Infrared properties of the integrands of loop amplitudes

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I: **Introduction**

II: **The story at NLO**

III: **NNLO and beyond**

The infrared structure of loop amplitudes

$\mathcal{A}_n^{(l)}$: Amplitude with n external particles and l loops.

After integration over the loop momenta, the infrared divergent parts can be isolated in insertion operators $\mathbf{I}_n^{(l)}$:

$$\begin{aligned}\mathcal{A}_n^{(1)} &= \mathbf{I}_n^{(1)}\mathcal{A}_n^{(0)} + \mathcal{F}_n^{(1)} \\ \mathcal{A}_n^{(2)} &= \mathbf{I}_n^{(2)}\mathcal{A}_n^{(0)} + \mathbf{I}_n^{(1)}\mathcal{A}_n^{(1)} + \mathcal{F}_n^{(2)}\end{aligned}$$

Catani, '98

We also understand the generalisation to higher loops. Significant effort has been spent in recent years on computing the ingredients for $\mathbf{I}_n^{(3)}$ and $\mathbf{I}_n^{(4)}$.

Becher, Neubert, '09; Gardi, Magnea, '09; Almelid, Duhr, Gardi, '15; Grozin, Henn, Stahlhofen, '17; Boels, Huber, Yang, '17; Moch, Ruijl, Ueda, Vermaseren, Vogt, '18; Lee, Smirnov, Smirnov, Steinhauser, '19; Henn, Peraro, Stahlhofen, Wasser, '19; Henn, Korchemsky, Mistlberger, '19; von Manteuffel, Panzer, Schabinger, '20;

Theme of this talk

What is the structure of $\mathbf{I}_n^{(l)}$ before loop momentum integration?

Are there integrands $\mathcal{G}_n^{(l)}$ and $\mathcal{G}_{n,\mathrm{IR}}^{(l)}$

$$\begin{aligned}\mathcal{A}_n^{(l)} &= \int d^D k_1 \ldots d^D k_l \, \mathcal{G}_n^{(l)} \\ \mathbf{I}_n^{(l)} &= \int d^D k_1 \ldots d^D k_l \, \mathcal{G}_{n,\mathrm{IR}}^{(l)}\end{aligned}$$

such that

$$\int d^D k_1 \ldots d^D k_l \left(\mathcal{G}_n^{(l)} - \mathcal{G}_{n,\mathrm{IR}}^{(l)} \right)$$

is locally integrable in any infrared limit?

Locally integrable

The concept of locally integrable is best explained by a counter example:

$$F = \int_0^1 dx\, x^\varepsilon (1-x)^\varepsilon \left(x + \frac{1}{x} - \frac{1}{1-x} \right)$$

The integrand has singularities at $x = 0$ and $x = 1$, which are regulated by $x^\varepsilon (1-x)^\varepsilon$. The integral is finite and yields

$$F = \frac{\Gamma(1+\varepsilon)\Gamma(2+\varepsilon)}{\Gamma(3+2\varepsilon)} = \frac{1}{2} + O(\varepsilon)\,.$$

However this does not imply that we can remove the regulator before integration. Removing the regulator gives a non-integrable integrand.

Motivation

As the number of external particles increases, analytic calculations of loop amplitudes may no longer be feasible.

We have to resort to numerical methods.

Goal: Purely numerical calculations at higher orders.

Part II

The story at NLO

Numerical NLO QCD calculations

$$\int_{n+1} d\sigma^{\mathrm{R}} + \int_{n} d\sigma^{\mathrm{V}} = \underbrace{\int_{n+1} \left(d\sigma^{\mathrm{R}} - d\sigma^{\mathrm{A}}_{\mathrm{R}}\right)}_{\text{convergent}} + \underbrace{\int_{n} (\mathbf{I}+\mathbf{L}) \otimes d\sigma^{B}}_{\text{finite}} + \underbrace{\int_{n+\mathrm{loop}} \left(d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}\right)}_{\text{convergent}}$$

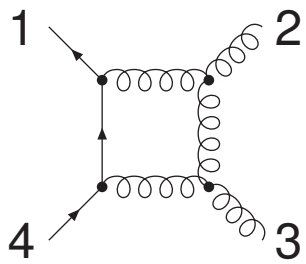
- In the last term $d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}$ the Monte Carlo integration is over a phase space integral of n final state particles plus a 4-dimensional loop integral.
- All explicit poles cancel in the combination $\mathbf{I}+\mathbf{L}$.
- Divergences of one-loop amplitudes related to IR-divergences (soft and collinear) and to UV-divergences.
- The IR-subtraction terms can be formulated at the level of amplitudes.

