

GAUGE THEORY for BEGINNERS

I - Gauge Invariance in Classical Field Theory:

basic object : Lagrangian density $\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$
 ↴ fields

action: $S = \int_{-\infty}^{+\infty} dt \cdot L(t) = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i)$ ($i=1, \dots, n$)

Hamilton Principle \Rightarrow Euler Equations:

$$\delta \int_{t_1}^{t_2} L(t) dt = 0 \quad \Rightarrow \quad \frac{\delta \mathcal{L}}{\delta \dot{\phi}_i} = \frac{\partial}{\partial x^\mu} \left(\frac{\delta \mathcal{L}}{\delta (\frac{\partial \phi_i}{\partial x^\mu})} \right)$$

"as a trick": $\varepsilon(t)$: small perturb. of the $\phi_i(t)$'s such that: $\varepsilon(t_1) = \varepsilon(t_2) = 0$

$$\delta S = \int_{t_1}^{t_2} [L(\phi_i + \varepsilon, \dot{\phi}_i + \dot{\varepsilon}) - L(\phi_i, \dot{\phi}_i)] dt = \int_{t_1}^{t_2} \left(\varepsilon \frac{\partial L}{\partial \phi_i} + \dot{\varepsilon} \frac{\partial L}{\partial \dot{\phi}_i} \right) dt$$

integ. by parts: $\delta \mathcal{L} = \left[\varepsilon \frac{\partial L}{\partial \phi_i} \right]_{t_1}^{t_2} + \int \varepsilon \left(\frac{\partial L}{\partial \phi_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_i} \right) dt$

\downarrow \downarrow

$= 0$ at boundaries $= 0$ since $\delta S = 0$
 for all feasible perturb
 $\varepsilon(t)$

to be generalized to $\phi_i(\vec{x}, t) \dots$

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Gauge transformation "Idea" comes from observing that, to each continuous symmetry of the Lagrangian under study corresponds a conservation law: Emmy NOETHER (1915-1918)

An example: let \mathcal{L} indep. of x^0

$$\delta \phi_i(x^0, \vec{x}) = \phi_i(x^0 + \varepsilon, \vec{x}) - \phi_i(x) = \varepsilon \frac{\partial \phi_i}{\partial x^0}$$

$$\delta(\partial_\mu \phi_i) = \varepsilon \partial_\mu \left(\frac{\partial \phi_i}{\partial x^0} \right) \quad \text{and } \delta \mathcal{L} = \varepsilon \frac{\partial \mathcal{L}}{\partial x^0}$$

$$\text{but } \delta \mathcal{L} = \sum_i \left[\frac{\delta \mathcal{L}}{\delta \phi_i} \delta \phi_i + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \cdot \delta(\partial_\mu \phi_i) \right]$$

$$= \varepsilon \sum_i \left[\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \partial_\mu \left(\frac{\partial \phi_i}{\partial x^0} \right) \right]$$

$$\frac{\partial \mathcal{L}}{\partial x^0} = \partial_\mu \left[\sum_i \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0} \right) \right]$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial x^0} \left[\mathcal{L} - \sum_i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0} \right]}_{\delta \mathcal{L} : \text{Hamilt. density}} = \vec{\nabla} \cdot \sum_i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial x^0}$$

Since $\phi_i|_{x^0 \rightarrow 0} \rightarrow 0$ as $|\vec{x}| \gg$

$$\boxed{\frac{\partial}{\partial x^0}} \int \delta \mathcal{L} d^3x = \frac{\partial}{\partial t} H = 0$$

"constant of motion"

In the same way, in a theory which is Lorentz invariant energy and angular momentum are conserved. (\mathcal{L} being a ~~scalar~~ Lorentz scalar density).

3 But one can look at conservat. laws which are not space-time symmetries

Ex: electric charge.

Suppose ϕ_i of charge q_i and \mathcal{L} neutral

- define: $\phi_i(x) \rightarrow \exp(-iq_i\theta) \phi_i(x)$
 $\text{U}(1)$ group.

