

# Multi-Regge Limit of the Non-Planar Two-Loop Five-Particle Amplitudes in $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ Supergravity

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Based on work in collaboration with  
S. Caron-Huot, J. Henn, Y. Zhang, S. Zoia

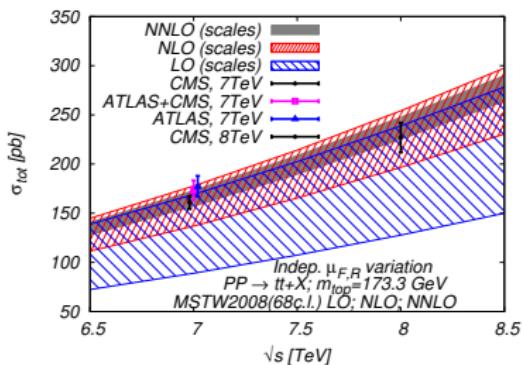
4<sup>th</sup> March 2020, Paris

# Outline

- multi-particle production at the LHC
- Two-loop five-particle massless scattering
- Beyond the symbol of the  $\mathcal{N} = 4$  super-Yang-Mills amplitude
- Pentagon functions: analytics and numerics
- Pentagon functions in the multi-Regge asymptotics
- Non-planar *ergo* Non-single-valued
- A quick look at the  $\mathcal{N} = 8$  sugra amplitude

# Towards an era of precision collider measurements

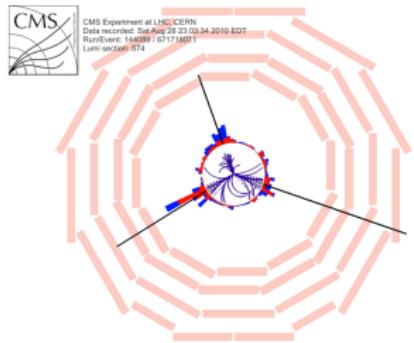
- Proton-proton collisions at LHC, huge background of strong-interaction processes
- High-energy scattering of hadrons is described by perturbative QCD



- Leading Order (LO) – large ambiguities
- Next-to-LO (NLO) – first reliable approximation
- NNLO – first precision order
- Ever improving experimental precision at the LHC
- NNLO predictions are required to fully exploit the LHC data

# Multi-jet processes at NNLO

- State of the art: two-to-two processes at NNLO



- Multi-jet processes are important for phenomenology:
  - ◊  $\alpha_s$  determination
  - ◊ tests of Standard Model
  - ◊ search for new physics

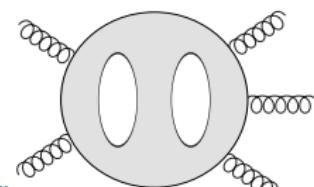
First  $2 \rightarrow 3$  NNLO calculation:  
3 $\gamma$ -production (planar)

[Chawdhry ,Czakon, Mitov, Poncelet '19]

- Three jets: double virtual corrections (two-loop five-particle amplitudes) are major bottleneck

process	known	desired
$pp \rightarrow 2$ jets	NNLO <sub>QCD</sub>	
	NLO <sub>QCD</sub> + NLO <sub>EW</sub>	
$pp \rightarrow 3$ jets	NLO <sub>QCD</sub> + NLO <sub>EW</sub>	NNLO <sub>QCD</sub>

Table I.2: Precision wish list: jet final states.



# Recent progress in calculation of the two-loop five-particle amplitudes

- All QCD amplitudes in the planar limit are known analytically [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov '19] [Abreu, Dormans, Febres Cordero, Ita, Page '18]  
Previous numerical [Badger, Brønnum-Hansen, Hartanto, Peraro '17][Abreu, Cordero, Ita, Page, Zeng '17] [Abreu, Cordero, Ita, Page, Sotnikov '18][Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro '18] and analytical results [Gehrmann, Henn, Lo Presti '15][Dunbar, Perkins '16] [Badger, Brønnum-Hansen, Hartanto, Peraro '18] in the planar approximation
- Full-color  $\mathcal{N} = 4$  super-Yang-Mills and  $\mathcal{N} = 8$  supergravity amplitudes [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18 '19][Abreu, Dixon, Herrmann, Page, Zeng '18 '19]  
⇒ Very first two-loop five-particle amplitude at the symbol level
- Full-color five-gluon all-plus helicity amplitude [Badger, D.C., Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19] [Dunbar, Godwin, Perkins, Strong '19]  
⇒ Very first complete analytic two-loop five-particle amplitude!

# Further immediate goals in the two-loop five-particle scattering

Particle Phenomenology and double-virtual corrections for 3-jet production:

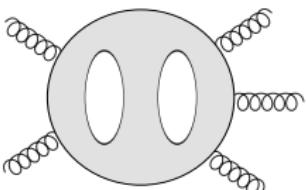
- All full-color five-point two-loop QCD amplitudes?
- Complete analytic expressions for the QCD amplitudes?
- Fast and high-precision numerical evaluation of the amplitudes?

In-depth study of  $\mathcal{N} = 4$  sYM and  $\mathcal{N} = 8$  supergravity amplitudes

[this talk]

- Beyond-symbol analytic expressions
- Simplifications in certain kinematic limits (multi-regge asymptotics)
- Novel analytic structures in the non-planar sector!

