# PHYSICAL PROJECTORS FOR MULTILOOP SCATTERING AMPLITUDES

**The IR in QFT**
**Paris 02/03/2020**

based on work with T. Peraro and F. Caola
[arXiv:1906.03298 , arXiv:20xx.xxxxx]

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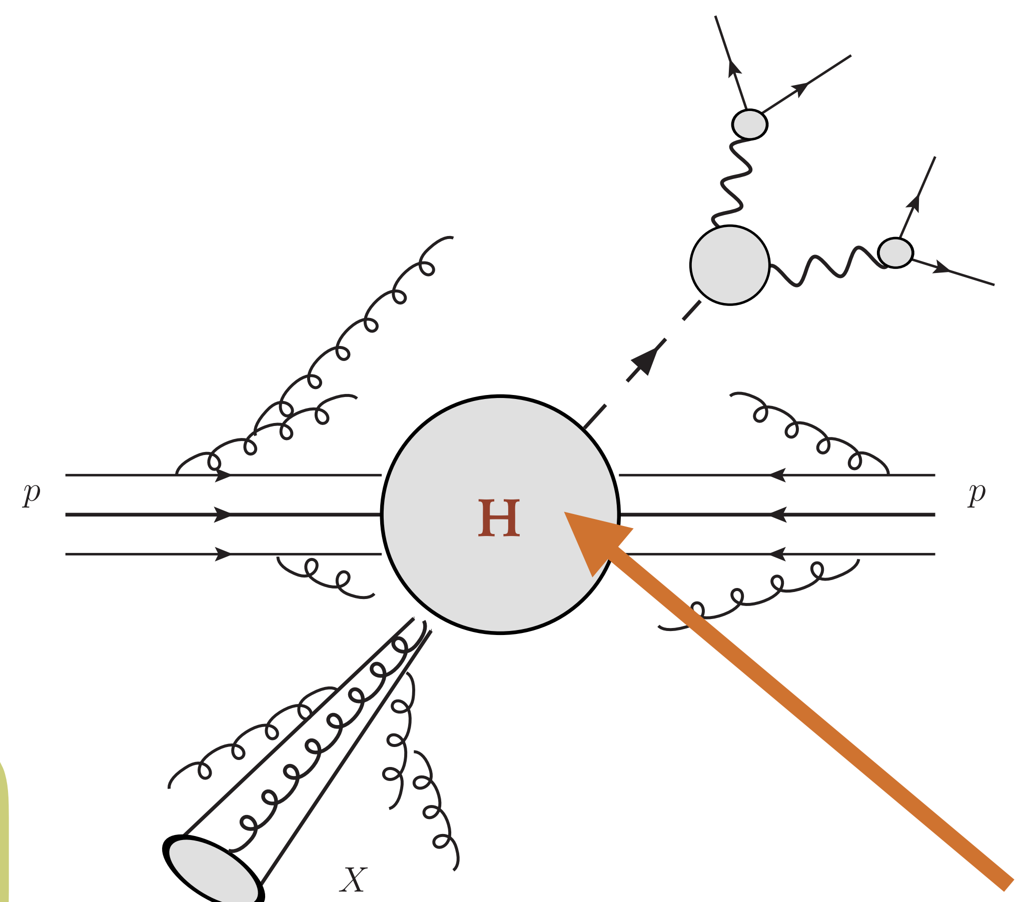
# SCATTERING AMPLITUDES IN QFT

Scattering amplitudes are one of the main ingredients to extract physical predictions from QFT

$$pp \to HX \to l_1\bar{l}_1 + l_2\bar{l}_2 + X$$

**Factorisation** of long and short range physics




**Non-perturbative** physics modelled in PDFs etc…

<u>HARD SCATTERING</u>

scattering amplitudes!

# SCATTERING AMPLITUDES IN QFT

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

Typically, **scattering amplitudes** can only be computed as **perturbative series** in the coupling constant(s), which involves expansion in ***number of loops*** and ***number of legs***

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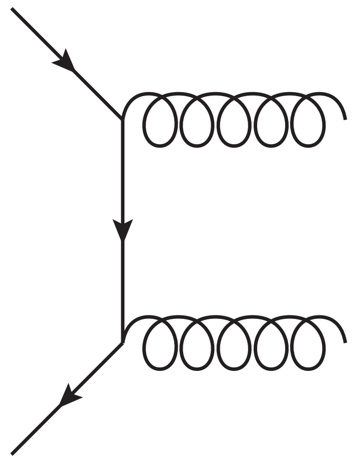

