# Expansion by regions

Vladimir A. Smirnov

Skobeltsyn Institute of Nuclear Physics of Moscow State University

Paris, March 2-7, 2020

Expanding a given Feynman integral in a given limit, where kinematic invariants and/or masses which essentially differ in scale.

Expanding a given Feynman integral in a given limit, where kinematic invariants and/or masses which essentially differ in scale.

Let us, first, keep in mind an integral $G_\Gamma(q^2, m^2)$ depending on two scales, e.g., $q^2$ and $m^2$, and let the limit be $t = -m^2/q^2 \to 0$.

Expanding a given Feynman integral in a given limit, where kinematic invariants and/or masses which essentially differ in scale.

Let us, first, keep in mind an integral $G_\Gamma(q^2, m^2)$ depending on two scales, e.g., $q^2$ and $m^2$, and let the limit be $t = -m^2/q^2 \to 0$.

Experience tells us that the expansion at $t \to 0$ has the form

$$G_\Gamma(x, \varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{2h} c_{n,k}(\varepsilon) \log^k t \; t^n ,$$

Expanding a given Feynman integral in a given limit, where kinematic invariants and/or masses which essentially differ in scale.

Let us, first, keep in mind an integral $G_\Gamma(q^2, m^2)$ depending on two scales, e.g., $q^2$ and $m^2$, and let the limit be $t = -m^2/q^2 \to 0$.

Experience tells us that the expansion at $t \to 0$ has the form

$$G_\Gamma(x, \varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{2h} c_{n,k}(\varepsilon) \log^k t \; t^n ,$$

where $h$ is the number of loops and $\varepsilon = (4-d)/2$.

Expanding a given Feynman integral in a given limit, where kinematic invariants and/or masses which essentially differ in scale.

Let us, first, keep in mind an integral $G_\Gamma(q^2, m^2)$ depending on two scales, e.g., $q^2$ and $m^2$, and let the limit be $t = -m^2/q^2 \to 0$.

Experience tells us that the expansion at $t \to 0$ has the form

$$G_\Gamma(x, \varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{2h} c_{n,k}(\varepsilon) \log^k t \; t^n,$$

where $h$ is the number of loops and $\varepsilon = (4-d)/2$.

The expansion is often called asymptotic, i.e. the remainder of expansion after keeping terms up to $t^N$ is $o(t^N)$.

It is very useful to consider expansion at general $\varepsilon$,

$$G_\Gamma(x,\varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{h} \sum_{j=0}^{h} c'_{n,j,k}(\varepsilon) \log^k t \; t^{n-j\varepsilon} \; .$$

It is very useful to consider expansion at general $\varepsilon$,

$$G_\Gamma(x,\varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{h} \sum_{j=0}^{h} c'_{n,j,k}(\varepsilon) \log^k t \; t^{n-j\varepsilon} \; .$$

There are various methods to obtain an expansion of a given Feynman integral, e.g., using a MB-representation.

It is very useful to consider expansion at general $\varepsilon$,

$$G_\Gamma(x,\varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{h} \sum_{j=0}^{h} c'_{n,j,k}(\varepsilon) \log^k t \; t^{n-j\varepsilon} \; .$$

There are various methods to obtain an expansion of a given Feynman integral, e.g., using a MB-representation.

There are, however, two *general* strategies, *expansion by subgraphs* and *expansion by regions*, which provide a result in this form for any given Feynman integral, where coefficients are expressed either in graph-theoretical language, or in the language of polytopes associated with a given integral.

Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expanding a given Feynman integral in a given limit.

Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expanding a given Feynman integral in a given limit.

In the 'physical' language:

Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expanding a given Feynman integral in a given limit.

In the 'physical' language:

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a series with respect to the parameters that are considered there small.

Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expanding a given Feynman integral in a given limit.

In the 'physical' language:

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a series with respect to the parameters that are considered there small.
- Integrate the integrand, expanded in this way in each region, over the *whole integration domain* of the loop momenta.

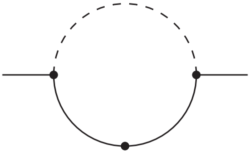
Expansion by regions [M. Beneke & V.S.'98]
introduced and applied in the case of threshold expansion.

Expanding a given Feynman integral in a given limit.

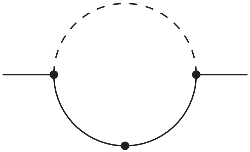
In the 'physical' language:

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a series with respect to the parameters that are considered there small.
- Integrate the integrand, expanded in this way in each region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

# A simple example

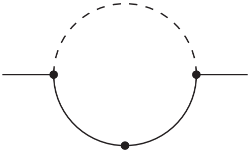


## A simple example



$$G(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2}$$

A simple example



$$G(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2}$$

with $d = 4 - 2\varepsilon$ in the limit $m^2/q^2 \to 0$.

Two relevant regions: $k \sim q$ and $k \sim m$
(large and small loop momenta)

Two relevant regions: $k \sim q$ and $k \sim m$
(large and small loop momenta)

$k \sim q$:

$$\frac{1}{(k^2-m^2)^2} \sim \frac{1}{(k^2)^2} + \dots$$

$$\frac{1}{(q-k)^2} \quad \text{unexpanded}$$

Two relevant regions: $k \sim q$ and $k \sim m$
(large and small loop momenta)

$k \sim q$:

$$\frac{1}{(k^2-m^2)^2} \sim \frac{1}{(k^2)^2} + \ldots$$

$$\frac{1}{(q-k)^2} \quad \text{unexpanded}$$

$k \sim m$:

$$\frac{1}{(k^2-m^2)^2} \quad \text{unexpanded}$$

$$\frac{1}{(q-k)^2} \sim \frac{1}{q^2} + \ldots$$

$$G(q^2, m^2; d) \sim \int \frac{\mathrm{d}^d k}{(k^2)^2 (q-k)^2} + \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2} + \ldots$$

$$G(q^2, m^2; d) \sim \int \frac{\mathrm{d}^d k}{(k^2)^2 (q-k)^2} + \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2} + \ldots$$

$$= \mathrm{i}\pi^{d/2} \left( \frac{\Gamma(1-\varepsilon)^2 \Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)(-q^2)^{1+\varepsilon}} + \frac{\Gamma(\varepsilon)}{q^2 (m^2)^\varepsilon} + \ldots \right)$$

$$G(q^2, m^2; d) \sim \int \frac{\mathrm{d}^d k}{(k^2)^2 (q-k)^2} + \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2} + \dots$$

$$= \mathrm{i}\pi^{d/2} \left( \frac{\Gamma(1-\varepsilon)^2 \Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)(-q^2)^{1+\varepsilon}} + \frac{\Gamma(\varepsilon)}{q^2 (m^2)^\varepsilon} + \dots \right)$$

$$= \mathrm{i}\pi^{d/2} \left( \log\left( \frac{-q^2}{m^2} \right) + \dots \right)$$

[M. Beneke '98, V.S. 'Applied asymptotic expansions in momenta and masses', 2002]:
a toy example of a one-parametric integral

[M. Beneke '98, V.S. 'Applied asymptotic expansions in momenta and masses', 2002]:
a toy example of a one-parametric integral

$$G(q, m, \varepsilon) = \int_0^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)} \mathrm{d}k \equiv \int_0^\infty I(q, m, \varepsilon, k) \mathrm{d}k,$$

with $m, q > 0$, in the limit $m/q \to 0$.