# $\mathcal{N} = 4$ super-Yang-Mills is an ideal theoretical laboratory



$$= \text{Color} \otimes \epsilon \otimes \begin{matrix} \text{Rational} \\ \text{factors} \end{matrix} \otimes \begin{matrix} \text{Pentagon} \\ \text{functions} \end{matrix}$$

	$\mathcal{N} = 4$ sYM amplitude	QCD amplitudes
Master Integrals	✓	✓
Pentagon Functions	✓	✓
Rational Factors	simple	complicated
Uniform Transcendentality	✓	
LHC physics		✓

# Structure of the full-color two-loop five-point $\mathcal{N} = 4$ sYM amplitude

- Perturbative expansion of the amplitude

$$\mathcal{A}_5 = g^3 \delta^8(Q) \delta^4(P) \left[ A_5^{(0)} + g^2 A_5^{(1)} + g^4 A_5^{(2)} + \dots \right]$$

$Q$  – total supercharge,  $P$  – total momentum

- $SU(N_c)$  color decomposition at two loops

$$A_5^{(2)} = \sum_{\lambda=1}^{12} \left( N_c^2 A_\lambda^{(2,0)} + A_\lambda^{(2,2)} \right) \mathcal{T}_\lambda + \sum_{\lambda=13}^{22} N_c A_\lambda^{(2,1)} \mathcal{T}_\lambda$$

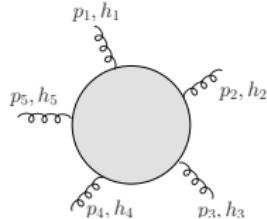
Twelve single traces       planar       not independent       nonplanar

$$\mathcal{T}_1 = \text{Tr}(12345) - \text{Tr}(54321), \dots, \mathcal{T}_{12} = \text{Tr}(13542) - \text{Tr}(12453)$$

Ten double traces

$$\mathcal{T}_{13} = \text{Tr}(12) [\text{Tr}(345) - \text{Tr}(543)], \dots, \mathcal{T}_{22} = \text{Tr}(52) [\text{Tr}(134) - \text{Tr}(431)]$$

with  $i \equiv T^{a_i}$



# Planar scattering

Planar five-points amplitudes are known at all loop orders

[Anastasiou, Bern, Dixon, Kosower '03][Bern, Dixon, Smirnov '05]

[Cachazo, Spradlin, Volovich '06][Bern, Czakon, Kosower, Roiban, Smirnov '06]

$$\mathcal{M}_5(\epsilon) = \exp\left(\mathcal{M}_5^{(1)}(\epsilon)\right), \quad \mathcal{M}_5 \equiv \frac{A_5}{A_5^{(0)}}$$

due to duality of MHV amplitudes and polygonal Wilson Loops [Alday, Maldacena '07]

[Drummond, Korchesmky, Sokatchev '07]

and the dual-conformal symmetry

[Drummond, Henn, Korchesmky, Sokatchev '07]

Nonplanar corrections are much more complicated

# Structure of the full-color two-loop five-point $\mathcal{N} = 4$ sYM amplitude

- Rational factors are Parke-Taylor factors

[Arkani-Hamed, Bourjaily, Cachazo, Trnka '14]

$$\text{PT}(\rho_1, \dots, \rho_5) = \frac{1}{\langle \rho_1 \rho_2 \rangle \langle \rho_2 \rho_3 \rangle \dots \langle \rho_5 \rho_1 \rangle}, \quad \sigma_\mu^{\alpha \dot{\alpha}} p_i^\mu \equiv \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \quad \langle ij \rangle \equiv \lambda_i^\alpha \lambda_{j\alpha}$$

- Six linearly independent Parke-Taylor factors

$$\text{PT}_1, \dots, \text{PT}_6$$

- Color-stripped amplitudes have a very simple structure!

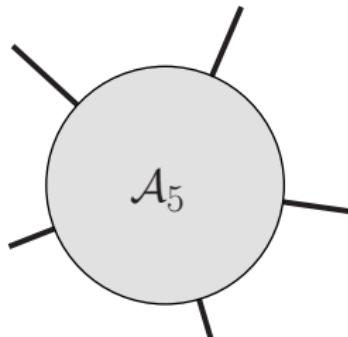
$$A_\lambda^{(2,k)} = \sum_{i=1}^6 \sum_j a_{\lambda,ij}^{(k)} \text{PT}_i \mathcal{I}_j^{\text{pure}} \quad , \quad \mathcal{I}^{\text{pure}} = \frac{1}{\epsilon^4} \sum_{w=0}^{\infty} \epsilon^w h^{(w)}(p)$$

$h^{(w)}(p)$  – weight- $w$  pure transcendental functions of the kinematics – weight- $w$  pentagon functions

# Factorization of the Infrared divergences

Universal factorization of the IR poles

$$\mathcal{A}_5(\epsilon) = \mathcal{Z}(\epsilon) \mathcal{A}_5^f(\epsilon)$$



**Hard function** is finite

$$\mathcal{H}_5 = \lim_{\epsilon \rightarrow 0} \mathcal{A}_5^f(\epsilon)$$

Exponentiation of IR divergences

$$\mathcal{Z}(\epsilon) = \exp(\mathbf{R})$$

$$\mathbf{R} = \frac{f_1(g)}{\epsilon^2} \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j + \frac{f_2(g)}{\epsilon} \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \log \left( -\frac{s_{ij}}{\mu^2} \right)$$

[Catani '98][Sterman, Tejeda-Yeomans '03][Dixon, Magnea, Sterman '08]