Z. Nagy, D. Soper, '03; M. Assadsolimani, S. Becker, D. Götz, Ch. Reuschle, Ch. Schwan, S.W., '09

Primitive amplitudes

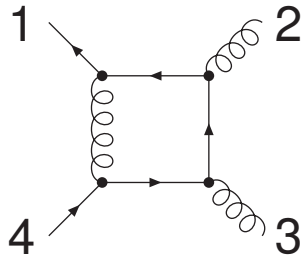
Colour-decomposition of one-loop amplitudes:

$$\mathcal{A}^{(1)} = \sum_j C_j A_j^{(1)}.$$



Primitive amplitudes distinguished by:

- fixed cyclic ordering
- definite routing of the fermion lines
- particle content circulating in the loop



Z. Bern, L. Dixon, D. Kosower, '95

The infrared subtraction terms for the virtual corrections

Local unintegrated form:

$$G^{(1)}_{\text{soft+coll}} = -4\pi\alpha_s i \sum_{i\in I_g} \left(\frac{4p_i p_{i+1}}{k_{i-1}^2 k_i^2 k_{i+1}^2} - 2\frac{S_i g^{UV}_{i-1,i}}{k_{i-1}^2 k_i^2} - 2\frac{S_{i+1} g^{UV}_{i,i+1}}{k_i^2 k_{i+1}^2} \right) A_i^{(0)}.$$

with $S_q = 1$, $S_g = 1/2$. The function $g^{UV}_{i,j}$ provides damping in the UV-region:

$$\lim_{k\to\infty} g^{UV}_{i,j} = O\left(k^{-2}\right), \qquad \lim_{k_i||k_j} g^{UV}_{i,j} = 1.$$

Integrated form:

$$S_\varepsilon^{-1}\mu^{2\varepsilon}\int \frac{d^Dk}{(2\pi)^D} G^{(1)}_{\text{soft+coll}} = \frac{\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \sum_{i\in I_g} \left[\frac{2}{\varepsilon^2}\left(\frac{-2p_i\cdot p_{i+1}}{\mu^2}\right)^{-\varepsilon} + \left(\frac{2}{\varepsilon}+2\right)(S_i+S_{i+1})\left(\frac{\mu_c^2}{\mu^2}\right)^{-\varepsilon} \right] A_i^{(0)}$$
$$+O(\varepsilon),$$

M. Assadsolimani, S. Becker, S.W., '09

UV-subtraction terms

In a fixed direction in loop momentum space the amplitude has up to quadratic UV-divergences.

Only the integration over the angles reduces this to a logarithmic divergence.

For a local subtraction term we have to match the quadratic, linear and logarithmic divergence.

The subtraction terms have the form of counter-terms for propagators and vertices.

The complete UV-subtraction term can be calculated recursively.

S. Becker, Ch. Reuschle, S.W., '10

Contour deformation

With the subtraction terms for UV- and IR-singularities one removes

- UV divergences
- Pinch singularities due to soft or collinear partons

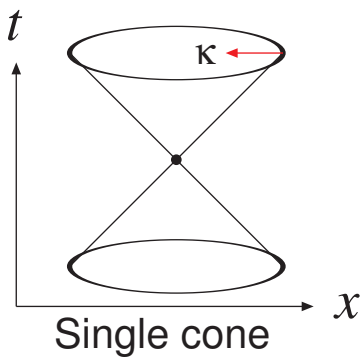
Still remains:

- Singularities in the integrand, where a deformation into the complex plane of the contour is possible.
- Pinch singularities for exceptional configurations of the external momenta (thresholds, anomalous thresholds ...)

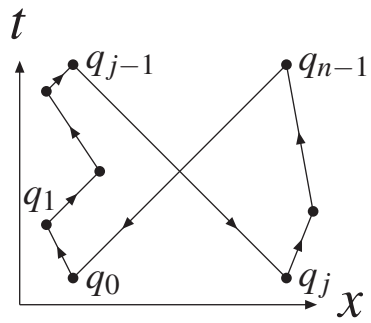
Contour deformation

Deformation of the loop momentum:

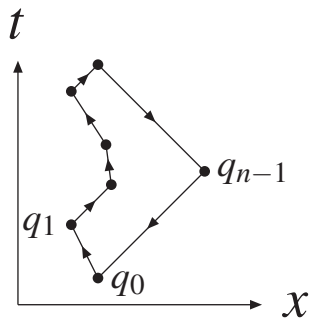
$$k_{\mathbb{C}} \quad = \quad k_{\mathbb{R}} + i\kappa$$



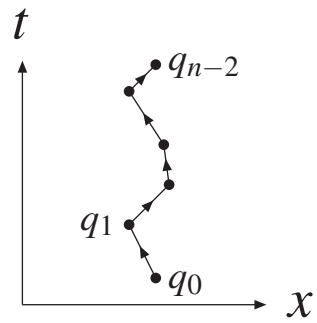
For n cones draw only the origins of the cones:



generic with 2 initial partons



initial partons adjacent



no initial partons

Gong, Nagy, Soper, ’08; Becker, S.W., ’12

Cancellations at the integrand level

$$\int_{n+1} d\sigma^{\mathrm{R}} + \int_{n} d\sigma^{\mathrm{V}} = \int_{n+1} \left(d\sigma^{\mathrm{R}} - d\sigma^{\mathrm{A}}_{\mathrm{R}}\right) + \underbrace{\int_{n} (\mathbf{I}+\mathbf{L}) \otimes d\sigma^{B}}_{\text{numerical integrable?}} + \int_{n+\mathrm{loop}} \left(d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}\right)$$

- At NLO both $d\sigma^{\mathrm{A}}_{\mathrm{R}}$ and $d\sigma^{\mathrm{A}}_{\mathrm{V}}$ are easily integrated analytically.
- This is no longer true at NNLO and beyond.