(if $\phi_1(x) \dots \phi_n(x)$ in $\mathcal{L} \Rightarrow q_1 + \dots + q_n = 0$)

then $\partial_\mu \phi_i(x) \rightarrow \exp(-iq_i\theta) \partial_\mu \phi_i(x)$

("global" gauge transf.)

$$\text{infinit. } \delta \phi_i = -i\varepsilon q_i \phi_i$$

$$\text{and } \delta \mathcal{L} = 0 \Rightarrow \frac{\partial}{\partial x^\mu} \left[\underbrace{\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} q_i \phi_i}_{J^\mu} \right] = 0 \quad (\text{see p.1})$$

$\Rightarrow J^\mu$ = conserved current

$$\partial_\mu J^\mu = 0.$$

Note:

Actually, the q_i 's are the eigenvalues of infin. oper. Q .

In quantum theory: $\delta \phi_i = -i\varepsilon [Q, \phi_i] = -i\varepsilon q_i \phi_i$

$$\text{and } Q = \int d^3x J_0(\vec{x}, t) \quad \text{w.t.e. } \frac{\partial Q}{\partial x^\mu} = 0.$$

But, one can have more than one - not space-time-conserved quant.

Ex: with $SU(2)$

$$\vec{\phi} \Rightarrow \exp(-i \vec{L} \cdot \vec{\theta}) \vec{\phi}$$

↳ matricial repres. of $SU(2)$

$$[L_x^i, L_y^j] = i\varepsilon_{ijk} L_z^k$$

"Local" gauge transformations:

It is well known that electrodynamics has a symmetry larger than a global gauge symmetry, i.e.:

$$\phi_i(x) \rightarrow \phi'_i(x) = \exp(-iq_i\Theta(x)) \phi_i(x)$$

x -dependent

we note: $\delta\phi_i = -iq_i\Theta(x)\phi_i(x)$

and $\partial_\mu\phi_i \rightarrow \partial_\mu\phi'_i = \exp(-iq_i\Theta(x))\partial_\mu\phi_i(x) - iq_i(\partial_\mu\Theta(x))\exp(iq_i\Theta(x))\phi_i(x)$

$\cancel{\phi_i(x)}$

$\cancel{\Theta(x)}$

And we know that "all works" (i.e. covariance restored) by introducing photon field following a "minimal coupling"

$$\boxed{\partial_\mu\phi_i \rightarrow (\partial_\mu - ie q_i A_\mu)\phi_i}$$

A_μ spin 1 field
1st example of gauge boson.

and A_μ must transform s.t. ω to be $U(1)$ invariant.

i.e.: $A_\mu \rightarrow A'_\mu$ $\phi_0 \rightarrow \phi'_0$

with $(\partial_\mu - ie q_i A'_\mu)\phi'_0(x) = \exp(-ie q_i \Theta(x)) (\partial_\mu - ie q_i A_\mu)\phi_0(x)$

$$\Rightarrow \boxed{A'_\mu(x) = -\frac{1}{e} \partial_\mu \Theta(x) + A_\mu(x)}$$

or: $\boxed{\delta A_\mu(x) = -\frac{1}{e} \partial_\mu \Theta(x)}$

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Rh: in addition to term coupling γ to charged particle
there is the kinetic energy term coupling A_μ to itself

$$\text{with: } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\text{with } \partial F_{\mu\nu} = 0 \quad (\text{from above})$$

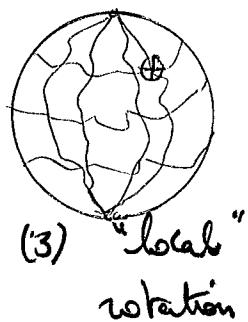
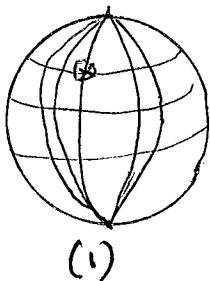
$$\text{that is: } L_{\text{kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(factor $-\frac{1}{4}$ to get a good normal. force in Euler eqns.)

Note: a mass term $-\frac{1}{2} m^2 A_\mu A^\mu$ would violate local gauge inv^g

Of course, photon has not been discovered by imposing local gauge inv^g ! Gauge transform. came as a property of Maxwell eqns. (gauge inv^g in electrodynamics allow to derive Ward identities proving in fact renormalizability).