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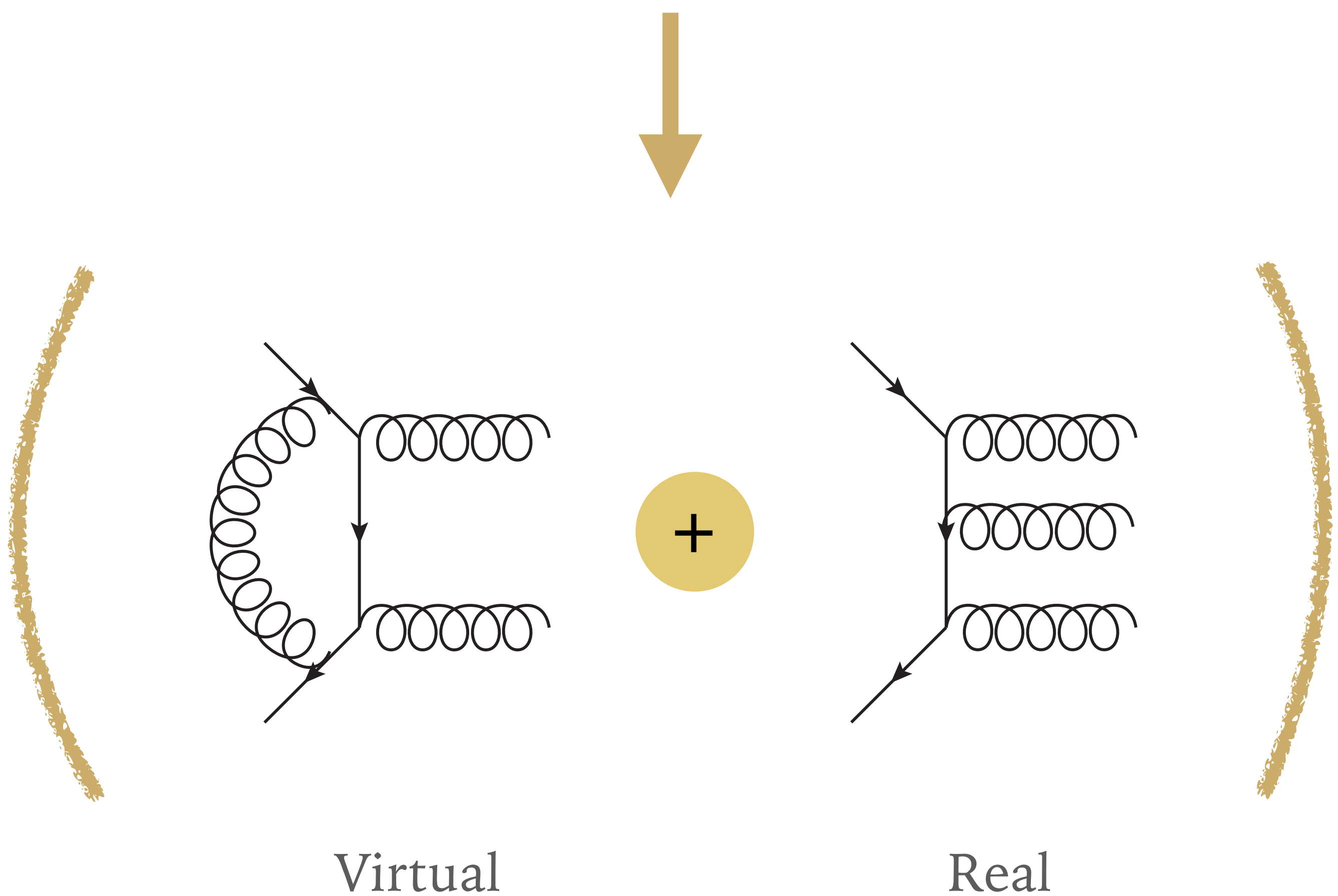