$$\int_{n} (\mathbf{I}+\mathbf{L}) = \int_{n} \left[\int_{1} d\sigma^{\mathrm{A}}_{\mathrm{R}} + \int_{\mathrm{loop}} d\sigma^{\mathrm{A}}_{\mathrm{V}} + d\sigma^{\mathrm{V}}_{\mathrm{CT}} + d\sigma^{\mathrm{C}} \right].$$

- Unresolved phase space is $(D-1)$-dimensional.
- Loop momentum space is D-dimensional
- $d\sigma^{\mathrm{V}}_{\mathrm{CT}}$ counterterm from renormalisation
- $d\sigma^{\mathrm{C}}$ counterterm from factorisation

Loop-tree duality

A cyclic-ordered one-loop amplitude

$$A_n = \int \frac{d^D k}{(2\pi)^D} \frac{P(k)}{\prod_{j=1}^{n} \left(k_j^2 - m_j^2 + i\delta\right)}.$$

can be written with Cauchy's theorem as

$$A_n = -i \sum_{i=1}^{n} \int \frac{d^{D-1} k}{(2\pi)^{D-1}\, 2k_i^0} \frac{P(k)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left[k_j^2 - m_j^2 - i\delta\left(k_j^0 - k_i^0\right)\right]} \Bigg|_{k_i^0 = \sqrt{\vec{k}_i^2 + m_i^2}},$$

Note the modified $i\delta$-prescription!

Catani, Gleisberg, Krauss, Rodrigo, Winter, '08

Maps

We need to relate the real unresolved phase space and the loop integration in the loop-tree duality approach:

Given a set $\{p_1, p_2, ..., p_n\}$ of external momenta and an on-shell loop momentum k there is an invertible map

$$\{p_1, p_2, ..., p_n\} \times \{k\} \quad \rightarrow \quad \{p'_1, p'_2, ..., p'_n, p'_{n+1}\}$$

Remark:

$$\{p'_1, p'_2, ..., p'_n, p'_{n+1}\} \quad \rightarrow \quad \{p_1, p_2, ..., p_n\}$$

is the standard Catani-Seymour projection.

Sborlini, Driencourt-Mangin, Hernandez-Pinto, German; Seth, S.W.

Collinear singularities

Problem with collinear singularities:

$d\sigma_{\mathrm{R}}^{\mathrm{A}}$: both partons have transverse polarisations, divergence in $g \to q\bar{q}$,

$d\sigma_{\mathrm{V}}^{\mathrm{A}}$: one parton has longitudinal polarisation, no divergence in $g \to q\bar{q}$.

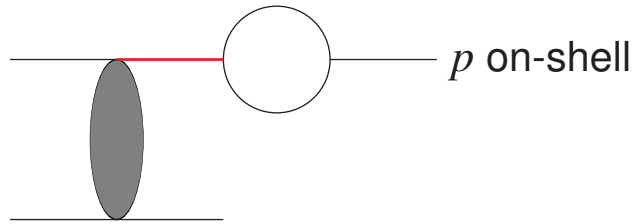
Solution: Take field renormalisation constants into account:

$$Z_2 = 1 = 1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\varepsilon_{\mathrm{IR}}} - \frac{1}{\varepsilon_{\mathrm{UV}}} \right)$$

$$Z_3 = 1 = 1 + \frac{\alpha_s}{4\pi} (2C_A - \beta_0) \left(\frac{1}{\varepsilon_{\mathrm{IR}}} - \frac{1}{\varepsilon_{\mathrm{UV}}} \right)$$

Field renormalisation

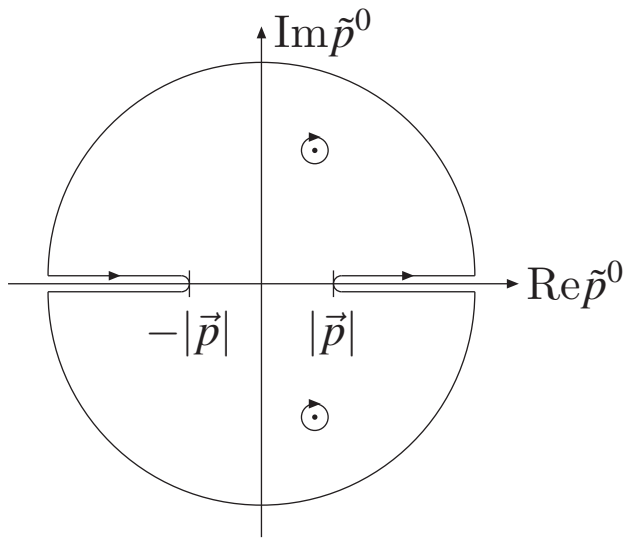
Field renormalisation constants derived from self-energies.



Problem: Internal on-shell propagator.