Remarks on local gauge transf.:



While rotating the sphere does not change, but surface is modified
The transformations induce tensions, forces between pts on the sphere
One can think that forces at the basis of matter behaviour are of this kind.

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Yang & Mills (1954) generalize the $U(1)$ case to $SU(2)$
 "isotropic spin".

$$[L_i, L_j] = i \epsilon_{ijk} L_k$$

$$\phi(x) \rightarrow \phi'(x) = \exp(-i \vec{L} \cdot \vec{\theta}(x)) \phi(x) \\ = U(\theta(x)) \phi(x).$$

$$\partial_\mu \phi \rightarrow U(\theta) \partial_\mu \phi + (\partial_\mu U(\theta)) \phi$$

Introducing the "covariant derivative"

$$D_\mu \phi = (\partial_\mu - ig \vec{A}_\mu \cdot \vec{L}) \phi$$

cstrg analogous to "e"
 now 3 "gauge bosons"

such that $D_\mu \phi \rightarrow U(\theta) D_\mu \phi$

How A_μ will be transformed?

$$D_\mu' \phi' = \partial_\mu \phi' - ig \vec{A}'_\mu \cdot \vec{L} \cdot \phi' = U(\theta) D_\mu \phi$$

$$\text{i.e. } U [\partial_\mu - ig \vec{A}_\mu \cdot \vec{L}] \phi = (\partial_\mu U) \phi + U \partial_\mu \phi - ig \vec{A}'_\mu \cdot \vec{L} \cdot U \phi$$

$$\Rightarrow \boxed{\vec{A}'_\mu \cdot \vec{L} = U \vec{A}_\mu \cdot \vec{L} U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}}$$

Exercise: prove these transf. satisfy group action.

$$\vec{A}'_\mu \vec{L} \xrightarrow{U} \vec{A}_\mu \vec{L} \xrightarrow{V} \vec{A}''_\mu \vec{L}$$

$$\text{with } \vec{A}''_\mu \vec{L} = (UV) \vec{A}_\mu \vec{L} (VU)^{-1} - \frac{i}{g} [\partial_\mu (UV)] (VU)^{-1}$$

7/ Infin^t

with $U(\theta) = \exp(-i\vec{\theta} \cdot \vec{L})$

$$\boxed{U(\theta) \vec{A}_j \cdot \vec{L} U(\theta)^{-1} = (1 - i\vec{\theta} \cdot \vec{L})(\vec{A}_j \cdot \vec{L})(1 + i\vec{\theta} \cdot \vec{L})} \\ = \vec{A}_j \cdot \vec{L} - \epsilon_{ijk} \theta^i A_j^i L^k + \dots$$

$$\boxed{[\partial_\mu, U(\theta)] U(\theta)^{-1} = -i \partial_\mu \theta^i \cdot \vec{L}^i \quad \text{in Lie algebra!}}$$

$$\Rightarrow \delta A_j^i \cdot \vec{L}^i = -\frac{1}{g} \partial_\mu \theta^i \cdot \vec{L}^i - \epsilon_{ijk} A_j^i \theta^j L^k$$

that is $\boxed{\delta A_j^i = -\frac{1}{g} \partial_\mu \theta^i + \epsilon_{ijk} \theta^j A_j^k}$

Note: the transform. of A_j^i do not depend on the refis. but only on the commut. relations (struct. const.)

RQ: we recover the e.m. case $\delta A_\mu = -\frac{1}{g} \partial_\mu \theta$ with U(1) group!

What about $F_{\mu\nu}$?

Define now

$$\vec{F}_{\mu\nu} \cdot \vec{L} = \partial_\mu \vec{A}_\nu \cdot \vec{L} - \partial_\nu \vec{A}_\mu \cdot \vec{L} - ig [\vec{A}_\mu \cdot \vec{L}, \vec{A}_\nu \cdot \vec{L}]$$

$$\text{or } F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g \epsilon_{ijk} A_j^i A_k^\nu$$

Group action: $\delta F_{\mu\nu}^i = \epsilon_{ijk} \theta^j F_{\mu\nu}^k$

that is also:

$$\boxed{\vec{F}_{\mu\nu} \cdot \vec{L} \rightarrow U(\theta) \vec{F}_{\mu\nu} \cdot \vec{L} U(\theta)^{-1}}$$

and $\vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu}$ scalar under $SU(2)$.