[M. Beneke '98, V.S. 'Applied asymptotic expansions in momenta and masses', 2002]:
a toy example of a one-parametric integral

$$G(q,m,\varepsilon) = \int_0^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)} \mathrm{d}k \equiv \int_0^\infty I(q,m,\varepsilon,k)\mathrm{d}k,$$

with $m, q > 0$, in the limit $m/q \to 0$. In fact, partial fractions

$$\frac{1}{(k+m)(k+q)} \to \frac{1}{q-m}\left(\frac{1}{k+m} - \frac{1}{k+q}\right)$$

[M. Beneke '98, V.S. 'Applied asymptotic expansions in momenta and masses', 2002]:
a toy example of a one-parametric integral

$$G(q,m,\varepsilon) = \int_0^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)}\mathrm{d}k \equiv \int_0^\infty I(q,m,\varepsilon,k)\mathrm{d}k,$$

with $m, q > 0$, in the limit $m/q \to 0$. In fact, partial fractions

$$\frac{1}{(k+m)(k+q)} \to \frac{1}{q-m}\left(\frac{1}{k+m} - \frac{1}{k+q}\right)$$

provide a simple explicit result

$$\Gamma(1-\varepsilon)\Gamma(\varepsilon)\frac{q^{-\varepsilon} - m^{-\varepsilon}}{m-q}$$

which can then simply be expanded at $m/q \to 0$.

[M. Beneke '98, V.S. 'Applied asymptotic expansions in momenta and masses', 2002]:
a toy example of a one-parametric integral

$$G(q,m,\varepsilon) = \int_0^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)} \mathrm{d}k \equiv \int_0^\infty I(q,m,\varepsilon,k)\mathrm{d}k,$$

with $m, q > 0$, in the limit $m/q \to 0$. In fact, partial fractions

$$\frac{1}{(k+m)(k+q)} \to \frac{1}{q-m}\left(\frac{1}{k+m} - \frac{1}{k+q}\right)$$

provide a simple explicit result

$$\Gamma(1-\varepsilon)\Gamma(\varepsilon)\frac{q^{-\varepsilon} - m^{-\varepsilon}}{m-q}$$

which can then simply be expanded at $m/q \to 0$.

Two relevant regions: $k \sim q$ and $k \sim m$

Two relevant regions: $k \sim q$ and $k \sim m$

$k \sim q$:

$$\frac{1}{k+m} \sim \frac{1}{k} + \ldots$$

$$\frac{1}{k+q} \quad \text{unexpanded}$$

Two relevant regions: $k \sim q$ and $k \sim m$

$k \sim q$:

$$\frac{1}{k+m} \sim \frac{1}{k} + \ldots$$

$$\frac{1}{k+q} \quad \text{unexpanded}$$

$k \sim m$:

$$\frac{1}{k+m} \quad \text{unexpanded}$$

$$\frac{1}{k+q} \sim \frac{1}{q} + \ldots$$

Two relevant regions: $k \sim q$ and $k \sim m$

$k \sim q$:

$$\frac{1}{k+m} \sim \frac{1}{k} + \ldots$$

$$\frac{1}{k+q} \quad \text{unexpanded}$$

$k \sim m$:

$$\frac{1}{k+m} \quad \text{unexpanded}$$

$$\frac{1}{k+q} \sim \frac{1}{q} + \ldots$$

$$G(q, m, \varepsilon) \sim \int_0^\infty \frac{k^{-1-\varepsilon}}{k+q} \mathrm{d}k + \frac{1}{q} \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathrm{d}k + \ldots$$

$$G \to G_s + G_l \equiv \int_0^{\Lambda} I(q, m, \varepsilon, k)\mathrm{d}k + \int_{\Lambda}^{\infty} I(q, m, \varepsilon, k)\mathrm{d}k$$

where $m < \Lambda < q$.

$$G \to G_s + G_l \equiv \int_0^\Lambda I(q, m, \varepsilon, k)\mathrm{d}k + \int_\Lambda^\infty I(q, m, \varepsilon, k)\mathrm{d}k$$

where $m < \Lambda < q$.

$$G_l = \int_\Lambda^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)}\mathrm{d}k \sim \int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m \frac{1}{k+m}\mathrm{d}k$$

where $\mathcal{T}_x f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)} x^n$,

$$G \to G_s + G_l \equiv \int_0^\Lambda I(q, m, \varepsilon, k)\mathrm{d}k + \int_\Lambda^\infty I(q, m, \varepsilon, k)\mathrm{d}k$$

where $m < \Lambda < q$.

$$G_l = \int_\Lambda^\infty \frac{k^{-\varepsilon}}{(k+m)(k+q)}\mathrm{d}k \sim \int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m \frac{1}{k+m}\mathrm{d}k$$

where $\mathcal{T}_x f(x) = \sum_{n=0}^\infty \frac{1}{n!} f^{(n)} x^n$, so that

$$G_l \sim \int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+q}\left(\frac{1}{k} - \frac{m}{k^2} + \dots\right) .$$

$$G \to G_s + G_l \equiv \int_0^{\Lambda} I(q,m,\varepsilon,k)\mathrm{d}k + \int_{\Lambda}^{\infty} I(q,m,\varepsilon,k)\mathrm{d}k$$

where $m < \Lambda < q$.

$$G_l = \int_{\Lambda}^{\infty} \frac{k^{-\varepsilon}}{(k+m)(k+q)}\mathrm{d}k \sim \int_{\Lambda}^{\infty} \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m \frac{1}{k+m}\mathrm{d}k$$

where $\mathcal{T}_x f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)} x^n$, so that

$$G_l \sim \int_{\Lambda}^{\infty} \frac{k^{-\varepsilon}}{k+q}\left(\frac{1}{k} - \frac{m}{k^2} + \dots\right) .$$

Here one can change the order of integration and Taylor expansion.