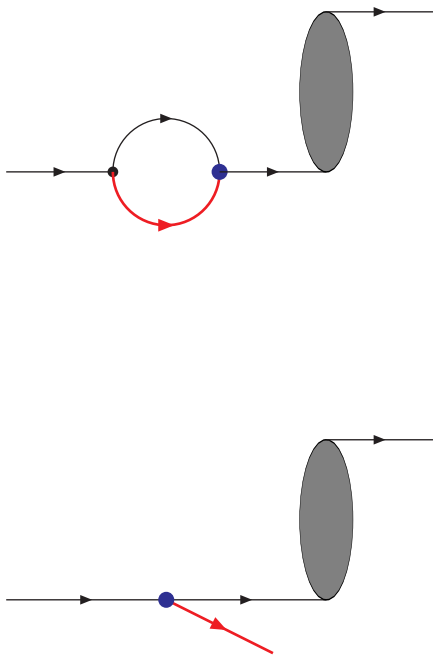
Solution: Use dispersion relation.

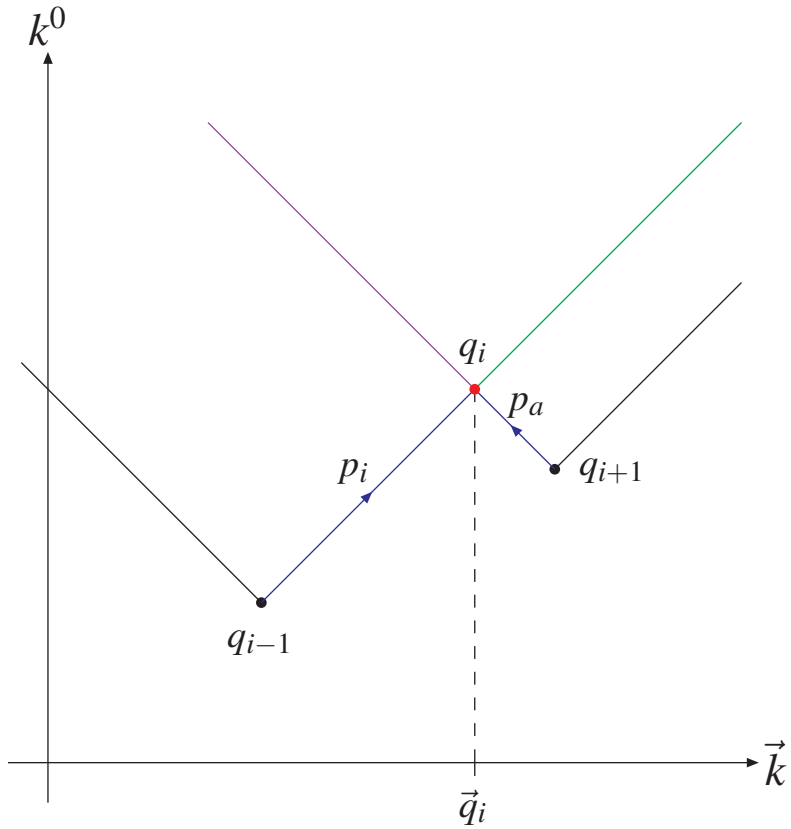
Soper, '01; Seth, S.W., '16



Initial-state collinear singularities

Problem: For initial-state collinear singularities the regions do not match.





Initial-state collinear singularities

We still have to include the counterterm from factorisation.

$$d\sigma^{\mathrm{C}} = \frac{\alpha_s}{4\pi} \int_0^1 dx_a \, \frac{2}{\varepsilon} \left(\frac{\mu_F^2}{\mu^2} \right)^{-\varepsilon} P^{a'a}(x_a) \, d\sigma^{\mathrm{B}}(\ldots, x_a p'_a, \ldots) \, .$$

Example of splitting function:

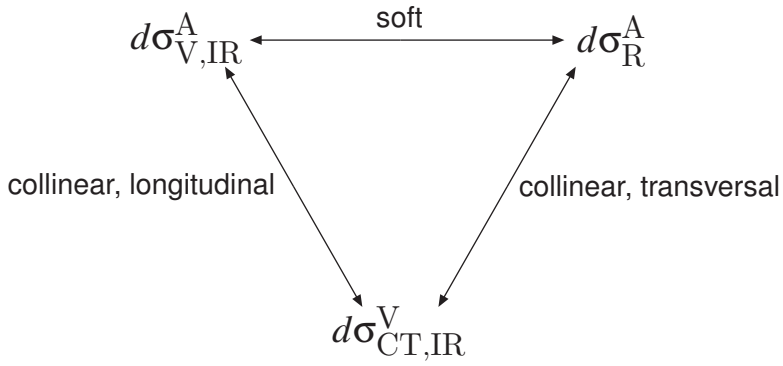
$$P^{gg} = 2C_A \left[\frac{1}{1-x}\bigg|_+ + \frac{1-x}{x} - 1 + x(1-x) \right] + \frac{\beta_0}{2} \delta(1-x) \, .$$

Solution: Unintegrated representation of the collinear subtraction term $d\sigma^{\mathrm{C}}$.