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Summary:

$(\phi_i, \partial_\mu \phi_i)$ invariant under G simple
with $[T_i, T_j] = i \epsilon_{ijk} T_k$

$$\phi \rightarrow \exp(-i\vec{\theta} \cdot \vec{L}) \phi \quad \theta^i(x) \hookrightarrow \text{func. of } x.$$

- add to each T_i a "bosonic field" A_μ^i .

- write \mathcal{L} as $* \mathcal{L} = \mathcal{L}_0 + \mathcal{L}(\phi_i, (\partial_\mu - ig \vec{A}_\mu \cdot \vec{L}) \phi_i)$

$$* \quad d_\omega = -\frac{i}{g} \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu}$$

with $\vec{F}_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g \epsilon_{ijk} A_\mu^j A_\nu^k$

the gauge bosons transforming under G as:

$$* * \quad \vec{L} \cdot \vec{A}_\mu \rightarrow U(\theta) \vec{L} \cdot \vec{A}_\mu U(\theta)^{-1} - \frac{i}{g} [\partial_\mu U(\theta)] \cdot U(\theta)^{-1}$$

and: $\vec{F}_{\mu\nu} \cdot \vec{L} \rightarrow U(\theta) \vec{F}_{\mu\nu} \cdot \vec{L} U(\theta)^{-1} \quad (\vec{F}_{\mu\nu} \text{ G-scalar})$

Note: If $G = G_1 \times G_2$

one can have g_1 and g_2 (\neq) for G_1 and G_2 respectively

(i.e. only 1 coupling constant for each simple group or U(1) group)

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Spontaneous Symmetry Breaking

a) Let us start with the simpliest case:

ϕ : scalar field (complex)

$$\mathcal{L} = \partial^\mu \phi^*(x) \partial_\mu \phi(x) - m^2 \phi^*(x) \phi(x)$$

Euler-Lagrange Equa.: $\partial_\mu \nabla^\mu \phi(x) + m^2 \phi(x) = 0$ Klein-Gordon.

$U(1)$ covariance: $\phi(x) \rightarrow e^{i\theta} \phi(x)$

Hamilton. density

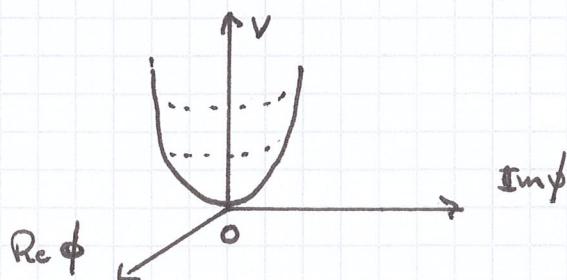
$$\mathcal{H} = \partial_0 \phi^*(x) \partial_0 \phi(x) + \sum_j \partial_j \phi^*(x) \partial_j \phi(x) + V(\phi)$$

$$V(\phi) = m^2 \phi^* \phi$$

gives the system energy.

Fundamental state (= vacuum) corresponds to minimal energy level

First two terms $\geq 0 \Rightarrow V$ is min. for $\phi=0$



solut. unique

and $U(1)$ invariant

b) BUT consider: $V(\phi) = m^2 \phi^*(x) \phi(x) + \lambda [\phi^*(x) \phi(x)]^2$

\uparrow
 $U(1)$ invat

and keep theory renormalizable.