Add and subtract the integral over $(0, \Lambda)$ which is by definition understood as the sum of integrals of the Taylor-expanded integrand:

$$G_l \sim \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k - \int_0^\Lambda \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k$$

where each integral is evaluated in the corresponding domain of $\varepsilon$ where it is convergent and then the result it continued analytically to a given domain, i.e. a vicinity of $\varepsilon = 0$.

Add and subtract the integral over $(0, \Lambda)$ which is by definition understood as the sum of integrals of the Taylor-expanded integrand:

$$G_l \sim \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k - \int_0^\Lambda \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k$$

where each integral is evaluated in the corresponding domain of $\varepsilon$ where it is convergent and then the result it continued analytically to a given domain, i.e. a vicinity of $\varepsilon = 0$. Similarly,

$$G_s \sim \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k - \int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k$$

'Additional' pieces:

$$-\int_0^\Lambda \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m \frac{1}{k+m}\mathrm{d}k = -\sum_{n=0}^{\infty}(-1)^n m^n \int_0^\Lambda \frac{k^{-\varepsilon-n-1}}{k+q}\mathrm{d}k$$

$$= -\sum_{n,l=0}^{\infty}(-1)^{n+l} m^n q^{-l-1} \int_0^\Lambda k^{-\varepsilon-n+l-1}\mathrm{d}k$$

'Additional' pieces:

$$-\int_0^\Lambda \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m\frac{1}{k+m}\mathrm{d}k = -\sum_{n=0}^{\infty}(-1)^n m^n \int_0^\Lambda \frac{k^{-\varepsilon-n-1}}{k+q}\mathrm{d}k$$

$$= -\sum_{n,l=0}^{\infty}(-1)^{n+l} m^n q^{-l-1} \int_0^\Lambda k^{-\varepsilon-n+l-1}\mathrm{d}k$$

$$-\int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+m}\mathcal{T}_k\frac{1}{k+q}\mathrm{d}k = -\sum_{l=0}^{\infty}(-1)^l q^{-l-1} \int_\Lambda^\infty \frac{k^{l-\varepsilon}}{k+m}\mathrm{d}k$$

$$= -\sum_{n,l=0}^{\infty}(-1)^{n+l} m^n q^{-l-1} \int_\Lambda^\infty k^{-\varepsilon-n+l-1}\mathrm{d}k$$

The additional pieces cancel each other because

$$\int_0^\Lambda k^{-\varepsilon-n+l-1} \mathrm{d}k = \Lambda^{-\varepsilon-n+l}\,, \quad \int_\Lambda^\infty k^{-\varepsilon-n+l-1} \mathrm{d}k = -\Lambda^{-\varepsilon-n+l}$$

The additional pieces cancel each other because

$$\int_0^\Lambda k^{-\varepsilon-n+l-1}\mathrm{d}k = \Lambda^{-\varepsilon-n+l}\,, \quad \int_\Lambda^\infty k^{-\varepsilon-n+l-1}\mathrm{d}k = -\Lambda^{-\varepsilon-n+l}$$

We did not refer to the zero value of scaleless integrals

$$\int_0^\infty k^\lambda \mathrm{d}k = 0.$$

We arrive at the expansion $G \sim M_1 G + M_2 G$ with

$$M_1 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k,$$

$$M_2 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k \ .$$

We arrive at the expansion $G \sim M_1 G + M_2 G$ with

$$M_1 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k,$$
$$M_2 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k\ .$$

Each resulting integral is evaluated in the corresponding domain of $\varepsilon$ where it is convergent, with a subsequent analytic continuation to the initial domain, i.e. a vicinity of $\varepsilon = 0$.

We arrive at the expansion $G \sim M_1 G + M_2 G$ with

$$M_1 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k,$$
$$M_2 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k \,.$$

Each resulting integral is evaluated in the corresponding domain of $\varepsilon$ where it is convergent, with a subsequent analytic continuation to the initial domain, i.e. a vicinity of $\varepsilon = 0$.

The remainder can be described as

$$R^n G = (1 - M_1^n)(1 - M_2^n) G$$
$$= \int_0^\infty k^{-\varepsilon} \left[ (1 - \mathcal{T}_m^n) \frac{1}{k+m} \right] \left[ (1 - \mathcal{T}_k^n) \frac{1}{k+q} \right] \mathrm{d}k$$

Obtaining expansion from the remainder in a ‘physical way’

Obtaining expansion from the remainder in a 'physical way'

$$1 = 1 - R^n + R^n = 1 - (1 - M_1^n)(1 - M_2^n) + R^n$$

Obtaining expansion from the remainder in a 'physical way'

$$1 = 1 - R^n + R^n = 1 - (1 - M_1^n)(1 - M_2^n) + R^n$$

where

$$1 - (1 - M_1^n)(1 - M_2^n) = M_1^n + M_2^n - M_1^n M_2^n$$

Obtaining expansion from the remainder in a 'physical way'

$$1 = 1 - R^n + R^n = 1 - (1 - M_1^n)(1 - M_2^n) + R^n$$

where

$$1 - (1 - M_1^n)(1 - M_2^n) = M_1^n + M_2^n - M_1^n M_2^n$$

Set scaleless integrals in $M_1^n M_2^n$ to zero to obtain

$$G \sim M_1^n G + M_2^n G + R^n G$$

Obtaining expansion from the remainder in a mathematical way. Let $M_i^n = \sum_{j=0}^{n} M_i^{(j)}$ for $i = 1, 2$.

Obtaining expansion from the remainder in a mathematical way. Let $M_i^n = \sum_{j=0}^n M_i^{(j)}$ for $i = 1, 2$. Then

$$M_1^n + M_2^n - M_1^n M_2^n = \sum_{j=0}^{n} (1 - M_2^{j-1}) M_1^{(j)} + \sum_{j=0}^{n} (1 - M_1^j) M_2^{(j)} \,.$$

Obtaining expansion from the remainder in a mathematical way. Let $M_i^n = \sum_{j=0}^n M_i^{(j)}$ for $i = 1, 2$. Then

$$M_1^n + M_2^n - M_1^n M_2^n = \sum_{j=0}^{n} (1 - M_2^{j-1}) M_1^{(j)} + \sum_{j=0}^{n} (1 - M_1^{j}) M_2^{(j)} \,.$$

Let $\mathrm{Re}\varepsilon < 0$.