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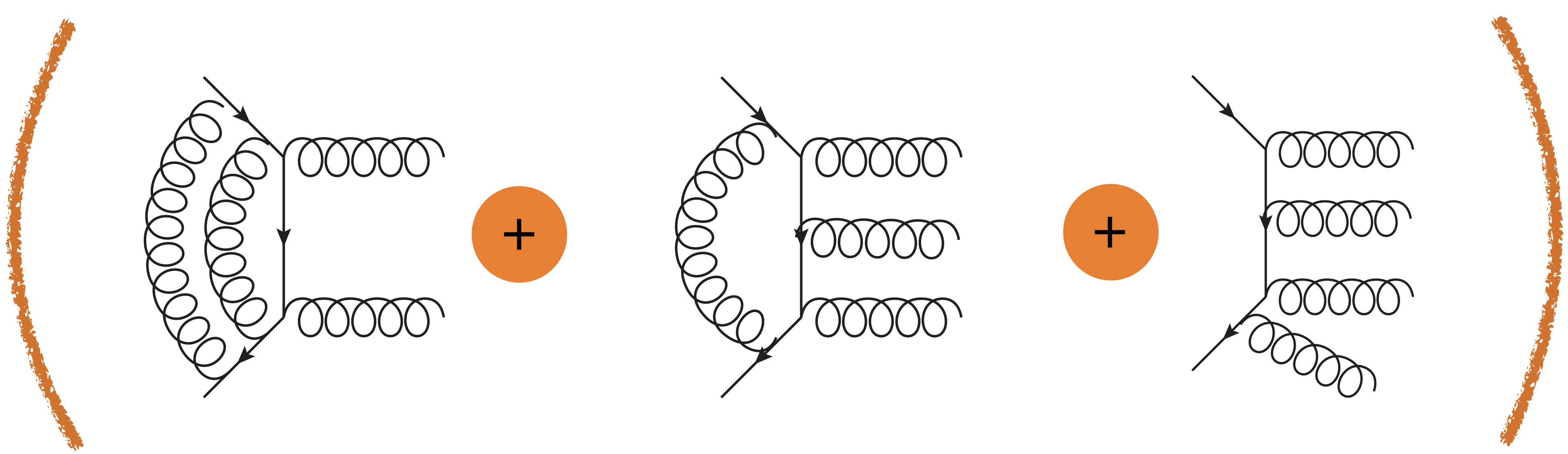


Double Virtual Real Virtual Double Real

# INCREASING COMPLEXITY

Complexity increases (obviously) from many points of view

1. **combinatorial:** number of objects to compute increases
2. **analytical:** new mathematical structures appear and must be understood
3. **structure:** *splitting into different ingredients introduces spurious IR poles, whose cancellation in physical observables becomes more and more cumbersome… (see L. Magnea's talk)*

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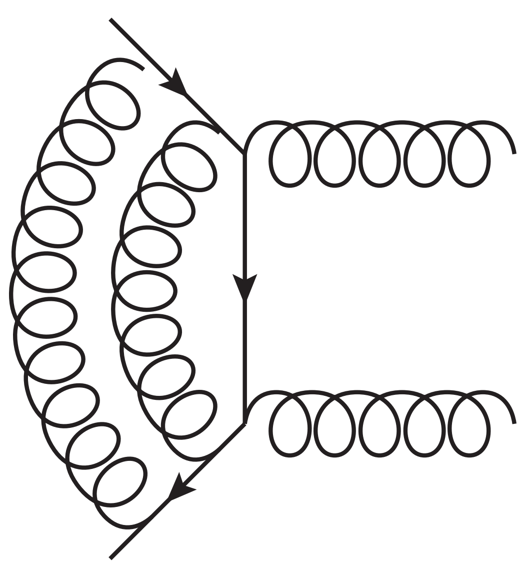
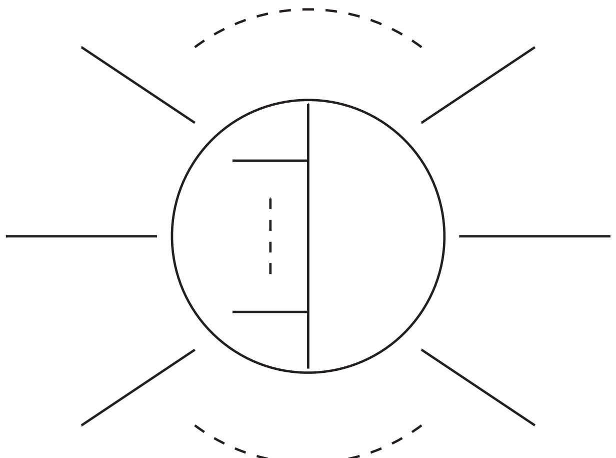


**My focus here:**

how to organise calculation of multi-loop (virtual) scattering amplitudes

# MULTILOOP SCATTERING AMPLITUDES: THE STANDARD WAY

One way to go about it: **standard approach *(divide et impera)***



$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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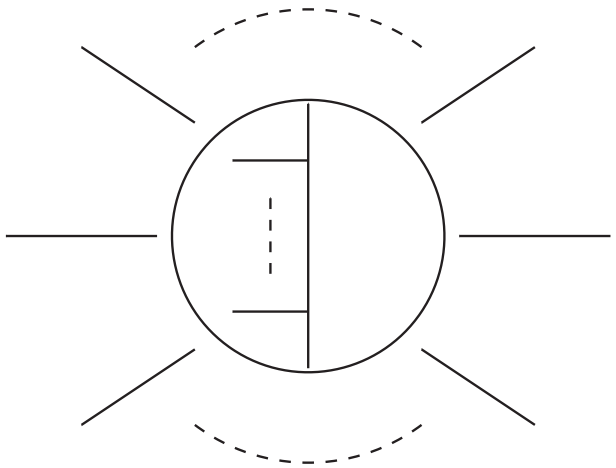