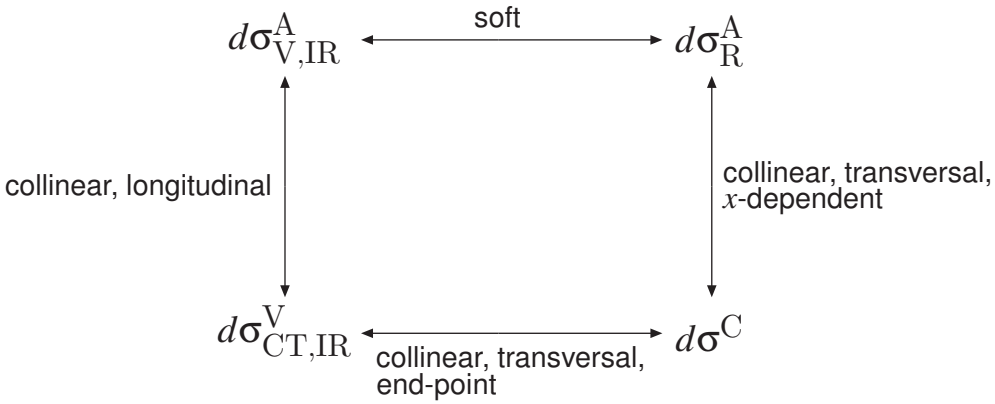
- x-dependent part matches on real contribution
- end-point part matches on virtual contribution

Cancellations of infrared singularities

Only final-state particles:



With initial-state particles:



Part III

NNLO and beyond

Goal

In D spacetime dimensions an l-loop amplitude with n external particles involves

$$D \cdot l$$

integrations.

Have also real emission contributions with fewer loops and more external particles, down to 0 loops and $n+l$ external particles. These involve

$$(D-1) \cdot l$$

integrations beyond the integrations for the Born contribution.

We would like to cancel all divergences at the integrand level, take $D=4$ and integrate numerically.
We don't want to work with individual graphs, but with amplitude-like objects.

Loop-tree duality

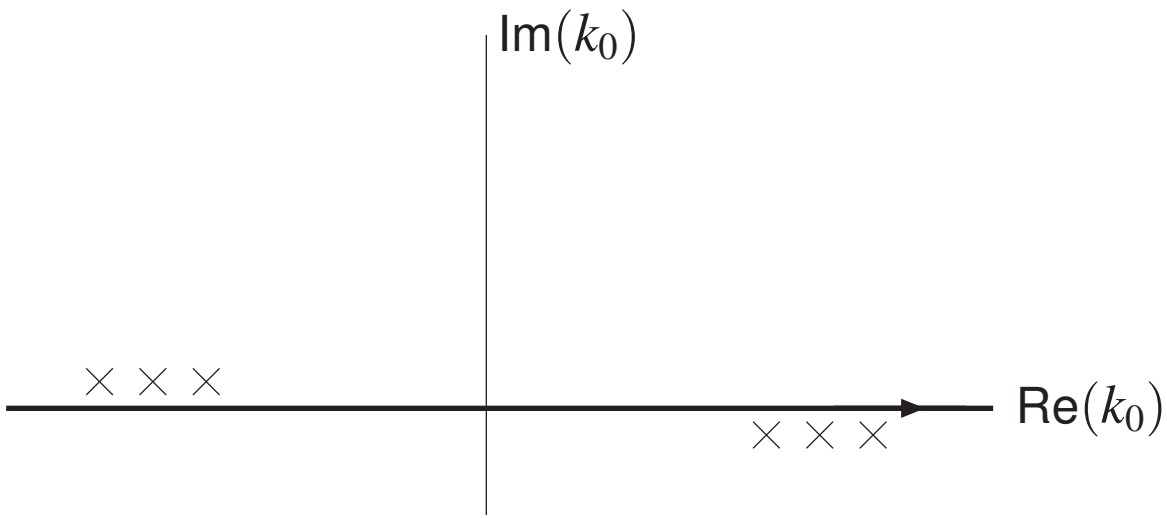
For each loop, do the energy integration with the help of Cauchy's residue theorem.

This leaves

$$(D-1)\cdot l$$

integrations at l-loops.

Can close the contour below or above.

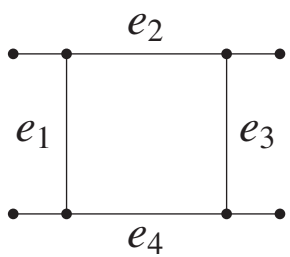


Loop-tree duality beyond one-loop

- Modified causal $i\delta$-prescription
- Absence of higher poles in the on-shell scheme
- Combinatorial factors
- From graphs to amplitude-like objects

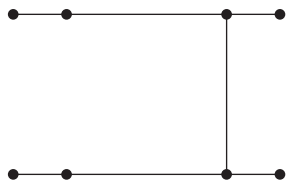
Spanning trees and cut trees

Spanning tree: Sub-graph of Γ, which contains all the vertices and is a connected tree graph.

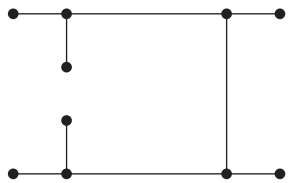


Obtained by deleting l internal edges.

Denote by $\sigma = \{\sigma_1, ..., \sigma_l\}$ the set of indices of the deleted edges and by $\mathcal{C}_\Gamma$ the set of all such sets of indices.



Cut tree: Each σ defines also a cut graph, obtained by cutting each of the l internal edges e_{σ_j} into two half-edges.