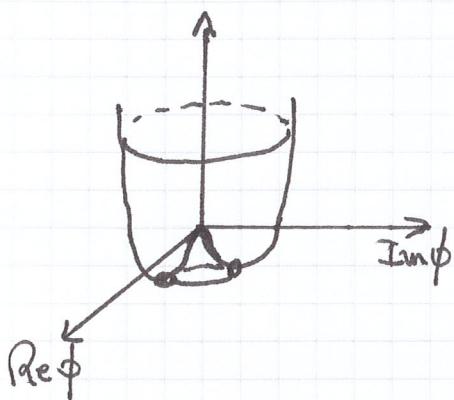
(10)

$$\phi = \rho e^{i\theta}$$

$$V = m^2 \rho^2 + \lambda \rho^4$$

$$\frac{\partial V}{\partial \rho} = 0 \Rightarrow m^2 + 2\lambda \rho^2 = 0$$

$$\text{i.e. } \phi(x) = e^{i\theta} \sqrt{-\frac{m^2}{2\lambda}} \quad (\star)$$



Vacuum is degenerate.

Let us choose one vacuum, for ex: $\phi_0 = \sqrt{-\frac{m^2}{2\lambda}}$

and define a new field

$$\tilde{\phi}(x) = \phi(x) - \phi_0$$

s.t. vacuum corresponds to $\tilde{\phi}(x) = 0$

with : $\tilde{\phi}(x) = \frac{1}{\sqrt{2}} (\zeta(x) + i\eta(x))$ ζ and η real.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi) = \frac{1}{2} (\partial_\mu \eta \cdot \partial^\mu \eta + \partial_\mu \zeta \cdot \partial^\mu \zeta) - 2m^2 \zeta^2(x) + 0 \eta^2(x)$$

+ other terms ...

to field $\zeta(x)$ corresponds a particle of mass $\sqrt{-2m^2}$

$\eta(x)$ " zero mass = Goldstone Boson.

Conclusion: If \mathcal{L} has a symmetry which is not a symmetry of the vacuum, ζ a meson boson.

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\Rightarrow "Spontaneous" breaking of the symmetry.

with some symmetry of the \mathcal{L} which is not a symmetry of physical states.

Analogy with "roulette": the equa. of the roulette and the ball invariant by rotation around the axis roulette. But at the end of the game, the ball always stands in an asymmetric position.

Remark:

- the solut $\phi = 0$ is unstable.
- the fundamental state is degenerate.
- critical threshold ($m^2 = 0$).

c) Now, let ϕ be a n -real component field $\phi = (\phi_1, \dots, \phi_n)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \mu^2 \phi \cdot \phi - \frac{\lambda}{4} (\phi \cdot \phi)^2$$

$O(n)$ invariance.

if $\mu^2 < 0$: "imag" of minima at $v = \sqrt{-\mu^2} \hat{\phi}$

let us choose $\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \leftarrow n^{\text{th comp}}^+$

$\langle \phi \rangle$ invar under $O(n-1) \subset O(n)$.

Nbr of "broken" generators: $\frac{m}{2}(n-1) - \frac{1}{2}(n-1)(n-2) = n-1$

\downarrow \uparrow
in $O(n)$ in $O(n-1)$

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If L_0 generators in $O(n)$, the "broken" one will be L_{in}
 $i=1,..n-1$

Defining: η and ξ_i ($i=1,..n-1$) such that:

$$\phi = \exp\left(\frac{\xi}{r} \cdot L_{in}\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r+\eta \end{pmatrix} = \begin{pmatrix} \xi_1 + \dots & \leftarrow \text{higher} \\ \xi_2 + \dots & \leftarrow \text{order} \\ \vdots & \text{terms} \\ r+\eta & \end{pmatrix}$$

(generalizing: $\tilde{\phi} + \phi_0 = \phi$)

$$\text{then: } \mathcal{L} = \frac{1}{2} (\partial^\mu \eta \partial_\mu \eta + \partial^\mu \xi_0 \partial_\mu \xi^0) + \text{higher order terms...}$$

$$-\frac{1}{2} \mu^2 (r+\eta)^2 - \frac{1}{4} \lambda (r+\eta)^4$$

\Rightarrow the field η acquires a mass: $-\mu^2$

the fields ξ_i ($i=1..n-1$) are massless!

Consequence: The number of massless Goldstone which show up
 is equal to the number of "broken" generators.

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Higgs Mechanism:

Consider \mathcal{L} with spontaneous breaking
and local gauge symmetry.