$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

**Standard steps:**

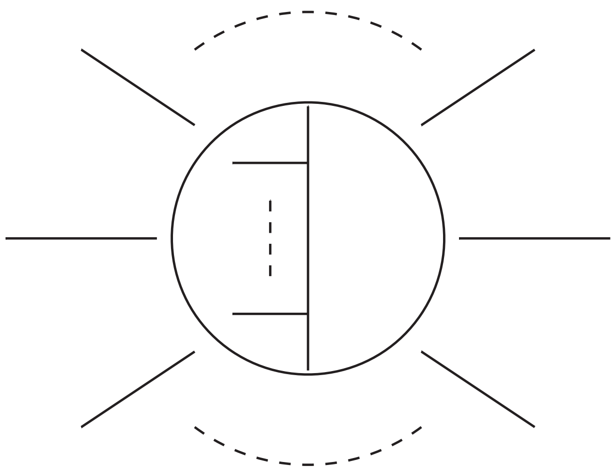
1) Obtain *somehow* the **integrand** (From Feynman diagrams, Unitarity, …?)

2) Somehow reduce this integrand to a **basis** of **integrals** to compute *(T. Peraro's talk)*

3) Compute the **integrals** *(for once I will NOT talk about that!)*

# WHAT ABOUT THE INTEGRAND?

First problem is ***"getting the integrand"***:



$$= \sum \textit{Feynman Diagrams} \;\rightarrow\; ?$$

**Problems:**

- Number of diagrams ***grows factorially*** *(not a real problem though, at least for reasonable processes in QCD…)*

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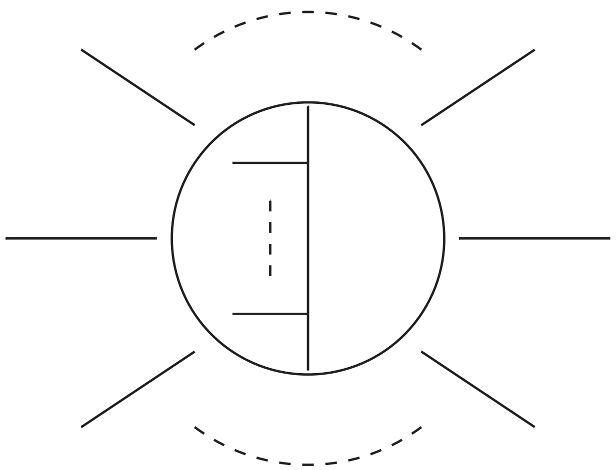

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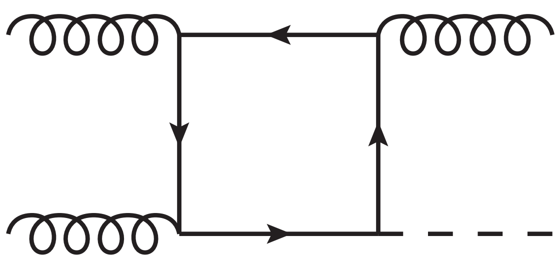
**Problems:**

- Number of diagrams ***grows factorially*** *(not a real problem though, at least for reasonable processes in QCD…)*
- **More serious problem(s):** *"tensor decomposition"*

# TENSOR DECOMPOSITION

$\mathcal{M}_{gg\to Hg} \sim$ 

Strip it of Lorentz and Dirac structures

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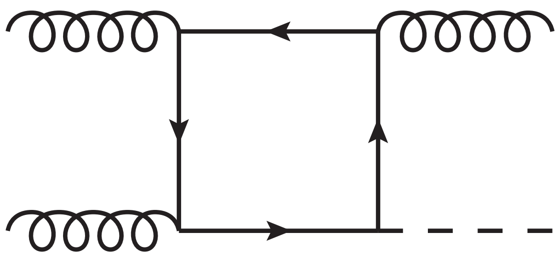
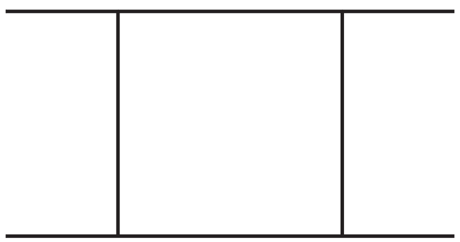


**Strip it of Lorentz and Dirac structures**

$$\sim \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} =$$



**Scalar Feynman Integrals** are what we know how to compute

I will talk about an ANCIENT method to do this: ***the projector - form factor method***