The $2l$ half-edges become external lines and the cut graph is a tree graph with $n+2l$ external lines.

l-fold residue

Consider an l-loop graph Γ. Choose an orientation for each internal edge. This defines positive energy / negative energy:

$$k_j^2 - m_j^2 + i\delta \quad = \quad \left(E_j - \sqrt{\vec{k}_j^2 + m_j^2 - i\delta}\right)\left(E_j + \sqrt{\vec{k}_j^2 + m_j^2 - i\delta}\right)$$

$\mathcal{C}_\Gamma$ set of all spanning trees / cut trees.

$\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges

$\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

$$\mathrm{Cut}\left(\sigma_1^{\alpha_1}, ..., \sigma_l^{\alpha_l}\right) \quad = \quad (-i)^l \left(\prod_{j=1}^{l} \alpha_j\right) \mathrm{res}\,(...)$$

Modified causal $i\delta$-prescription

All uncut propagators have a modified $i\delta$-prescription:

$$\frac{1}{\prod\limits_{j\notin\sigma}\left(k_j^2-m_j^2+i s_j(\sigma)\delta\right)}, \qquad s_j(\sigma) = \sum_{a\in\{j\}\cup\pi}\frac{E_j}{E_a}.$$

The set σ defines a cut tree. Cutting in addition edge e_j will give a two-forest (T_1, T_2).

We orient the external momenta of T_1 such that all momenta are outgoing.

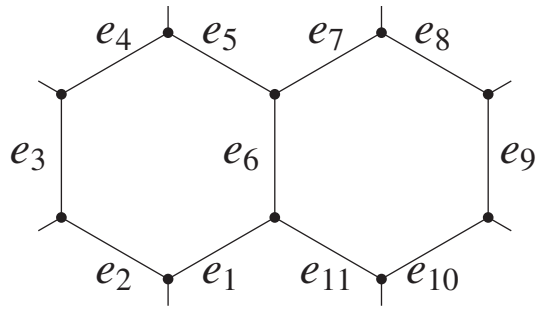
Let π be the set of indices corresponding to the external edges of T_1 which come from cutting the edges e_{σ_i}.

The set π may contain an index twice, this is the case if both half-edges of a cut edge belong to T_1.

R. Runkel, Z. Szőr, J.P. Vesga, S.W., '19

Example

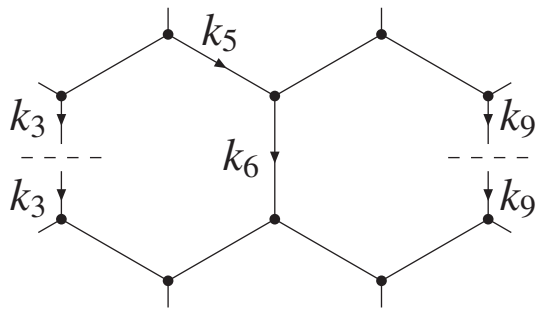
Two-loop eight-point graph.



Consider the cut $\sigma = (3, 9)$.

Then

$$
\begin{aligned}
s_5(\sigma) &= \frac{E_3 + E_5}{E_3} \\
s_6(\sigma) &= \frac{E_3 E_6 + E_3 E_9 + E_6 E_9}{E_3 E_9}
\end{aligned}
$$



Absence of higher poles in the on-shell scheme

Self-energy insertion on internal lines lead to higher poles.
Have also UV-counterterms.

Some cuts are unproblematic, some other cuts correspond to residues of higher poles:

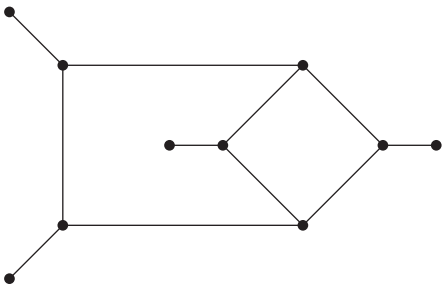
In the on-shell scheme we may choose an integral representation for the UV-counterterm sucht that the problematic residues are zero.

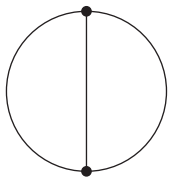
R. Baumeister, D. Mediger, J. Pečovnik, S.W. ’19

Chain graphs

Two propagators belong to the same chain, if their momenta differ only by a linear combination of the external momenta.

Chain graph: delete all external lines and choose one propagator for each chain as a representative.