**Hard function:** simpler than amplitude, truly new piece of information

# Factorization of the Infrared divergences

Universal factorization of the IR poles

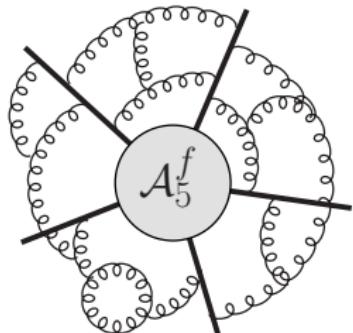
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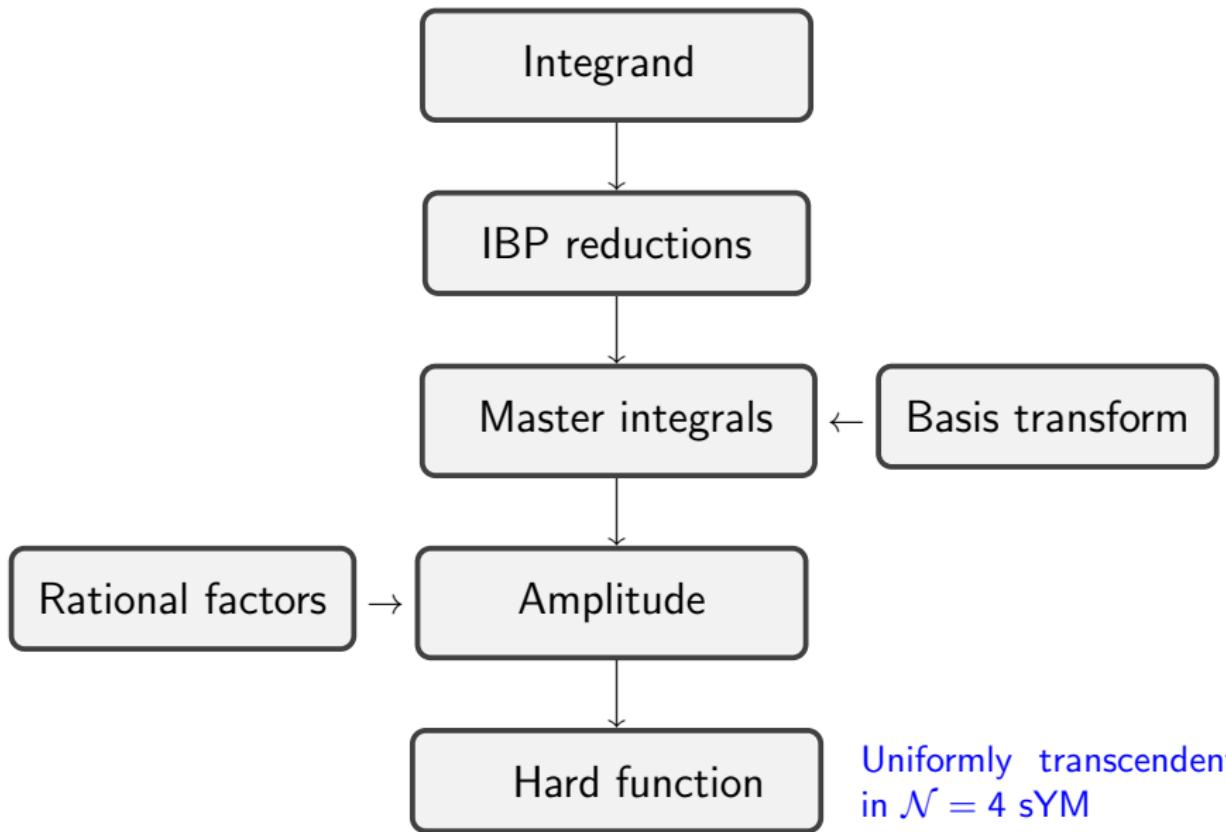


$$\mathbf{R} = \frac{f_1(g)}{\epsilon^2} \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j + \frac{f_2(g)}{\epsilon} \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \log \left( -\frac{s_{ij}}{\mu^2} \right)$$

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**Hard function:** simpler than amplitude, truly new piece of information

# Amplitude calculation workflow



# Integrand of the two-loop five-point $\mathcal{N} = 4$ sYM amplitude

$$\mathcal{A}_5^{(2)} = \sum_{S_5} \left\{ \begin{array}{l} \frac{1}{2} \text{(a)} \\ + \frac{1}{4} \text{(b)} \\ + \frac{1}{4} \text{(c)} \\ + \frac{1}{2} \text{(d)} \\ + \frac{1}{4} \text{(e)} \\ + \frac{1}{4} \text{(f)} \end{array} \right\}$$

The equation shows the sum of six Feynman diagrams labeled (a) through (f). Each diagram is a two-loop five-point function with external legs numbered 1 through 5. The diagrams represent different topologies of the two loops.

- (a) A pentagonal-like structure with a central horizontal loop.
- (b) A pentagonal-like structure with a central vertical loop.
- (c) A rectangle with a central vertical line segment.
- (d) A rectangle with two internal vertical lines.
- (e) A pentagonal-like structure with a central horizontal loop.
- (f) A pentagonal-like structure with a central vertical loop.

[Carrasco, Johansson '11]

Numerators are degree one in the loop momenta

# Kinematics of five-particle scattering

Massless particles:  $p_i^2 = 0$

Mandelstam invariants:

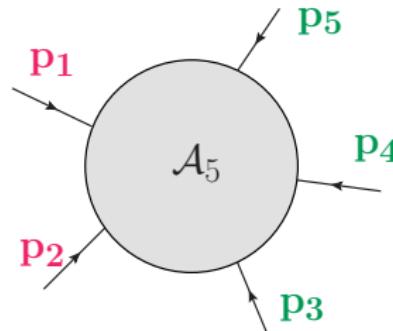
$$s_{ij} = (p_i + p_j)^2$$

Five independent:

$$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$$

One pseudo-scalar (parity-odd):

$$\epsilon_5 \equiv 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4} p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}$$



Physical scattering region  $12 \rightarrow 345$

$$s_{12}, s_{34}, s_{45}, s_{35} \geq 0$$

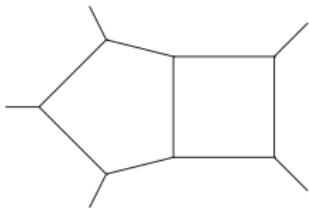
$$s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25} \leq 0$$

$$(\epsilon_5)^2 \leq 0$$

Mandelstam variables do not completely specify the kinematics

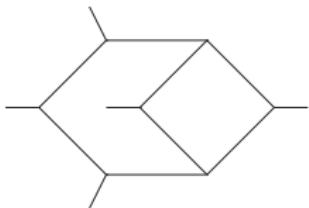
$$\epsilon_5 = \pm \sqrt{s_{12}^2 s_{15}^2 - 2s_{12}^2 s_{15} s_{23} + s_{12}^2 s_{23}^2 + \dots + s_{34}^2 s_{45}^2}$$

# The master integral families for massless two-loop five-particle scattering



[Gehrman, Henn, Lo Presti '15, '18]

[Papadopoulos, Tommasini, Wever '15]

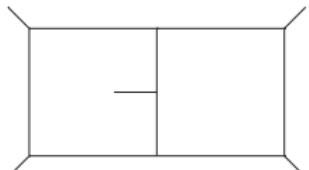


[D.C., Mitev, Henn '17]

[Boehm, Georgoudis, Larsen, Schoenemann, Zhang '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrman, Henn, Lo Presti, Mitev, Wasser '18]



[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrman, Henn, Wasser, Zhang, Zoia '18]

[Badger, D.C., Gehrman, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19]