# THE PROJECTOR-FORM FACTOR METHOD

The idea is very simple:

1. Use **Lorentz invariance**, **gauge invariance** (and any other allowed **symmetries**) to ***parametrise*** the scattering amplitude at ***any number of loops*** in terms of <u>tensor structures</u> and <u>scalar form factors</u>
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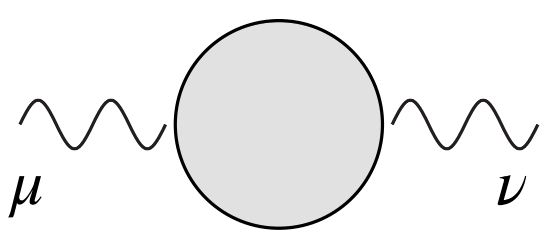

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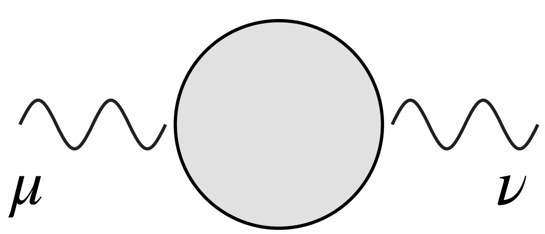


$$= \sum_{i=1}^{n} F_i \, T_i^{\mu\nu} = \left( F_1(p,m^2)\, p^\mu p^\nu + F_2(p,m^2)\, g^{\mu\nu} \right)$$

**Lorentz Invariance**

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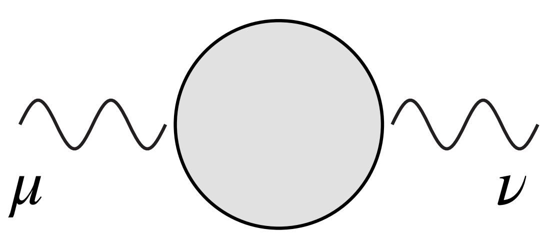


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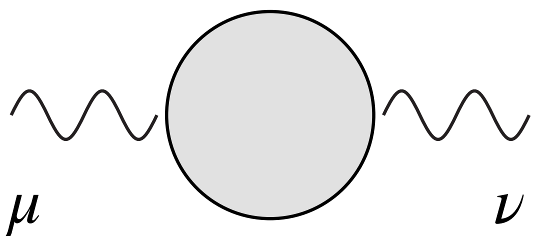
**Lorentz Invariance**

$$= \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) F(p, m^2)$$

**Gauge Invariance!**

# THE PROJECTOR-FORM FACTOR METHOD




$$= \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) F(p, m^2) = \Pi^{\mu\nu}$$

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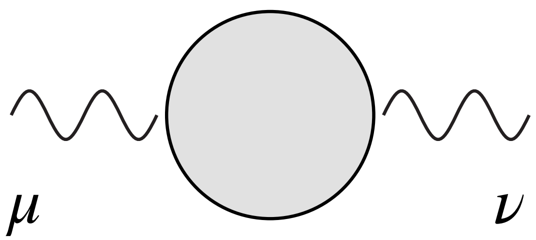


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To extract $F(p, m^2)$ I define a projector operator $P_{\mu\nu} = C(d, p, m^2)\left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$

I can then determine the coefficient $C(d, p, m^2)$ by imposing $P_{\mu\nu}\Pi^{\mu\nu} = F(p, m^2)$

We find $$P_{\mu\nu} = \frac{1}{d-1}\left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

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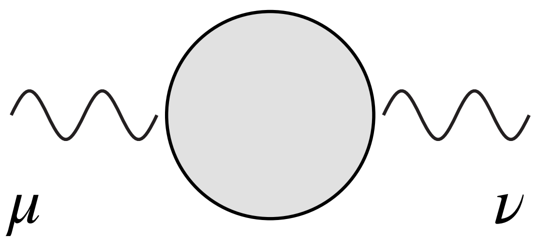


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Now at any number of loops, the form factor $F(p,m^2)$ can be obtained by generating Feynman diagrams and applying the projector $P_{\mu\nu}$ on each of them *(or on clever combinations of them… or on any other representation you might have)*

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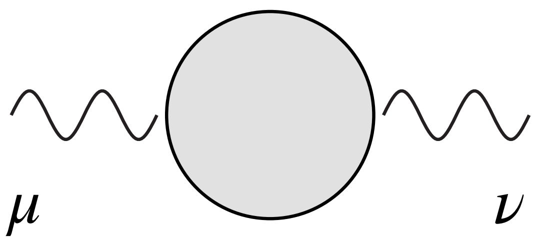