Obtaining expansion from the remainder in a mathematical way. Let $M_i^n = \sum_{j=0}^n M_i^{(j)}$ for $i = 1, 2$. Then

$$M_1^n + M_2^n - M_1^n M_2^n = \sum_{j=0}^{n} (1 - M_2^{j-1}) M_1^{(j)} + \sum_{j=0}^{n} (1 - M_1^j) M_2^{(j)} \ .$$

Let $\mathrm{Re}\varepsilon < 0$. Then

$$\int_0^\infty k^{-\varepsilon} \left[ (1 - \mathcal{T}_k^{j-1}) \frac{1}{k+q} \right] \mathcal{T}_m^{(j)}) \frac{1}{k+m} \mathrm{d}k$$
$$\sim m^j \int_0^\infty k^{-\varepsilon-j-1} \left[ (1 - \mathcal{T}_k^{j-1}) \frac{1}{k+q} \right]$$

$$\int_0^\infty k^{-\varepsilon-j-1}\left[(1-\mathcal{T}_k^{j-1})\frac{1}{k+q}\right]$$

$$\int_0^\infty k^{-\varepsilon-j-1}\left[(1-\mathcal{T}_k^{j-1})\frac{1}{k+q}\right]$$

is nothing but the analytic continuation of the integral

$$\int_0^\infty k^{-\varepsilon-j-1}\frac{1}{k+q}$$

from $0 < -\mathrm{Re}\varepsilon < 1$ to $j < -\mathrm{Re}\varepsilon < j+1$.

$$\int_0^\infty k^{-\varepsilon-j-1}\left[(1-\mathcal{T}_k^{j-1})\frac{1}{k+q}\right]$$

is nothing but the analytic continuation of the integral

$$\int_0^\infty k^{-\varepsilon-j-1}\frac{1}{k+q}$$

from $0 < -\mathrm{Re}\varepsilon < 1$ to $j < -\mathrm{Re}\varepsilon < j+1$.

Like in the case of the analytic continuation of the distribution $x_+^\lambda$ from $\mathrm{Re}\lambda > -1$ to the whole complex plane [I.M. Gelfand '55], i.e. for integrals

$$\int_0^\infty x^\lambda \phi(x)\mathrm{d}x$$

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

An indirect proof [V.S.'90] of expansion by regions for limits typical of Euclidean space (where one has two different regions which can be called large and small).

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

An indirect proof [V.S.'90] of expansion by regions for limits typical of Euclidean space (where one has two different regions which can be called large and small).

Expansion by subgraphs [K.G. Chetyrkin'88, S. Gorishny'89],

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

An indirect proof [V.S.'90] of expansion by regions for limits typical of Euclidean space (where one has two different regions which can be called large and small).

Expansion by subgraphs [K.G. Chetyrkin'88, S. Gorishny'89], for example, in the off-shell large-momentum limit, i.e. where a momentum $Q$ is considered large and momenta $q_i$ as well as the masses $m_j$ are small,

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

An indirect proof [V.S.'90] of expansion by regions for limits typical of Euclidean space (where one has two different regions which can be called large and small).

Expansion by subgraphs [K.G. Chetyrkin'88, S. Gorishny'89], for example, in the off-shell large-momentum limit, i.e. where a momentum $Q$ is considered large and momenta $q_i$ as well as the masses $m_j$ are small,

$$G_\Gamma \sim \sum_\gamma G_{\Gamma/\gamma} \circ \mathcal{T}_{q_\gamma, m_\gamma} G_\gamma$$

How to find relevant regions?

How to find relevant regions?

For limits typical of Euclidean space, these are regions of large (hard) and small (soft) momenta.

How to find relevant regions?

For limits typical of Euclidean space, these are regions of large (hard) and small (soft) momenta.

For the Regge limit and various versions of the Sudakov limit, these are hard, soft, 1-collinear, . . . , ultrasoft regions.

How to find relevant regions?

For limits typical of Euclidean space, these are regions of large (hard) and small (soft) momenta.

For the Regge limit and various versions of the Sudakov limit, these are hard, soft, 1-collinear, ..., ultrasoft regions.

For the threshold limit $y = m^2 - q^2/4 \to 0$, one has

$$
\begin{aligned}
&\text{(hard)}, && k_0 \sim \sqrt{q^2}\,,\ \vec{k} \sim \sqrt{q^2}\,,\\
&\text{(soft)}, && k_0 \sim \sqrt{y}\,,\ \vec{k} \sim \sqrt{y}\,,\\
&\text{(potential)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim \sqrt{y}\,,\\
&\text{(ultrasoft)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim y/\sqrt{q^2}\,.
\end{aligned}
$$

where $q = (q_0, \vec{0})$.

Expansion by regions in Feynman parameters [V.S.'99], also formulated in the physical language.

Expansion by regions in Feynman parameters [V.S.'99], also formulated in the physical language.
Feynman parametric representation for a Feynman integral with propagators $1/(-p^2 + m_l^2 - i0)$

Expansion by regions in Feynman parameters [V.S.'99], also formulated in the physical language.

Feynman parametric representation for a Feynman integral with propagators $1/(-p^2 + m_l^2 - i0)$

$$\int_0^\infty \ldots \int_0^\infty \delta\left(\sum x_i - 1\right) U^{n-(h+1)d/2} F^{hd/2-n} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Expansion by regions in Feynman parameters [V.S.'99], also formulated in the physical language.
Feynman parametric representation for a Feynman integral with propagators $1/(-p^2+m_l^2-i0)$

$$\int_0^\infty \ldots \int_0^\infty \delta\left(\sum x_i - 1\right)\, U^{n-(h+1)d/2} F^{hd/2-n} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

where $n$ is the number of lines (edges), $h$ is the number of loops (independent circuits) of the graph,

$$F = -V + U \sum m_l^2 x_l \ ,$$

Expansion by regions in Feynman parameters [V.S.'99], also formulated in the physical language.
Feynman parametric representation for a Feynman integral with propagators $1/(-p^2 + m_l^2 - i0)$

$$\int_0^\infty \ldots \int_0^\infty \delta\left(\sum x_i - 1\right) \; U^{n-(h+1)d/2} F^{hd/2-n} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

where $n$ is the number of lines (edges), $h$ is the number of loops (independent circuits) of the graph,

$$F = -V + U \sum m_l^2 x_l \; ,$$

and $U$ and $V$ are two basic functions
(Symanzik polynomials, or graph polynomials).