All master integrals evaluate to pentagon functions

# Pentagon functions

- Proposed in [D.C., Mitev, Henn '17]
- Confirmed in [Abreu, Dixon, Herrmann, Page, Zeng '18]  
[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]



Chen iterated integrals along path  $\gamma$ :

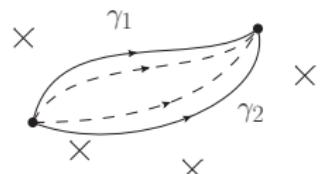
$$\int_0^1 dt_1 \partial_{t_1} \log W_{i_1}(s(t_1)) \int_0^{t_1} dt_2 \partial_{t_2} \log W_{i_2}(s(t_2)) \dots \int_0^{t_{n-1}} dt_n \partial_{t_n} \log W_{i_n}(s(t_n))$$

dlog-form on the path  $\gamma^*(d \log W_i(s)) = \partial_{t_i} \log W(s(t_i)) dt_i$

$\{W_i(s)\}_{i=1}^{31}$  – functions of energies and scattering angles

$n$  – transcendental weight

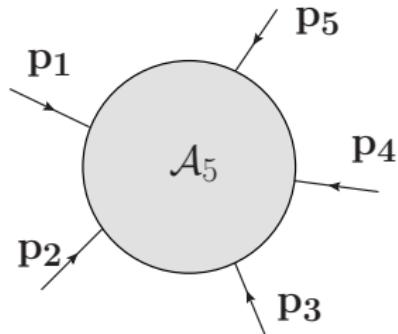
Homotopy invariance  $\gamma_1 \sim \gamma_2 \Rightarrow I(\gamma_1) = I(\gamma_2)$



# Pentagon alphabet

$$31\text{-letter alphabet } \mathbb{A} = \left\{ W_j(s) \right\}_{j=1}^{31}$$

$W_1, \dots, W_5$	$2 p_1 \cdot p_2$	$+(4)$
$W_6, \dots, W_{10}$	$2 p_4 \cdot (p_3 + p_5)$	$+(4)$
$W_{11}, \dots, W_{15}$	$2 p_3 \cdot (p_4 + p_5)$	$+(4)$
$W_{16}, \dots, W_{20}$	$2 p_1 \cdot p_3$	$+(4)$
$W_{21}, \dots, W_{25}$	$2 p_3 \cdot (p_1 + p_4)$	$+(4)$
$W_{26}, \dots, W_{30}$	$\frac{\text{tr}[(1-\gamma_5)p_1p_2p_4p_5]}{\text{tr}[(1+\gamma_5)p_1p_2p_4p_5]}$	$+(4)$
$W_{31}$	$\epsilon_5$	



- The alphabet splits into orbits of  $\mathbb{Z}_5$
- Invariance under  $S_5$
- 26 parity-even and 5 parity-odd letters
- Zero loci of letters: branch points of the master integrals

# Master integrals from differential equations

Change of master integral basis  $\vec{f}$  enormously simplifies DE [Henn '13]

$$d\vec{f}(s, \epsilon) = \epsilon d\tilde{A}(s) \vec{f}(s, \epsilon)$$

$$d\tilde{A}(s) = \sum_{i=1}^{31} a_i d \log W_i(s)$$

letters of the alphabet

rational matrices

Solution has uniform transcendentality

$$\vec{f}(s, \epsilon) = \text{Pexp}\left(\epsilon \int_{\gamma} d\tilde{A}(s)\right) \vec{f}(s_0, \epsilon) \implies \frac{1}{\epsilon^4} \sum_{w=0}^{\infty} \epsilon^w \vec{h}^{(w)}(s)$$

w-fold iterated integral

Construction of the canonical basis:

- ◊ Algorithm to find 4D dlog integrals [Wasser '16]
- ◊ D-dimensional leading singularities based on the Baikov parametrization

[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]

# Boundary constants for the differential equations

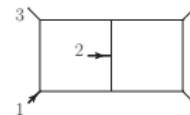
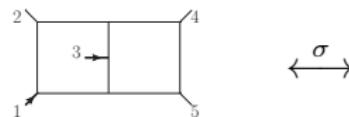
Boundary constants  $\vec{f}(s_0, \epsilon)$  uniquely specify solution of the DE

- Absence of spurious singularities in Feynman integrals
- No Euclidean region for nonplanar topologies! Work in physical  $s_{12}$ -channel
- Boundary constants for all  $5!$  permutations of the master integrals

$$\sigma \left( \begin{array}{c} \text{Diagram 1} \\ \text{(nonplanar)} \end{array} \right) \quad \sigma \left( \begin{array}{c} \text{Diagram 2} \\ \text{(nonplanar)} \end{array} \right) \quad \sigma \left( \begin{array}{c} \text{Diagram 3} \\ \text{(planar)} \end{array} \right), \quad \sigma \in \mathcal{S}_5$$

- A number of nontrivial checks: weight-drop of the all-plus amplitude
- Permutation of the channels  $\Leftrightarrow$  Permutation of the particles

$s_{13}$ -channel



$s_{12}$ -channel

- ✓ Boundary constants in one physical channel cover the whole physical region
- ✓ No need for analytic continuations
- ✓ Completely algorithmic approach
- ✓ Analytic expressions for the boundary constants

# Iterated integrals in terms of familiar functions

Choose boundary point in the physical  $s_{12}$ -channel

$$s_0 : \quad s_{12} = 3, \quad s_{23} = -1, \quad s_{34} = 1, \quad s_{45} = 1, \quad s_{15} = -1, \quad \epsilon_5 = 3i\sqrt{3}$$

- Weight 1

$$\int_{s_0 \rightarrow s} d \log W_1 = \log \left( \frac{s_{12}}{3} \right), \quad \int_{s_0 \rightarrow s} d \log W_2 = \log(-s_{23}), \dots$$

- Weight 2

$$\begin{aligned} \int_{s_0 \rightarrow s} d \log \left( \frac{W_1}{W_3} \right) \circ d \log \left( \frac{W_{13}}{W_1} \right) &= \text{Li}_2 \left( 1 - \frac{s_{34}}{s_{12}} \right) - \log(3) \log \left( 1 - \frac{s_{35}}{s_{12}} \right) \\ &\quad + \log(3) \log(2/3) - \text{Li}_2(2/3) \end{aligned}$$

Parity-odd weight-2 functions in the nonplanar sector!