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All algebra has to be performed in **d space-time dimensions** to be able to use the method in **CDR** (Conventional Dimensional Regularisation)

# THE PROJECTOR-FORM FACTOR METHOD

**Works in general, no restrictions of any kinds in principle:**

1. Pick your favourite process
2. Use Lorentz + gauge + any symmetry (parity, Bose etc…) to find minimal set of tensor structures in d space-time dimensions
3. Derive projectors operators to single out corresponding form factors
4. Apply these projectors on your favourite representation for the scattering amplitude

$$\mathscr{A} = \sum_j F_j T_j \quad \rightarrow \quad M_{ij} = \sum_{pol} T_i T_j^\dagger$$

$$\mathscr{P}_j = \sum_k \left(M^{-1}\right)_{jk} T_k^\dagger \quad \rightarrow \quad \boxed{\mathscr{P}_j \mathscr{A} = F_j}$$

# PROBLEMS WITH PROJECTORS

$$P_{\mu\nu} = \frac{1}{d-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

Seems neat. Where are the issues?

Let's have a look at a less simple example: **massless quark scattering** $q\bar{q} \to Q\bar{Q}$

Studied **up to 2 loops** first by N. Glover in hep-ph/0401119

$$0 \to q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) + Q(p_3, \lambda_3) + \bar{Q}(p_4, \lambda_4)$$

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What is the most general d-dimensional tensor structure?

$$\bar{u}(p_1)\Gamma^{\mu_1,\ldots,\mu_n} u(p_2)\ \bar{u}(p_3)\Gamma_{\mu_1,\ldots,\mu_n} u(p_4)$$

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**Problem**: $\gamma$-algebra is not closed in d-dimensions!

In principle at arbitrary loops I can build arbitrary fermion lines with arbitrary numbers of  matrices and they will all be independent!

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Forget about n loops then. Let's follow Glover @ **2 loops**:

$$\mathscr{A}^{(2l)}_{qqQQ} = \sum_{j=1}^{6} A_j \, D_j$$

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$$\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1} u(p_4), \\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4), \\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_4), \\
\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5} u(p_4).
\end{aligned}$$

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$$
\begin{aligned}
\mathcal{P}(A_2) = & \frac{1}{32 s_{13}^2 s_{23}^2 s_{12}^2 (d-5)(d-7)(d-3)(d-4)} \times \Big( \\
& - s_{13}(35 s_{23}^2 d^3 - 55 s_{13} s_{23} d^3 + 1046 s_{13} s_{23} d^2 - 1872 s_{13}^2 d + 2432 s_{13}^2 - 454 s_{23}^2 d^2 \\
& -6040 s_{13} s_{23} d - 2688 s_{23}^2 + 368 s_{13}^2 d^2 + 1928 s_{23}^2 d - 20 s_{13}^2 d^3 + 11136 s_{13} s_{23}) \mathcal{D}_1^\dagger \\
& + 2 s_{13}(-2 s_{13}^2 d^2 - 9 s_{13} s_{23} d^2 + 142 s_{13} s_{23} d - 448 s_{13} s_{23} + 7 s_{23}^2 d^2 + 136 s_{23}^2 - 48 s_{13}^2 \\
& +28 s_{13}^2 d - 62 s_{23}^2 d) \mathcal{D}_3^\dagger \\
& + (-340 s_{13}^2 d^3 + 11008 s_{13}^2 - 740 s_{13} s_{23} d^3 + 44032 s_{13} s_{23} - 260 s_{23}^2 d^3 - 4144 s_{23}^2 d + 3712 s_{23}^2 \\
& +15 s_{13}^2 d^4 + 2852 s_{13}^2 d^2 - 28864 s_{13} s_{23} d + 1604 s_{23}^2 d^2 + 6944 s_{13} s_{23} d^2 - 9968 s_{13}^2 d \\
& +30 s_{13} s_{23} d^4 + 15 s_{23}^2 d^4) \mathcal{D}_2^\dagger \\
& - s_{13} s_{23} (12 s_{13} + s_{23} d - 4 s_{23} - s_{13} d) \mathcal{D}_5^\dagger \\
& + (-6 s_{23}^2 d + 24 s_{13}^2 + 2 s_{13} s_{23} d^2 - 40 s_{13} s_{23} d - 14 s_{13}^2 d + s_{13}^2 d^2 + 8 s_{23}^2 + s_{23}^2 d^2 + 192 s_{13} s_{23}) \mathcal{D}_6^\dagger \\
& - 2(5 s_{13}^2 d^3 + 5 s_{23}^2 d^3 + 10 s_{13} s_{23} d^3 - 240 s_{13} s_{23} d^2 - 100 s_{13}^2 d^2 - 56 s_{23}^2 d^2 + 580 s_{13}^2 d \\
& +1832 s_{13} s_{23} d + 196 s_{23}^2 d - 208 s_{23}^2 - 800 s_{13}^2 - 4224 s_{13} s_{23}) \mathcal{D}_4^\dagger \Big),
\end{aligned}
$$

with growth of number of tensors, the inversion can become extremely expensive!