One can consider quite general limits for a Feynman integral which depends on external momenta $q_i$ and masses and is a scalar function of kinematic invariants and squares of masses, $s_i$, and assume that each $s_i$ has certain scaling $\rho^{\kappa_i}$ where $\rho$ is a small parameter.

One can consider quite general limits for a Feynman integral which depends on external momenta $q_i$ and masses and is a scalar function of kinematic invariants and squares of masses, $s_i$, and assume that each $s_i$ has certain scaling $\rho^{\kappa_i}$ where $\rho$ is a small parameter.

A region $\to$ scaling, i.e. $x_i \to \rho^{r_i} x_i$ where $\rho$ is a small parameter connected with a given limit.

A systematical procedure to find regions based on geometry of polytopes and implemented as a public computer code `asy.m` [A. Pak & A.V. Smirnov'10] which is now included in the code `FIESTA` [A.V. Smirnov'09-16]

A systematical procedure to find regions based on geometry of polytopes and implemented as a public computer code `asy.m` [A. Pak & A.V. Smirnov'10] which is now included in the code `FIESTA` [A.V. Smirnov'09-16]

Using this code one can not only find relevant regions but also evaluate numerically coefficients at powers and logarithms of the given expansion parameter.

A systematical procedure to find regions based on geometry of polytopes and implemented as a public computer code `asy.m` [A. Pak & A.V. Smirnov'10] which is now included in the code `FIESTA` [A.V. Smirnov'09-16]

Using this code one can not only find relevant regions but also evaluate numerically coefficients at powers and logarithms of the given expansion parameter.

Numerous applications have shown that the code `asy.m` works consistently even in cases where the function $F$ is not positive – see, e.g.
[J.M. Henn, K. Melnikov & V.S.'14; F. Caola, J.M. Henn, K. Melnikov & V.S.'14]

Generalizations of this procedure to some cases where terms of the function $F$ are negative
[B. Jantzen, A. Smirnov & V.S.'12]
Potential and Glauber regions.

Generalizations of this procedure to some cases where terms of the function $F$ are negative
[B. Jantzen, A. Smirnov & V.S.'12]
Potential and Glauber regions.

An example: one-loop diagram with two massive lines in the threshold limit $y = m^2 - q^2/4 \to 0$

$$F(q^2, y) = \mathrm{i}\pi^{d/2}\,\Gamma(\varepsilon)$$
$$\times \int_0^\infty \int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\left[\frac{q^2}{4}(\alpha_1-\alpha_2)^2 + y(\alpha_1+\alpha_2)^2 - \mathrm{i}0\right]^\varepsilon}$$

The code `asy.m` in its first version revealed only the contribution of the hard region, i.e. $\alpha_i \sim y^0$.

Decompose integration over $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, with equal contributions.

Decompose integration over $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, with equal contributions.

In the first domain, turn to new variables by
$\alpha_1 = \alpha_1'/2,\ \alpha_2 = \alpha_2' + \alpha_1'/2$ and arrive at

$$i\pi^{d/2} \frac{\Gamma(\varepsilon)}{2} \int_0^\infty \int_0^\infty \frac{(\alpha_1 + \alpha_2)^{2\varepsilon - 2}\ \delta\left(\alpha_1 + \alpha_2 - 1\right)\ \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2 + y(\alpha_1 + \alpha_2)^2 - \mathrm{i}0\right]^\varepsilon} \ .$$

Decompose integration over $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, with equal contributions.

In the first domain, turn to new variables by $\alpha_1 = \alpha_1'/2,\ \alpha_2 = \alpha_2' + \alpha_1'/2$ and arrive at

$$i\pi^{d/2} \frac{\Gamma(\varepsilon)}{2} \int_0^\infty \int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\ \delta\left(\alpha_1+\alpha_2-1\right)\ \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2 + y(\alpha_1+\alpha_2)^2 - \mathrm{i}0\right]^\varepsilon} .$$

Two regions: $(0,0)$ and $(0,1/2)$.

Decompose integration over $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, with equal contributions.

In the first domain, turn to new variables by
$\alpha_1 = \alpha_1'/2,\ \alpha_2 = \alpha_2' + \alpha_1'/2$ and arrive at

$$i\pi^{d/2}\frac{\Gamma(\varepsilon)}{2}\int_0^\infty\int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2+y(\alpha_1+\alpha_2)^2-\mathrm{i}0\right]^\varepsilon}\,.$$

Two regions: $(0,0)$ and $(0,1/2)$. The second one, with
$\alpha_1 \sim y^0, \alpha_2 \sim \sqrt{y}$ gives

$$i\pi^{d/2}\frac{\Gamma(\varepsilon)}{2}\int_0^\infty \frac{\mathrm{d}\alpha_2}{\left(\frac{q^2}{4}\alpha_2^2+y\right)^\varepsilon}\,,$$

[T. Semenova, A. Smirnov & V.S.'19]:
Let us use the parametric representation of Lee and Pomeransky [R.N. Lee and A.A. Pomeransky'13]

[T. Semenova, A. Smirnov & V.S.'19]:
Let us use the parametric representation of Lee and Pomeransky [R.N. Lee and A.A. Pomeransky'13]

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \;,$$

where $\delta = d/2 = 2 - \varepsilon$ and $P = U + F$.

[T. Semenova, A. Smirnov & V.S.'19]:
Let us use the parametric representation of Lee and Pomeransky [R.N. Lee and A.A. Pomeransky'13]

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

where $\delta = d/2 = 2 - \varepsilon$ and $P = U + F$.

Feynman parametric representation can be obtained from it by inserting $1 = \int \delta(\sum_i x_i - \eta) \mathrm{d}\eta$, scaling $x \to \eta x$ and integrating over $\eta$.

Let $t$ be the small parameter, e.g. $-m^2/q^2$ for the Sudakov limit or $(p_1 + p_3)^2/(p_1 + p_2)^2$ for the Regge limit.

Let $t$ be the small parameter, e.g. $-m^2/q^2$ for the Sudakov limit or $(p_1+p_3)^2/(p_1+p_2)^2$ for the Regge limit.

Let $P$ be a polynomial with positive coefficients,

$$P(x_1,\ldots,x_n,t)=\sum_{w\in S} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w_{n+1}} \ ,$$

where $S$ is a finite set of points $w=(w_1,...,w_{n+1})$.

Let $t$ be the small parameter, e.g. $-m^2/q^2$ for the Sudakov limit or $(p_1+p_3)^2/(p_1+p_2)^2$ for the Regge limit.

Let $P$ be a polynomial with positive coefficients,

$$P(x_1,\ldots,x_n,t) = \sum_{w\in S} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w_{n+1}} \ ,$$

where $S$ is a finite set of points $w=(w_1,...,w_{n+1})$.