[D.C., Henn, Mitev '17]

- Higher weights

- ◊ Rationalization of the alphabet and Goncharov polylogarithms
- ◊ Reduction to familiar functions  $\text{Li}_3, \text{Li}_4, \text{Li}_{2,2}$
- ◊ One-fold integral representations

# Benchmark numeric values of the hard function

Choose a random kinematic point in  $s_{12}$ -channel

$$s_{12} = \frac{13}{4}, \quad s_{23} = -\frac{9}{11}, \quad s_{34} = \frac{3}{2}, \quad s_{45} = \frac{3}{4}, \quad s_{15} = -\frac{2}{3}, \quad \epsilon_5 = i \frac{\sqrt{222767}}{264}$$

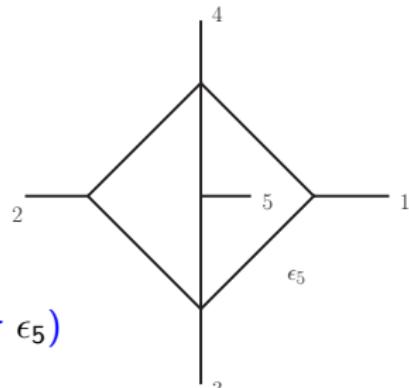
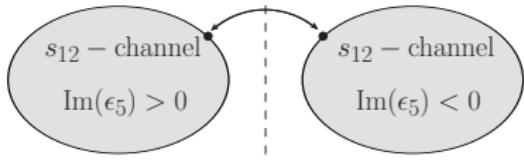
Double-trace color structures  $\mathcal{T}_{13}, \dots, \mathcal{T}_{22}$

	$N_c^2$	$N_c$	$N_c^0$
$\mathcal{T}_{13}$	0	$-125.2669 + 216.9434i$	0
$\mathcal{T}_{14}$	0	$-696.3813 - 209.4301i$	0
$\mathcal{T}_{15}$	0	$-344.4732 + 447.8376i$	0
$\mathcal{T}_{16}$	0	$-127.9880 + 116.6798i$	0
$\mathcal{T}_{17}$	0	$-444.5692 - 325.7655i$	0
$\mathcal{T}_{18}$	0	$-510.7351 - 321.1812i$	0
$\mathcal{T}_{19}$	0	$459.3389 + 210.4025i$	0
$\mathcal{T}_{20}$	0	$-120.7437 + 267.2953i$	0
$\mathcal{T}_{21}$	0	$711.4669 + 60.1616i$	0
$\mathcal{T}_{22}$	0	$-460.7431 - 329.6070i$	0

- ✓ Complete analytic control over amplitude
- ✓ Numeric evaluation in the physical region
- ✓ Multi-digit precision

# Singularities of the non-planar two-loop five-point Feynman integrals

Parity transformation  $\epsilon_5 = 4i\epsilon(p_1, p_2, p_3, p_4) \rightarrow -\epsilon_5$



Parity-odd pure integral (scalar integral with numerator  $\epsilon_5$ )

$$\mathcal{I} = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)$$

Discontinuity of  $f^{(2)}$  at  $\epsilon_5 = 0$

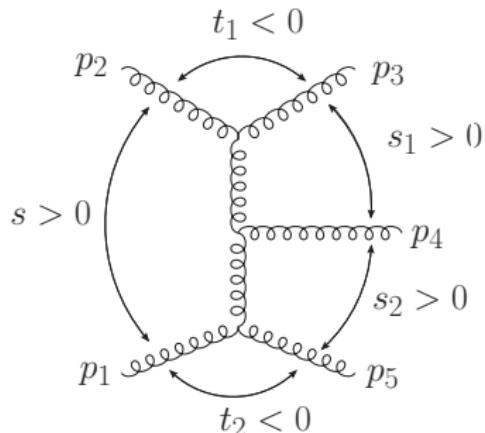
$$f^{(2)}|_{\text{Im}(\epsilon_5) \sim +0} = 12\pi^2 [\Theta(-a_{28}) - \Theta(-a_{26}) + \Theta(a_{30})] \neq 0$$

$$f^{(2)}|_{\text{Im}(\epsilon_5) \sim -0} = -12\pi^2 [\Theta(-a_{28}) - \Theta(-a_{26}) + \Theta(a_{30})] \neq 0$$

where  $a_{26} = \text{tr}(p_4 p_5 p_1 p_2), \dots$

Divergences at higher orders:  $f^{(3)}|_{\epsilon_5 \sim 0} \sim \log(\epsilon_5)$ ,  $f^{(4)}|_{\epsilon_5 \sim 0} \sim \log^2(\epsilon_5)$

# Multi-Regge kinematics



$$p_j \equiv (p_j^+, p_j^-, \mathbf{p}_j)$$

Strong ordering of rapidities

$$|p_3^+| \gg |p_4^+| \gg |p_5^+|$$

$$|p_3^-| \ll |p_4^-| \ll |p_5^-|$$

$$|\mathbf{p}_3| \simeq |\mathbf{p}_4| \simeq |\mathbf{p}_5|$$

Mandelstam invariants in MRK

$$s_{12} = \frac{s}{x^2}, \quad s_{23} = t_1, \quad s_{34} = \frac{s_1}{x}, \quad s_{45} = \frac{s_2}{x}, \quad s_{15} = t_2, \quad x \rightarrow 0$$

Kinematic variables in the multi-Regge limit:  $s, s_1, s_2, z, \bar{z}$  and  $x \rightarrow 0$

$$\frac{t_1 s}{s_1 s_2} = -z\bar{z}, \quad \frac{t_2 s}{s_1 s_2} = -(1-z)(1-\bar{z}), \quad \epsilon_5 = \frac{s_1 s_2 (z - \bar{z})}{x^2} + \mathcal{O}\left(\frac{1}{x}\right)$$