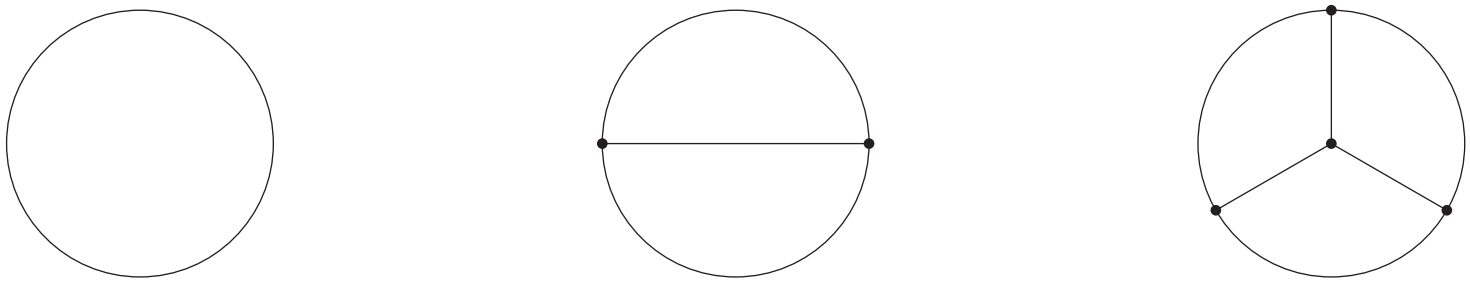




Kinoshita '62

Chain graphs

Up to three loops, all chain graphs are (sub-) topologies of



Combinatorial factors

Γ a graph with l loops and n external legs, $I_{l,n}$ the corresponding Feynman integral. Take l-fold residues:

$$I_{l,n} \quad = \quad \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\alpha=1}^{2^l} c_{\sigma\alpha} \, \mathrm{Cut}\,(\sigma, \alpha)$$

for some coefficients $c_{\sigma\alpha}$.

Recall:

- $\mathcal{C}_\Gamma$ set of all spanning trees / cut trees.
- $\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges
- $\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

Remark: The representation in terms of cuts is not unique. The sum of all residues in any subloop equals zero.

Loop-tree duality representation

$$I_{l,n} \quad = \quad \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\pi \in S_l} \sum_{\alpha=1}^{2^l} C_{\sigma\pi\alpha}^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}} \, \mathrm{Cut}\,(\sigma, \alpha)$$

- $\tilde{\sigma} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_l) \in \mathcal{C}_\Gamma$: indices of the chosen independent loop momenta
- $\tilde{\pi} = (\tilde{\pi}_1, ..., \tilde{\pi}_l) \in S_l$: order in which the integration are carried out
- $\tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_l) \in \{1, -1\}^l$: specifications whether the contour is closed below or above

- $\sigma = (\sigma_1, ..., \sigma_l) \in \mathcal{C}_\Gamma$: indices of the cut edges
- $\pi = (\pi_1, ..., \pi_l) \in S_l$: order in which the residues are picked up
- $\alpha = (\alpha_1, ..., \alpha_l) \in \{1, -1\}^l$: energy signs

Z. Capatti, V. Hirschi, D. Kermanschah, B. Ruijl, '19

Averaging

Sum over π and average over $\tilde{\sigma}$, $\tilde{\pi}$, $\tilde{\alpha}$. For a chain graph:

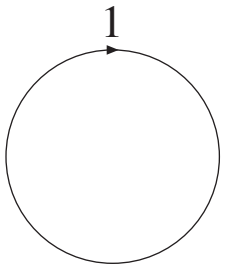
$$S_{\sigma\alpha} = \frac{1}{2^l l! \, |\mathcal{C}_\Gamma|} \sum_{\pi \in S_l} \sum_{\tilde{\sigma} \in \mathcal{C}_\Gamma} \sum_{\tilde{\pi} \in S_l} \sum_{\tilde{\alpha} \in \{1,-1\}^l} C_{\sigma\pi\alpha}^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}}$$

Then

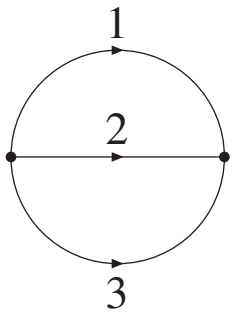
$$I_{l,n} = \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\alpha=1}^{2^l} S_{\sigma\alpha} \, \mathrm{Cut}\,(\sigma, \alpha)$$

with combinatorial factor $S_{\sigma\alpha}$.

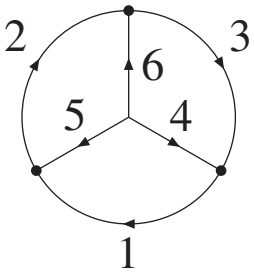
Examples



Cut	(1^+)	(1^-)
$S_{\sigma\alpha}$	$\frac{1}{2}$	$\frac{1}{2}$



Cut	$(1^+,2^+)$	$(1^+,2^-)$	$(1^-,2^+)$	$(1^-,2^-)$
$S_{\sigma\alpha}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$



Cut	$(1^+,2^+,3^+)$	$(1^+,2^+,3^-)$	$(1^+,2^-,3^+)$	$(1^+,2^-,3^-)$
$S_{\sigma\alpha}$	$\frac{3}{64}$	$\frac{29}{192}$	$\frac{29}{192}$	$\frac{29}{192}$

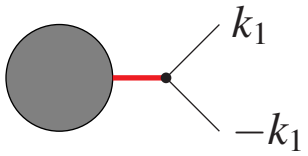
Cut	$(1^+,2^+,4^+)$	$(1^+,2^+,4^-)$	$(1^+,2^-,4^+)$	$(1^+,2^-,4^-)$
$S_{\sigma\alpha}$	$\frac{5}{96}$	$\frac{19}{192}$	$\frac{19}{192}$	$\frac{1}{4}$

From graphs to amplitude-like objects

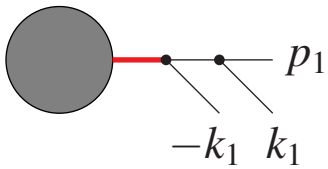
- UV-subtracted
- Regularised forward limit
- Minus signs for closed fermion loops
- Combinatorial factors

Regularised forward limit

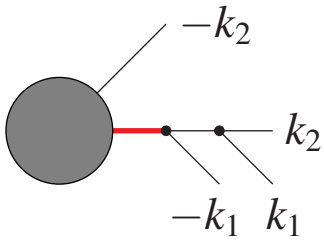
l-fold forward limit of tree-amplitude like objects: Exclude singular contributions.