The Newton polytope $\mathcal{N}_P$ of $P$ is the convex hull of the set $S$ in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ equipped with the scalar product $v\cdot w=\sum_{i=1}^{n+1} v_i w_i$.

Let $t$ be the small parameter, e.g. $-m^2/q^2$ for the Sudakov limit or $(p_1+p_3)^2/(p_1+p_2)^2$ for the Regge limit.

Let $P$ be a polynomial with positive coefficients,

$$P(x_1,\ldots,x_n,t)=\sum_{w\in S} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w_{n+1}} \ ,$$

where $S$ is a finite set of points $w=(w_1,...,w_{n+1})$.

The Newton polytope $\mathcal{N}_P$ of $P$ is the convex hull of the set $S$ in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ equipped with the scalar product $v\cdot w=\sum_{i=1}^{n+1} v_i w_i$.

A facet of $P$ is a face of maximal dimension, i.e. $n$.

The main conjecture.

The asymptotic expansion of

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

in the limit $t \to +0$ is given by

$$G(t,\varepsilon) \sim \sum_\gamma \int_0^\infty \ldots \int_0^\infty \left[ M_\gamma \left( P(x_1,\ldots,x_n,t) \right)^{-\delta} \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of $P$, for which the normal vectors $r^\gamma = (r_1^\gamma, \ldots, r_n^\gamma, r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$.

## The main conjecture.

The asymptotic expansion of

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

in the limit $t \to +0$ is given by

$$G(t,\varepsilon) \sim \sum_\gamma \int_0^\infty \ldots \int_0^\infty \left[ M_\gamma \left( P(x_1, \ldots, x_n, t) \right)^{-\delta} \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of $P$, for which the normal vectors $r^\gamma = (r_1^\gamma, \ldots, r_n^\gamma, r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$.
Let us call these facets *essential*.

## The main conjecture.

The asymptotic expansion of

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

in the limit $t \to +0$ is given by

$$G(t,\varepsilon) \sim \sum_\gamma \int_0^\infty \ldots \int_0^\infty \left[ M_\gamma \left( P(x_1, \ldots, x_n, t) \right)^{-\delta} \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of $P$, for which the normal vectors $r^\gamma = (r_1^\gamma, \ldots, r_n^\gamma, r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$.
Let us call these facets *essential*.

Let us normalize these vectors by $r_{n+1}^\gamma = 1$.

The contribution of a given essential facet is defined by the change of variables $x_i \to t^{r_i^\gamma} x_i$ in the integral and expanding the resulting integrand in powers of $t$.

The contribution of a given essential facet is defined by the change of variables $x_i \rightarrow t^{r_i^\gamma} x_i$ in the integral and expanding the resulting integrand in powers of $t$.
This leads to the following definitions.

The contribution of a given essential facet is defined by the change of variables $x_i \to t^{r_i^\gamma} x_i$ in the integral and expanding the resulting integrand in powers of $t$.
This leads to the following definitions.

For a given essential facet $\gamma$, let us define the polynomial

$$P^\gamma(x_1, \dots, x_n, t) = P(t^{r_1^\gamma} x_1, \dots, t^{r_n^\gamma} x_n, t) \equiv \sum_{w \in S} c_w x_1^{w_1} \dots x_n^{w_n} t^{w \cdot r^\gamma}$$

The contribution of a given essential facet is defined by the change of variables $x_i \to t^{r_i^\gamma} x_i$ in the integral and expanding the resulting integrand in powers of $t$.
This leads to the following definitions.

For a given essential facet $\gamma$, let us define the polynomial

$$P^\gamma(x_1, \ldots, x_n, t) = P(t^{r_1^\gamma} x_1, \ldots, t^{r_n^\gamma} x_n, t) \equiv \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w \cdot r^\gamma}$$

The scalar product $w \cdot r^\gamma$ is proportional to the projection of the point $w$ on the vector $r^\gamma$. For $w \in S$, it takes a minimal value for all the points belonging to the considered facet $w \in S \cap \gamma$. Let us denote it by $L(\gamma)$.

The polynomial $P^\gamma$ can be represented as

$$t^{L(\gamma)}\left(P_0^\gamma(x_1,\ldots,x_n)+P_1^\gamma(x_1,\ldots,x_n,t)\right)\ ,$$

The polynomial $P^\gamma$ can be represented as

$$t^{L(\gamma)}\left(P_0^\gamma(x_1,\ldots,x_n)+P_1^\gamma(x_1,\ldots,x_n,t)\right)\;,$$

where

$$\begin{aligned} P_0^\gamma(x_1,\ldots,x_n) &= \sum_{w\in S\cap\gamma} c_w x_1^{w_1}\ldots x_n^{w_n}\;, \\ P_1^\gamma(x_1,\ldots,x_n,t) &= \sum_{w\in S\setminus\gamma} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w\cdot r^\gamma - L(\gamma)}\;. \end{aligned}$$

The polynomial $P^\gamma$ can be represented as

$$t^{L(\gamma)}\left(P_0^\gamma(x_1,\ldots,x_n)+P_1^\gamma(x_1,\ldots,x_n,t)\right) \;,$$

where

$$\begin{aligned} P_0^\gamma(x_1,\ldots,x_n) &= \sum_{w\in S\cap\gamma} c_w x_1^{w_1}\ldots x_n^{w_n} \;, \\ P_1^\gamma(x_1,\ldots,x_n,t) &= \sum_{w\in S\setminus\gamma} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w\cdot r^\gamma - L(\gamma)} \;. \end{aligned}$$

The polynomial $P_0^\gamma$ is independent of $t$ while $P_1^\gamma$ can be represented as a linear combination of positive rational powers of $t$ with coefficients which are polynomials of $x$.

For a given facet $\gamma$, the operator $M_\gamma$ acts on the integrand as follows

$$
\begin{aligned}
& M_\gamma \left(P(x_1,\ldots,x_n,t)\right)^{-\delta} \\
= \quad & t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t \left(P_0^\gamma(x_1,\ldots,x_n) + P_1^\gamma(x_1,\ldots,x_n,t)\right)^{-\delta} \\
= \quad & t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \ \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} + \ldots
\end{aligned}
$$

For a given facet $\gamma$, the operator $M_\gamma$ acts on the integrand as follows

$$
\begin{aligned}
& M_\gamma \left( P(x_1, \ldots, x_n, t) \right)^{-\delta} \\
= \;& t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t \left( P_0^\gamma(x_1, \ldots, x_n) + P_1^\gamma(x_1, \ldots, x_n, t) \right)^{-\delta} \\
= \;& t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \, \left( P_0^\gamma(x_1, \ldots, x_n) \right)^{-\delta} + \ldots
\end{aligned}
$$

where $\mathcal{T}_t$ performs an asymptotic expansion in powers of $t$ at $t = 0$.