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Define the 6x6 matrix $M_{ij} = \sum_{pol} D_i D_j^\dagger$, its inverse will provide us with the relevant projectors

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& \quad +15 s_{13}^2 d^4 + 2852 s_{13}^2 d^2 - 28864 s_{13} \ldots \\
& \quad +30 s_{13} s_{23} d^4 + 15 s_{23}^2 d^4)\mathcal{D}_2^\dagger \\
& - s_{13} s_{23}(12 s_{13} + s_{23} d - 4 s_{23} - s_{13} d \ldots \\
& + (-6 s_{23}^2 d + 24 s_{13}^2 + 2 s_{13} s_{23} d^2 - 40 \ldots s_{13} s_{23})\mathcal{D}_6^\dagger \\
& - 2(5 s_{13}^2 d^3 + 5 s_{23}^2 d^3 + 10 s_{13} s_{23} d^3 - 240 s_{13} s_{23} d^2 - 100 s_{13}^2 d^2 - 56 s_{23}^2 d^2 + 580 s_{13}^2 d \\
& \quad +1832 s_{13} s_{23} d + 196 s_{23}^2 d - 208 s_{23}^2 - 800 s_{13}^2 - 4224 s_{13} s_{23})\mathcal{D}_4^\dagger \Big),
\end{aligned}
$$

**Artificial poles in $d \to 4$**

**They arise because the tensors we have chosen are actually NOT independent in d=4**

**Matrix not invertible in d=4**

# HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

What are we interested in are **helicity amplitudes**, in d=4 in **'t Hooft-Veltman scheme**

$$\mathscr{A} = \sum_{i=1}^{n} F_j T_j \longrightarrow \mathscr{A}(\lambda_1, \ldots, \lambda_E) = \sum_{i=1}^{n} F_j T_j(\lambda_1, \ldots, \lambda_E) = \sum_{j=1}^{m<n} \bar{F}_j \, S_j(\lambda_1, \ldots, \lambda_E)$$

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Combinations of original form factors

Helicity amplitudes, spinor products, momentum twistors…

"By definition", in 't Hooft-Veltman scheme ***there cannot be more independent form factors than independent helicity amplitudes***

Indeed for massless $q\bar{q} \to Q\bar{Q}$ there are 4 helicities, reduced to 2 by parity invariance!

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For $q\bar{q} \to Q\bar{Q}$, M is <u>not invertible in d=4</u>, but a **2x2 restriction of M is invertible**!

I can choose any 2 independent tensors, any other **<u>(with any number of $\gamma$ matrices)</u>**, will be linearly dependent in d=4!

# HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

Let then pick 2: $T_j = D_j\,,\;\; j = 1{,}2$ $\qquad M^{2\times 2}_{ij} = T_i^\dagger T_j\,,$

$$\left(M^{2\times 2}\right)^{-1}_{ij} = \frac{1}{d-3} X_{ij}$$

with

$$X_{ij} = \frac{1}{4\, s_{12}^2} \begin{pmatrix} 1 & \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} \\ \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} & \dfrac{(d-2)s_{12}^2 + 4s_{23}(s_{12}+s_{23})}{s_{23}^2(s_{12}+s_{23})^2} \end{pmatrix}$$

the matrix is smooth in $d \to 4$

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$$\overline{T}_i = T_i - \sum_{j=1}^{2}\left(\overline{P}_j T_i\right)\overline{T}_j\,,\quad \text{for} \quad i = 3,4,5,6,\ldots$$

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I am effectively **block-diagonalising the matrix**!

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New tensors are *smooth* linear combinations of the old ones:

$$\bar{T}_3 = \left(-3d - \frac{12 s_{23}}{s_{12}} - 4\right) \bar{T}_1 - \frac{24}{s_{12}} \bar{T}_2 + T_3$$

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And the new 6x6 inverse matrix becomes **block-diagonal**

$$\left(\bar{M}_{ij}\right)^{-1} = \begin{pmatrix} \frac{X_{ij}}{d-3} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \boldsymbol{R_{ij}} & \\ 0 & & & \end{pmatrix}$$

*$R_{ij}$ contains the complexity that we saw before, but actually NEVER need to even compute it!*

# HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

By construction, helicity amplitudes only receive contributions from two tensors!

