitude is **permutation invariant!**

It is not. The preferred direction is an artefact of our representation

Unfortunately, each summand has a different weight

⇒ The full amplitude has no clear symmetry

Summary of this part

- The one-loop all-plus amplitude is conformally invariant
- It exhibits certain signs of directional dual conformal symmetry

Obvious outlook: why is the **single-minus** one-loop amplitude **not** conformally invariant?

$$A_4^{(1,1)}(1^-, 2^+, 3^+, 4^+) \propto \frac{[24]^2(s+t)}{[12]\langle 23\rangle\langle 34\rangle[41]} \quad \xrightarrow{\hspace{1cm}} \quad k_\mu A_4^{(1,1)}(1^-, 2^+, 3^+, 4^+) \neq 0$$

[Bern, Kosower 1992]

Unique chance to understand how conformal symmetry is broken in a finite gauge-theory amplitude

What about divergent amplitudes?

Two quantum sources of symmetry breaking: UV and IR effects

UV breaking of conformal symmetry is regulated by the β function

One possibility: study the theory at the **fixed point** $\beta(g^*) = 0$

[Braun, Korchemsky, Müller 2003] [Braun, Manashov, Moch, Strohmaier 2017]

E.g.

$$\langle \varphi(x_1)\varphi(x_2) \rangle = N_2(g^*)(\mu^*)^{-2\gamma(\mu^*)} \left(\frac{1}{(x_1 - x_2)^2} \right)^{\ell^{can} + \gamma(\mu^*)}$$

What are the implications for **scattering amplitudes**?

Revealing dual conformal symmetry in planar $\mathcal{N} = 4$ sYM @ loop level

$\mathcal{N} = 4$ sYM is UV finite, but has IR divergences

Breakdown of dual superconformal symmetry in planar $\mathcal{N} = 4$ sYM is well understood

The “remainder function” is finite $\mathcal{R}_{k,n} = \frac{\mathcal{A}_n^{N^k MHV}}{\mathcal{A}_n^{ABDK/BDS}}$

[Anastasiou, Bern, Dixon, Kosower 2003]

[Bern, Dixon, Smirnov 2005]

and exactly dual conformal @ all loop orders $K^{\alpha\dot{\alpha}}\mathcal{R}_{k,n} = 0$

[Drummond, Henn, Korchemsky, Sokatchev 2008]

A conformal-friendly hard function?

IR and UV divergences factorise in a well-understood way

$$\mathcal{A} = Z_{UV} Z_{IR} \mathcal{A}_f$$

The definition of the finite part \mathcal{A}_f is not unique

Is there a suitable definition of a hard function that has simple transformation rules under conformal symmetry?

Summary

Studying conformal symmetry in momentum space is complicated, but rewarding

- Calculation of certain **loop integrals**
- Interplay with **on-shell diagrams** and **BCFW recursion**
- Conformal symmetry of the **one-loop all-plus amplitude**

Summary

Studying conformal symmetry in momentum space is complicated, but rewarding

- Calculation of certain **loop integrals**
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There is still a lot to uncover!

- Extend the study of conformal symmetry breaking in finite amplitudes to gauge theories (**one-loop single-minus amplitude?**)
- Is there a “**conformal-friendly**” definition of **hard function**?