For a given facet $\gamma$, the operator $M_\gamma$ acts on the integrand as follows

$$\begin{aligned} & M_\gamma \left(P(x_1,\ldots,x_n,t)\right)^{-\delta} \\ = \;& t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t \left(P_0^\gamma(x_1,\ldots,x_n) + P_1^\gamma(x_1,\ldots,x_n,t)\right)^{-\delta} \\ = \;& t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \; \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} + \ldots \end{aligned}$$

where $\mathcal{T}_t$ performs an asymptotic expansion in powers of $t$ at $t = 0$.

In particular, the LO term of a given facet $\gamma$

$$t^{-L(\gamma)\delta + \sum_{i=1}^n r_i^\gamma} \int_0^\infty \ldots \int_0^\infty \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \; .$$

An example:

$$G(t,\varepsilon) = \int_0^\infty (x^2 + x + t)^{\varepsilon - 1} \mathrm{d}x$$

in the limit $t \to 0$.

An example:

$$G(t,\varepsilon) = \int_0^\infty (x^2 + x + t)^{\varepsilon - 1} \mathrm{d}x$$

in the limit $t \to 0$.

$P(x,t) = \sum_{(w_1,w_2)\in S} c_{(w_1,w_2)} x^{w_1} t^{w_2}$

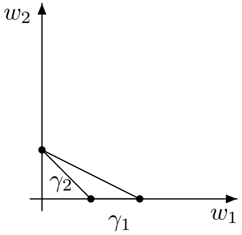
An example:

$$G(t,\varepsilon) = \int_0^\infty (x^2 + x + t)^{\varepsilon - 1} \mathrm{d}x$$

in the limit $t \to 0$.

$P(x,t) = \sum_{(w_1,w_2)\in S} c_{(w_1,w_2)} x^{w_1} t^{w_2}$

The Newton polytope (triangle)



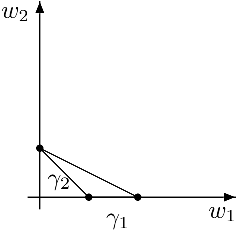
An example:

$$G(t,\varepsilon) = \int_0^\infty (x^2 + x + t)^{\varepsilon - 1} \mathrm{d}x$$

in the limit $t \to 0$.

$P(x,t) = \sum_{(w_1,w_2)\in S} c_{(w_1,w_2)} x^{w_1} t^{w_2}$

The Newton polytope (triangle)




Two essential facets $\gamma_1$ and $\gamma_2$ with the corresponding normal vectors $r_1 = (0,1)$ and $r_2 = (1,1)$.

$\gamma_1 \rightarrow$ expanding the integrand in $t$. L0 is given by

$$\int_0^\infty (x^2 + x)^{\varepsilon - 1} \mathrm{d}x = \frac{\Gamma(1 - 2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1 - \varepsilon)}$$

$\gamma_1 \to$ expanding the integrand in $t$. L0 is given by

$$\int_0^\infty (x^2+x)^{\varepsilon-1}\mathrm{d}x = \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-\varepsilon)}$$

$\gamma_2 \to t$ times the integral of the integrand with $x \to tx$ expanded in powers of $t$. L0 is given by

$$t^\varepsilon \int_0^\infty (x+1)^{\varepsilon-1}\mathrm{d}x = -\frac{t^\varepsilon}{\varepsilon}$$

$\gamma_1 \to$ expanding the integrand in $t$. L0 is given by

$$\int_0^\infty (x^2+x)^{\varepsilon-1}\mathrm{d}x = \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-\varepsilon)}$$

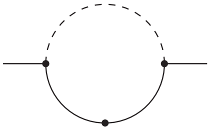
$\gamma_2 \to t$ times the integral of the integrand with $x \to tx$ expanded in powers of $t$. L0 is given by

$$t^\varepsilon \int_0^\infty (x+1)^{\varepsilon-1}\mathrm{d}x = -\frac{t^\varepsilon}{\varepsilon}$$

The sum of the contributions in the LO:

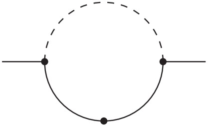
$$G(t,\varepsilon) \sim -\log t + O(\varepsilon)$$

## Another example



in the limit $m^2/q^2 \to 0$.

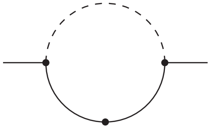
## Another example



in the limit $m^2/q^2 \to 0$.

$$G(t,\varepsilon) = \int_0^\infty (P(x_1, x_2, t))^{\varepsilon-2} x_1 \mathrm{d}x_1 \mathrm{d}x_2$$

Another example



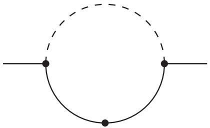
in the limit $m^2/q^2 \to 0$.

$$G(t,\varepsilon) = \int_0^\infty (P(x_1,x_2,t))^{\varepsilon-2} x_1 \mathrm{d}x_1 \mathrm{d}x_2$$

where

$$P = U + F = \sum_{w=(w_1,w_2,w_3)\in S} c_w x_1^{w_1} x_2^{w_2} t^{w_3} \ ,$$

## Another example



in the limit $m^2/q^2 \to 0$.

$$G(t,\varepsilon) = \int_0^\infty (P(x_1,x_2,t))^{\varepsilon-2} x_1 \mathrm{d}x_1 \mathrm{d}x_2$$

where

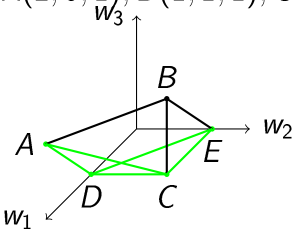
$$P = U + F = \sum_{w=(w_1,w_2,w_3)\in S} c_w x_1^{w_1} x_2^{w_2} t^{w_3} \;,$$

$$F = x_1(t(x_1+x_2)+x_2)\;, \quad U = x_1 + x_2$$

The vertices $A, B, C, D, E$ of the Newton polytope coincide with the set $S$

$$F = x_1(t(x_1 + x_2) + x_2)\ , \quad U = x_1 + x_2\ ;$$

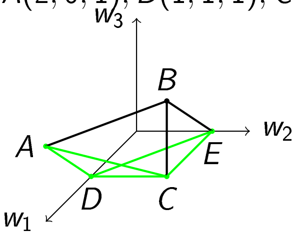
$$A(2,0,1), B(1,1,1), C(1,1,0), D(1,0,0), E(0,1,0).$$