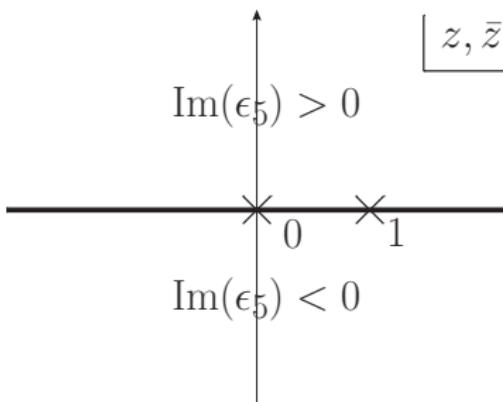
# Pentagon functions in the multi-Regge regime

Enormous simplifications of the pentagon alphabet

$$\left\{ W_j(s) \right\}_{j=1}^{31} \rightarrow \{x\}, \left\{ \frac{s_1 s_2}{s} \right\}, \{s_1, s_2, s_1 + s_2, s_1 - s_2\}, \{z, \bar{z}, 1-z, 1-\bar{z}, 1-z-\bar{z}\}$$

two-dimensional Harmonic Polylogarithms (2dHPL)

Symbol of nonplanar two-loop amplitude corrections vanishes in multi-Regge limit!  
We find multi-Regge asymptotics '**beyond the symbol**'



- Planar amplitudes in multi-Regge limit – **single-valued** 2dHPL  
[Dixon, Duhr, Pennington '12]
- Nonplanar amplitudes in multi-Regge limit – **not single-valued** 2dHPL

# Multi-Regge asymptotics of the pentagon functions

Canonical DE for pure integrals  $\vec{f}$  in MRK:  $y \equiv (s, s_1, s_2, z, \bar{z})$  and  $x \rightarrow 0$ ,

$$\begin{cases} \partial_x \vec{f}(x, y, \epsilon) = \epsilon A_x(x, y) \vec{f}(x, y, \epsilon) \\ \partial_y \vec{f}(x, y, \epsilon) = \epsilon A_y(x, y) \vec{f}(x, y, \epsilon) \end{cases}$$

matrix of the system in the d-log form  $dA \equiv A_x dx + A_y dy \equiv \sum_{j=1}^{31} a_j d \log W_j$

$$A_x(x, y) = \frac{A_0}{x} + \sum_{k \geq 0} x^k A_{k+1}(y), \quad A_y(x, y) = \mathcal{O}(1) \quad \text{at } x \rightarrow 0$$

Gauge transformation  $\vec{f} = T \vec{g}$

$$\begin{cases} \partial_x \vec{g}(x, y, \epsilon) = \epsilon \frac{A_0}{x} \vec{g}(x, y, \epsilon) \\ \partial_y \vec{g}(x, y, \epsilon) = B(x, y, \epsilon) \vec{g}(x, y, \epsilon) \end{cases}$$

Find  $T$  recursively solving

$$T^{-1}(\epsilon A_x T - \partial_x T) = \epsilon \frac{A_0}{x}, \quad T = 1 + \sum_{k \geq 1} \sum_{m \geq 1} x^k \epsilon^m T_{k,m}(y)$$

# Multi-Regge asymptotics of the pentagon functions

Easy to integrate the gauge-transformed system in  $(x, y)$ -plane

$$\begin{cases} \partial_x \vec{g}(x, y, \epsilon) = \epsilon \frac{A_0}{x} \vec{g}(x, y, \epsilon) \\ \partial_y \vec{g}(x, y, \epsilon) = B(x, y, \epsilon) \vec{g}(x, y, \epsilon) \end{cases}$$

$$\left. \begin{array}{l} B(x=0, y) = \epsilon A_y(x=0, y) \\ \vec{g}_0(\epsilon) \equiv g(y=y_0, \epsilon) \end{array} \right\} \quad \begin{array}{c} (0, y) \\ \longrightarrow \\ (x, y) \\ (0, y_0) \end{array}$$

$x$ -power corrections for pentagon functions at  $x \rightarrow 0$ :

$$\vec{f}(x, y, \epsilon) = T(x, y, \epsilon) x^{\epsilon A_0} \underbrace{\text{Pexp} \left[ \epsilon \int_{y_0}^y A_y(0, y') dy' \right] \vec{g}_0(\epsilon)}_{\text{2dHPL}(z, \bar{z}), \log(s), \log(s_1), \log(s_2)}$$

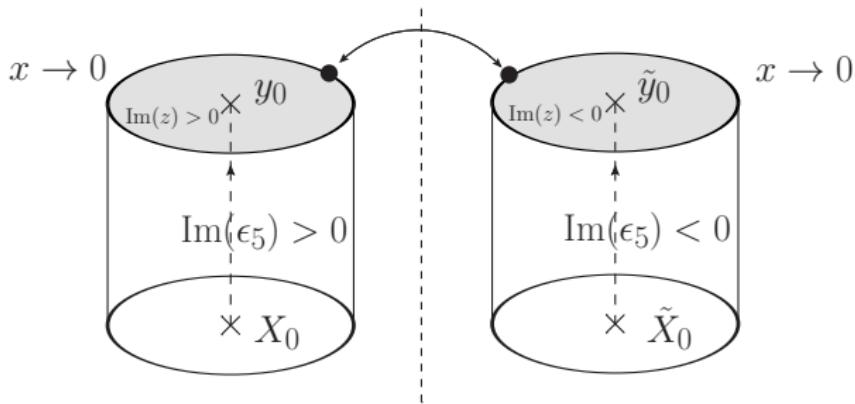
power corrections  $x^\#$

singular logarithms  $\log^\#(x)$

How to find the boundary constants  $\vec{g}_0(\epsilon)$  ?

# Boundary constants in the Multi-Regge asymptotics

Transport of the boundary constants:  $\vec{f}(X_0, \epsilon) \rightarrow \vec{g}_0(\epsilon)$



Boundary points are related by complex conjugation

$$X_0 : \epsilon_5 = 3\sqrt{3}i$$

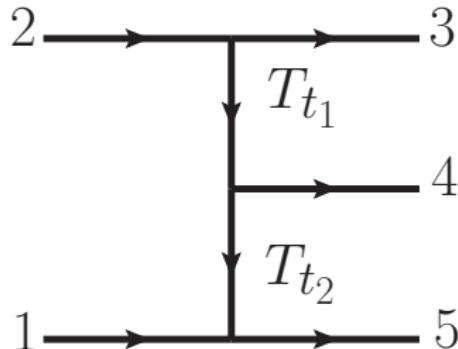
$$y_0 : z = e^{\frac{i\pi}{3}}, \bar{z} = e^{-\frac{i\pi}{3}}$$

$$\tilde{X}_0 : \epsilon_5 = -3\sqrt{3}i$$

$$\tilde{y}_0 : z = e^{-\frac{i\pi}{3}}, \bar{z} = e^{\frac{i\pi}{3}}$$

Concise analytic expressions for the boundary constants for all master integrals!