$\Rightarrow$ Tadpole



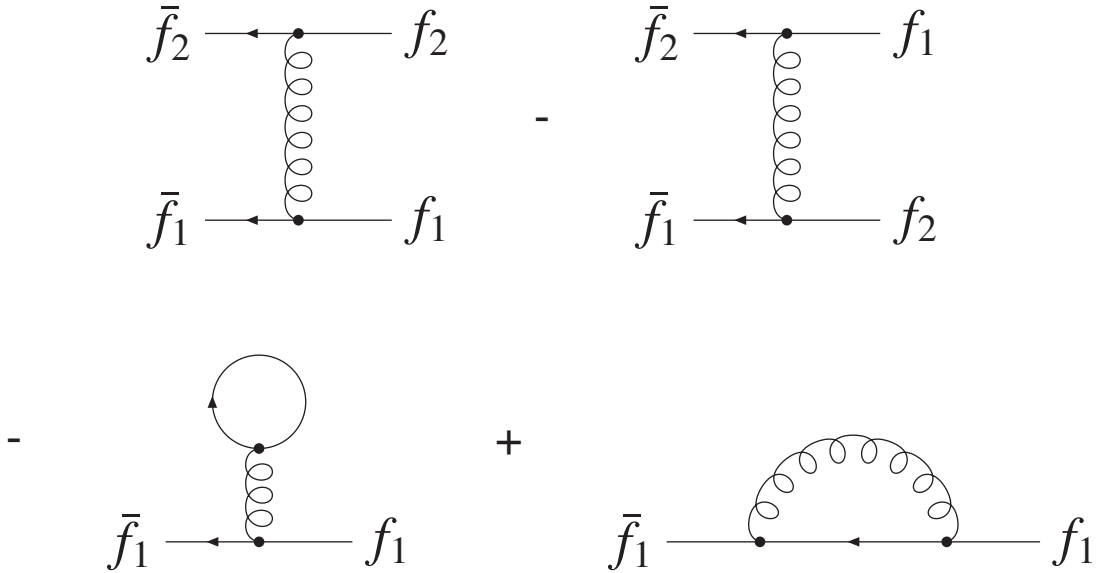
$\Rightarrow$ Self-energy insertion on an external line



$\Rightarrow$ Self-energy insertion on an internal line

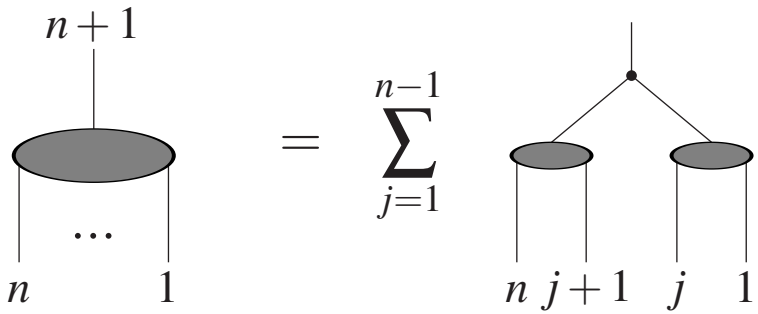
Minus signs for closed fermion loops from the forward limit of tree amplitudes

Solution: Include a minus sign for every forward limit of a fermion-antifermion pair.

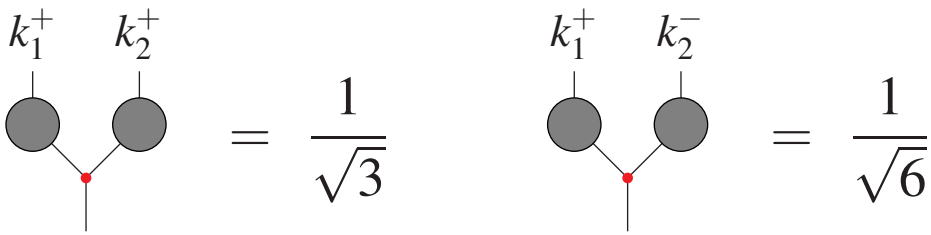


Combinatorial factors

Off-shell currents provide an efficient way to calculate amplitudes:



May incorporate combinatorial factors as effective Feynman rules:



Integrand of a UV-subtracted loop amplitude may be computed like a tree amplitude from off-shell recurrence relations.

Summary and outlook

The numerical approach:

- Cancellations at the integrand level
- Loop-tree duality
- Non-trivial cancellations between virtual, real, UV-counterterm and initial-state collinear factorisation term
- Contour deformation
- Integrands need to be computable at low cost