The vertices $A, B, C, D, E$ of the Newton polytope coincide with the set $S$

$$F = x_1(t(x_1 + x_2) + x_2)\ , \quad U = x_1 + x_2\ ;$$

$A(2,0,1), B(1,1,1), C(1,1,0), D(1,0,0), E(0,1,0).$

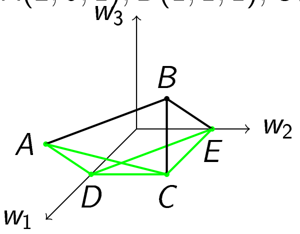



Two essential facets:

The vertices $A, B, C, D, E$ of the Newton polytope coincide with the set $S$

$$F = x_1(t(x_1 + x_2) + x_2)\ , \quad U = x_1 + x_2\ ;$$

$A(2,0,1), B(1,1,1), C(1,1,0), D(1,0,0), E(0,1,0).$



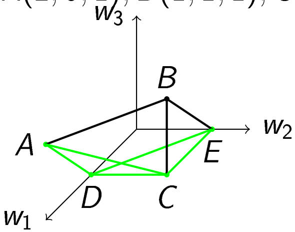

Two essential facets:

$CDE \in$ the plane $w_3 = 0$, with the normal vector $(0,0,1) \to$ expansion in $t$.

The vertices $A, B, C, D, E$ of the Newton polytope coincide with the set $S$

$$F = x_1(t(x_1 + x_2) + x_2)\ , \quad U = x_1 + x_2\ ;$$

$A(2,0,1), B(1,1,1), C(1,1,0), D(1,0,0), E(0,1,0).$




Two essential facets:

$CDE \in$ the plane $w_3 = 0$, with the normal vector $(0,0,1) \to$ expansion in $t$.

$ACD \in$ the plane $w_1 - w_3 = 1$, with the normal vector $(-1,0,1)$

$\to t^{-2} \int_0^\infty x_1 \left[x_1/t + x_2 + (x_1/t)(t(x_1/t + x_2)\right]^{\varepsilon-2} = \ldots$

**A typical feature** of results obtained within expansion by regions (or, subgraphs) is the appearance of poles in $\delta$ or $\varepsilon$ on the right-hand side: usually, they are infrared and ultraviolet but they can be also collinear.

**A typical feature** of results obtained within expansion by regions (or, subgraphs) is the appearance of poles in $\delta$ or $\varepsilon$ on the right-hand side: usually, they are infrared and ultraviolet but they can be also collinear.

The cancellation of these poles is a very natural check of the expansion procedure, i.e. the pole part of the sum of terms of the expansion should be equal to the pole part of the initial integral.

## Complications.

**1.** The contribution of each essential facet to the expansion is evaluated in the corresponding domain of $\delta$ where it is convergent and then the result it continued analytically to a desired domain.

## Complications.

**1.** The contribution of each essential facet to the expansion is evaluated in the corresponding domain of $\delta$ where it is convergent and then the result it continued analytically to a desired domain. Maybe, it will be natural to proceed with subtraction operators.

## Complications.

**1.** The contribution of each essential facet to the expansion is evaluated in the corresponding domain of $\delta$ where it is convergent and then the result it continued analytically to a desired domain. Maybe, it will be natural to proceed with subtraction operators.

**2.** One has to distinguish situations where contributions of individual facets are not regularized by the initial regularization parameter $\delta$. A natural way to proceed is to introduce auxiliary analytic regularization by inserting powers $x_i^{\lambda_i}$.
For Feynman integrals at Euclidean external momenta, Speer proved that the corresponding dimensionally and analytically regularized parametric integral is convergent in a non-empty domain of parameters $(\varepsilon, \lambda_1, \ldots, \lambda_n)$.
A generalization of Speer's theorem to the case of LP representation [T. Semenova, A. Smirnov & V.S.'19].

**Advantages of the new formulation.**

**1.** The degree of $P = U + F$ is less than the degree of $UF$. Therefore, the current version of asy is much more powerful.

**Advantages of the new formulation.**

**1.** The degree of $P = U + F$ is less than the degree of $UF$.
Therefore, the current version of `asy` is much more powerful.
Equivalence of expansion by regions for Feynman integrals based on the standard Feynman parametric representation and the LP representation (implemented in `FIESTA`) was proven [T. Semenova, A. Smirnov & V.S.'19]

**Advantages of the new formulation.**

**1.** The degree of $P = U + F$ is less than the degree of $UF$. Therefore, the current version of `asy` is much more powerful. Equivalence of expansion by regions for Feynman integrals based on the standard Feynman parametric representation and the LP representation (implemented in `FIESTA`) was proven [T. Semenova, A. Smirnov & V.S.'19]

**2.** The new formulation has more chances to be proven. A proof in a special case [T. Semenova, A. Smirnov & V.S.'19].

The leading contribution of a given essential facet.

The leading contribution of a given essential facet.

If the point $\left(\frac{1}{\delta}, \ldots, \frac{1}{\delta}\right) \in \mathbb{R}^n$ is inside $\pi(\Gamma)$ for some facet $\Gamma$ then the leading asymptotics is given by

$$t^{-L(\Gamma)\delta+\sum_i r_i^\Gamma} \int_0^\infty \ldots \int_0^\infty \left( \sum_{w \in \Gamma \cap S} c_w y_1^{w_1} \ldots y_n^{w_n} \right)^{-\delta} \mathrm{d}y_1 \ldots \mathrm{d}y_n$$

when $t \to +0$.

- Expansion by regions is a very important strategy successfully applied in numerous calculations.

- Expansion by regions is a very important strategy successfully applied in numerous calculations.
- The Lee–Pomeransky representation looks very natural to be used in proving expansion by regions.

- Expansion by regions is a very important strategy successfully applied in numerous calculations.
- The Lee–Pomeransky representation looks very natural to be used in proving expansion by regions.
- For the moment, expansion by regions still has the status of experimental mathematics. Hopefully, it will be mathematically justified.

- Expansion by regions is a very important strategy successfully applied in numerous calculations.
- The Lee–Pomeransky representation looks very natural to be used in proving expansion by regions.
- For the moment, expansion by regions still has the status of experimental mathematics. Hopefully, it will be mathematically justified.
- *Divide et impera*