# Color flow in the multi-Regge limit



$$\mathbf{T}_{t_1} = \mathbf{T}_2 + \mathbf{T}_3, \quad \mathbf{T}_{t_2} = -\mathbf{T}_1 - \mathbf{T}_5$$

$$[\mathbf{T}_{t_1}^2, \mathbf{T}_i] = [\mathbf{T}_{t_2}^2, \mathbf{T}_i] = 0$$

$$[\mathbf{T}_{t_1}^2, \mathbf{T}_{t_2}^2] = 0$$

Color exchanges of  $SU(N_c)$  representations in  $t$ -channels

$$\mathbf{8}_a \otimes \mathbf{8}_a = \mathbf{1} \oplus \mathbf{8}_a \oplus \mathbf{8}_s \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \oplus \mathbf{0}$$

22 color tensors (new color basis) diagonalize two  $t$ -channel Casimir operators

$$(\mathbf{1}, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{1}), (\mathbf{8}_a, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{8}_s), (\mathbf{8}_a, \mathbf{27}), \dots$$

# Multi-Regge limit of the two-loop hard function

We consider LL, NLL, NNLL approximation

$$\mathcal{H}^{(2)} = \text{PT}_1 \sum_{(r_1, r_2)} \mathcal{S}_{(r_1, r_2)} \sum_{k=0}^2 \log^k(x) h_{(r_1, r_2), k}^{(2)}(N_c, s, s_1, s_2, z, \bar{z}) + o(x^0)$$

Parke-Taylor  $\uparrow$  (r<sub>1</sub>, r<sub>2</sub>)  $\uparrow$  color  $\uparrow$  log<sup>k</sup>(x)  $\uparrow$  large logarithms

- Reggeized gluon:  $(r_1, r_2) = (\mathbf{8}_a, \mathbf{8}_a)$
- Only  $h_{(\mathbf{8}_a, \mathbf{8}_a)}^{(2)}$  does not vanish at the symbol level and does not vanish in LLA  
$$h_{(r_1, r_2), k}^{(2)} \sim \pi \times f^{(3-k)} \quad \text{or} \quad h_{(r_1, r_2), k}^{(2)} \sim \pi^2 \times f^{(2-k)}, \quad (r_1, r_2) \neq (\mathbf{8}_a, \mathbf{8}_a)$$
- Only HPL of weights 1,2,3
- Vanish at LLA and NLLA in color channels  
 $(\mathbf{8}_a, \mathbf{10}), (\mathbf{10}, \mathbf{8}_a), (\mathbf{8}_s, \mathbf{10}), (\mathbf{10}, \mathbf{8}_s), (\mathbf{1}, \mathbf{10}), (\mathbf{10}, \mathbf{1}), (\mathbf{27}, \mathbf{10}), (\mathbf{10}, \mathbf{27}), (\mathbf{10}, \mathbf{10})$

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- Vanishing at LLA, NLLA, NNLLA in color channels

$$(\mathbf{8}_s, \mathbf{10}), (\mathbf{10}, \mathbf{8}_s), (\mathbf{10}_-, \mathbf{10}_-)$$

- Single-valued HPL in color channels

$$(\mathbf{8}_a, \mathbf{8}_a), (\mathbf{8}_s, \mathbf{8}_s), (\mathbf{8}_a, \mathbf{8}_s), (\mathbf{8}_s, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{10}), (\mathbf{10}, \mathbf{8}_a), \\ (\mathbf{0}, \mathbf{10}), (\mathbf{10}, \mathbf{0}), (\mathbf{27}, \mathbf{10}), (\mathbf{10}, \mathbf{27}), (\mathbf{10}_+, \mathbf{10}_+)$$

- Non-single-valued 2dHPL in color channels

$$(\mathbf{0}, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{0}), (\mathbf{1}, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{1}), (\mathbf{27}, \mathbf{8}_a), (\mathbf{8}_a, \mathbf{27}), (\mathbf{0}, \mathbf{0}), (\mathbf{27}, \mathbf{27})$$

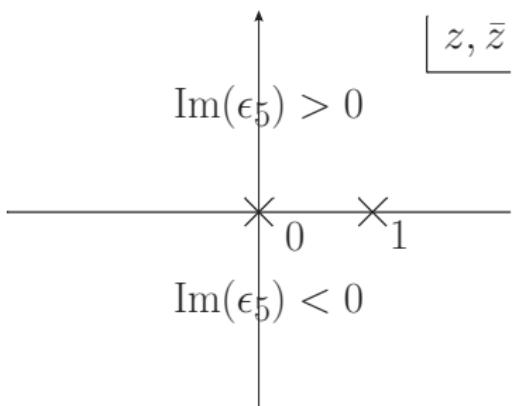
- Hard function is continuous in  $(z, \bar{z})$  but second derivatives jump across  $z = \bar{z}$

# Pentagon functions in the multi-Regge regime

Single-valued 2dHPLs

[Brown '04]

$$\log(z\bar{z}), \log(1-z)(1-\bar{z}), \text{Li}_2(z) - \text{Li}_2(\bar{z}) + \frac{1}{2} \log(z\bar{z}) [\log(1-z) - \log(1-\bar{z})], \dots$$



Single-valued 2dHPLs are not enough

Weight one example:

$$\log(z) - \log(\bar{z}), \quad \log(1-z) - \log(1-\bar{z})$$

Well-defined in both halves of the complex plane, but have discontinuity at  $z = \bar{z}$

Hard function:

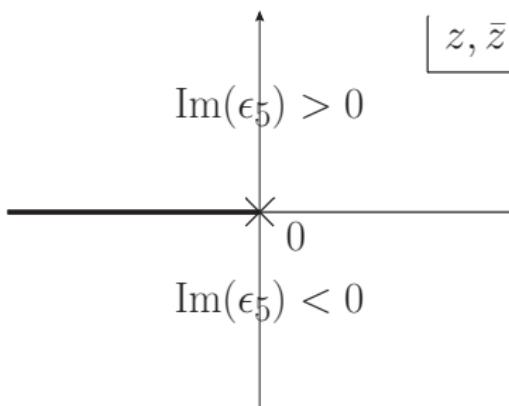
- Continuous at  $z = \bar{z}$
- Real analytic at  $\text{Im}(z) > 0$  and  $\text{Im}(z) < 0$

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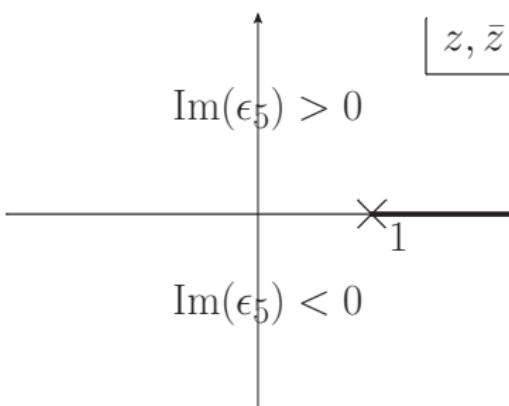
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# Two-loop five-particle $\mathcal{N} = 8$ supergravity amplitude

# (Super)gravity amplitudes

- No color structure. Intrinsically nonplanar amplitudes
- Full permutation symmetry  $S_5$
- Dimensionful coupling constant  $\kappa^2 = 32\pi G_N$

$$\mathcal{M}_5 = \kappa^3 \delta^4(P) \delta^{16}(Q) \left[ M_5^{(0)} + \kappa^2 M_5^{(1)} + \kappa^4 M_5^{(2)} + \dots \right]$$

  
supercharge conservation

- Simple IR structure.  $1/\epsilon$  pole per loop order. No collinear divergences [Weinberg '65] Soft divergences are one-loop exact

$$\mathcal{M}_5(s, \epsilon) = \exp\left(\frac{\sigma_5}{\epsilon}\right) \mathcal{M}_5^f(s, \epsilon), \quad \sigma_5(s) = \frac{\kappa^2}{4} \sum_{i < j} s_{ij} \log\left(-\frac{s_{ij}}{\mu^2}\right)$$

Finite hard function

$$\mathcal{F}_5(s) = \lim_{\epsilon \rightarrow 0} \mathcal{M}_5^f(s, \epsilon)$$

# Simplicity of the supergravity amplitude/hard function

- Color-kinematics duality of gauge-theory amplitude integrands and double copy

$$\text{gravity} = \text{gauge theory} \otimes \text{gauge theory}$$

Two-loop five-point integrand [Carrasco, Johansson '11] Numerators of degree two in loop momenta

- Uniform transcendental amplitude

$$M_5^{(2)} = \sum_i \sum_j b_{ij} r_i^{(2)} \mathcal{I}_j^{\text{pure}} , \quad \mathcal{I}^{\text{pure}} = \frac{1}{\epsilon^4} \sum_{w=0}^{\infty} \epsilon^w h^{(w)}(p)$$

$h^{(w)}(p)$  – weight- $w$  pentagon functions

- Rational prefactors  $r_i^{(\ell)}$  change with loop order. The hard function requires

$$s_{12} s_{23} s_{34} s_{45} \text{PT}(12345) \text{PT}(21435) \quad \text{and permutations}$$

# Analytics and numerics of the supergravity amplitudes

- Two-loop five-particle  $\mathcal{N} = 8$  supergravity amplitude was calculated at the symbol level  
[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '19][Abreu, Dixon, Herrmann, Page, Zeng '19]
- Find full analytic structure – beyond the symbol terms
- Benchmark numeric values

$$X_R : \quad s_{12} = \frac{13}{4}, \quad s_{23} = -\frac{9}{11}, \quad s_{34} = \frac{3}{2}, \quad s_{45} = \frac{3}{4}, \quad s_{15} = -\frac{2}{3}, \quad \epsilon_5 = i \frac{\sqrt{222767}}{264}$$

$$\left. \frac{\mathcal{F}^{(2)}}{(\text{PT}_1)^2} \right|_{X_R} = -1211.9365 - 215.6087i$$

# Multi-regge limit of the supergravity amplitude

- Rational factors diverge in multi-Regge limit  $x \rightarrow 0$

$$\frac{r_i^{(2)}}{M_5^{(0)}} = \mathcal{O}\left(\frac{1}{x^4}\right)$$

- Double series expansion

$$\frac{\mathcal{F}^{(2)}}{M_5^{(0)}} = \sum_{m=0}^4 \sum_{k=0}^4 x^{-m} F_{m,k}(s_1, s_2, s, z, \bar{z}) \log^k(x) + o(1)$$

Power corrections of the pentagon functions are required

- Poles in rational factors at  $z = \bar{z}$
- Nonplanar pentagon functions  $\implies$  Non-single-valued 2dHPL

$$\text{Li}_2(z) + \text{Li}_2(\bar{z}), \text{ Li}_3(z) - \text{Li}_3(\bar{z}), \text{ Li}_4(z) + \text{Li}_4(\bar{z}), \dots$$

- Analyticity in both halves of the complex plane

$$\text{Im}(\epsilon_5) > 0 \quad \text{and} \quad \text{Im}(\epsilon_5) < 0$$

- Genuine weight-4 2dHPL

# Conclusions

- Complete analytic expressions for two-loop five-particle amplitudes in  $\mathcal{N} = 4$  super-Yang-Mills and  $\mathcal{N} = 8$  supergravity
- Pentagon functions describe all massless five-particle two-loop amplitudes
- All master integrals are known analytically in the physical scattering region
- Numeric evaluation of the master integrals in the physical region
- New analytic structures in the nonplanar sector
- Multi-Regge limit of the supersymmetric amplitudes in  $N^k LL$  approximation
- Cross-checks through BFKL approach to high energy scattering in gauge theories