And **the projectors that we need to apply on the Feynman diagrams are much simpler**

$$\bar{P}_1 = \frac{1}{4s_{12}^2(d-3)}\left(T_1^\dagger + \frac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})}T_2^\dagger\right)$$

$$\bar{P}_2 = \frac{1}{4\,s_{12}^2(d-3)}\left(\frac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})}\,T_1^\dagger + \frac{(d-2)s_{12}^2+4s_{23}(s_{12}+s_{23})}{s_{23}^2(s_{12}+s_{23})^2}\,T_2^\dagger\right)$$

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Notice that

1. All manipulations are done in d dimensions, only **use d=4** to get rid of some tensors!!
2. **No spurious poles in d=4** in the new projectors
3. Number of tensors matches **<u>number of independent helicity amplitudes</u>**! Minimal complexity?

# HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The construction is very generic. For a given problem, assume there are n tensors in d-dimensions and that $j = 1,\ldots,m < n$ tensors are independent in d=4: $\bar{T}_j = T_j\,,\ \ j = 1,\ldots,m$

Define m Projectors $\bar{P}_j = \sum_{k=1}^{m} C_k \bar{T}_k$ and ***block-diagonalise*** the system of projectors:

$$\mathscr{A} = \sum_{i=1}^{n} F_j T_j \quad \longrightarrow \quad \mathscr{A} = \sum_{i=1}^{n} \bar{F}_j \bar{T}_j \qquad \begin{cases} \bar{T}_j = T_j\,,\ \ j = 1,\ldots,m \\ \bar{T}_j = T_j - \sum_{k=1}^{m} \bar{P}_k T_j\,,\ \ j = m+1,\ldots,n \end{cases}$$

Such that by construction

1. $\lim_{d\to 4} \bar{T}_j(\lambda_1,\ldots,\lambda_E) = 0\,,\ \ j = m+1,\ldots n$
2. projectors are block-diagonal

# APPLICATIONS TO 4-PARTICLE SCATTERING

1. Verified this construction for $gg \to gg$ *and* $q\bar{q} \to gg$
2. Verified that physical projectors reproduce the same helicity amplitudes at 2 loops

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Preliminary

$$gg \to gg$$ From 10 tensors in d-dimensions, to 8 in d=4 (8 indep. helicities)

$$q\bar{q} \to gg$$ From 5 tensors in d-dimensions, to 4 in d=4 (4 indep. helicities)

Projectors become **substantially simpler,** but one might argue improvement in number does not seem that impressive…

It becomes much more interesting from $n \geq 5$ particle scattering !

# APPLICATION TO N-PARTICLE SCATTERING

For $n \geq 5$ the method becomes even simpler, because momenta provide complete set of 4 vectors in d=4 dimensions!

Take the prototypical case of **5-gluon scattering:**

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5) \to 0$$

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**Standard d-dimensional approach:**

1. Rank-5 tensor out of $g^{\mu\nu}$, $p_i^\mu$, $i = 1,...,4$ contains **1724 tensor structures!**
2. Imposing **gauge invariance** reduced to **142 independent structures**
3. Projectors can *(painfully!)* be obtained inverting 142x142 matrix $\to \sim$ ***1GB of text file!***

# APPLICATION TO N-PARTICLE SCATTERING

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5) \to 0$$

Typical tensors will be like:

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} p_s^{\mu_5}$$

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} g^{\mu_4\mu_5}$$

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But since $p_1^\mu, \ldots, p_4^\mu$ are complete set in d=4, $g^{\mu\nu}$ **is not linear independent!**

I don't even need the construction that I have made for 4-point, I can drop all of them!

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These are (using gauge invariance) $2^5 = 32$ **independent tensors**: 32 helicity amplitudes!

New 32x32 matrix can be easily inverted: $\sim$ 500kb against 1GB !

**No spurious poles and no dependence on d!**

# APPLICATION TO N-PARTICLE SCATTERING

Similarly we can study:

1. 5-point scattering with **fermions** (***massless or massive***, of course)
2. n-point scattering with gluons shows even bigger simplifications:

**Further examples**

**Massive external legs:** H+4g, it requires 43 tensors in d-dimensions (1 scalar particle!)

In d=4 they becomes $2^4 = 16$ independent structures

**6-gluons** for example would entail tens of thousands of tensors in d dimensions

With this method only $2^6 = 64$ projectors are needed! It definitely scales much more nicely!

# CONCLUSIONS

1. Projector - form factor method is *ancient method* to compute scattering amplitudes
2. Strong point: very general
3. Weak point: too general, it works in CDR!
4. If we work in tHV (to compute helicity amplitudes) the method can be substantially simplified
5. Decrease of orders of magnitude in complexity!
6. Application to multi-loop and multi-leg processes not impossible anymore!