\Rightarrow exception to Goldstone

Simplest example : $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - g^2 \phi^* \phi - \lambda (\phi^* \phi)^2$

$$U(1) \text{ inv}^{\text{c}} : \phi(x) \rightarrow e^{-i\theta(x)} \phi(x)$$

Introducing A_μ :

$$\mathcal{L} = (\partial_\mu + ie A_\mu) \phi^* (\partial_\mu - ie A_\mu) \phi - g^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\text{with: } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{aligned} \mathcal{L} \text{ inv}^t \text{ under: } & \left| \begin{array}{l} \phi(x) \rightarrow e^{-i\theta(x)} \phi(x) \\ \phi^*(x) \rightarrow e^{i\theta(x)} \phi^*(x) \end{array} \right. \\ & A_\mu \rightarrow A_\mu - \frac{i}{e} \partial_\mu \theta(x) \end{aligned}$$

$$\text{Suppose } g^2 < 0 \quad \text{we define: } \langle \phi \rangle_0 = \frac{v}{\sqrt{2}} \quad v^2 = -\frac{\lambda}{g^2}$$

Now, instead of $\tilde{\phi} = \phi - \langle \phi \rangle_0$, let us choose:

$$\bullet \quad \phi(x) = \exp(i\beta/v) \left(\frac{v+\ell}{\sqrt{2}} \right) = \frac{1}{2} [v + \ell + i\beta + \text{higher terms}]$$

$\beta(x)$ is associated to $U(1)$

If no A_μ , β would be massless.

But with A_μ , things are $\neq !$

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$$\text{then: } \mathcal{L} = \frac{1}{2} (\partial_\mu \eta \cdot \partial^\mu \eta + \partial_\mu \zeta \cdot \partial^\mu \zeta) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$+ \frac{e^2 v^2}{c} A_\mu A^\mu - \text{irrelevant terms}$$

↑ ↑ ↑
 massive A_μ mixing massive η
 A_μ & ζ

Consider now the transform:

$$\begin{aligned} \phi &\rightarrow \phi' = \exp(-i \frac{\zeta(x)}{v}) \phi & \text{then } \phi' = \frac{1}{\sqrt{2}}(v+\eta) \\ A_\mu &\rightarrow A'_\mu = A_\mu - \frac{1}{ev} \partial_\mu \zeta \end{aligned}$$

\mathcal{L} invariant under such a transform:

$$\begin{aligned} \mathcal{L} &= (\partial_\mu + ie A'_\mu) \phi' (\partial_\mu - ie A'_\mu) \phi' - \frac{1}{2} v^2 (\phi'^* \phi') - \frac{\lambda}{4} (\phi'^* \phi')^2 \\ &\quad - \frac{1}{4} F'_{\mu\nu} F'_{\mu\nu} \\ &= \frac{1}{2} (\partial_\mu + ie A'_\mu) (v+\eta) (\partial_\mu - ie A'_\mu) (v+\eta) - \frac{1}{2} v^2 (v+\eta)^2 \\ &\quad - \frac{\lambda}{4} (v+\eta)^4 - \frac{1}{4} F'_{\mu\nu} F'_{\mu\nu} \end{aligned}$$

$$\text{with } F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

that is:

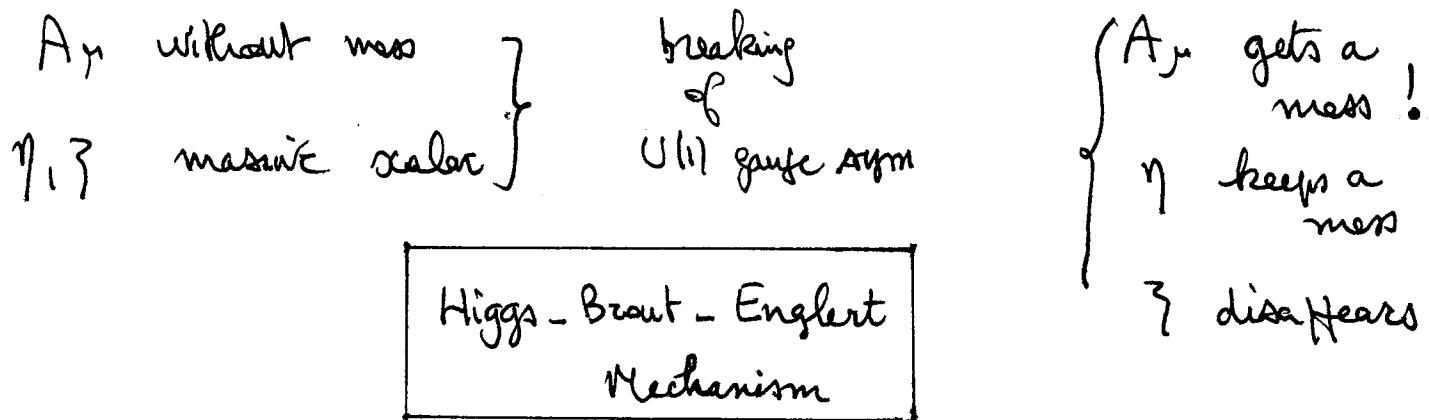
$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F'_{\mu\nu} F'_{\mu\nu} + \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + \frac{1}{2} e^2 v^2 A'_\mu \cdot A'_\mu - \frac{1}{2} \eta^2 (3\lambda v^2 + \eta^2) \\ &\quad + \frac{1}{2} e^2 A'_\mu A'_\mu \cdot \eta (2v+\eta) - \lambda v \eta^3 - \frac{\lambda}{4} \eta^4 \end{aligned}$$

⇒

η has acquired a mass: $\sqrt{3\lambda v^2 + \eta^2}$
 } disappeared!

A_μ got a mass $= ev$!

(15)

What happened?Non Abelian Case:ex. of $SU(2)$ scalars in the 3 dim. repres.

$$[T_i, T_j] = i \epsilon_{ijk} T_k \quad D_\mu \phi = (\partial_\mu - ig \vec{T} \cdot \vec{A}_\mu) \phi$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_i + g \sum_j \epsilon^{ijk} A_\mu^j \phi_k) (\partial^\mu \phi_i + g \sum_{jk} \epsilon^{ijk} A^\mu \phi_j) - V(\phi^2)$$

If min of V at $\langle \phi \rangle_0 = 0$: ordinary YM.If $\langle \phi \rangle_0 \neq 0$

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

only $T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ leaves $\langle \phi \rangle_0$ invariant.

$$\text{let } \phi = \exp\left(\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right) \begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix} = \langle \phi \rangle_0 + \begin{pmatrix} \eta \\ \eta \\ \eta \end{pmatrix} \dots$$

and use "local" gauge transform. $\phi \rightarrow \phi' = \exp\left(-\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right) \phi$

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$$\text{then } \vec{T} \cdot \vec{A}'_\mu = \exp\left(-\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right) \vec{T} \cdot \vec{A} \exp\left(\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right) \\ - \frac{i}{g} \partial_\mu \left[\exp\left(-\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right) \right] \cdot \exp\left(\frac{i}{v} (\beta_1 T_1 + \beta_2 T_2)\right)$$

$$\text{put into } \mathcal{L} \rightarrow \left(\partial_\mu - i g \vec{T} \cdot \vec{A}'_\mu \right) \phi'^* \left(\partial_\mu + i g \vec{T} \cdot \vec{A}'_\mu \right) \phi' + \dots$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \eta^\dagger \partial^\mu \eta + \frac{1}{2} g^2 v^2 \epsilon^{ijk} \epsilon^{lmn} A'^i_\mu A'^m_\nu - V((\eta + v)^2)$$

+ d.o.f. terms + ϕ -indep. terms.

β_1 and β_2 disappeared.

$$\text{and for } A_{\mu i} : \quad \frac{1}{2} g^2 v^2 \left[A'^i_\mu A'^{i*}_\nu + A'^i_\mu A'^{j*}_\nu \right]$$

T_3 is broken \rightarrow 1 massless vector boson A'^3_μ .
2 massive ones A'_μ and A''_μ

In the same way : let \mathcal{L} be invariant under $SU(n)$
 ϕ in the fundamental rep (complex rep)
 $\langle \phi \rangle$ invariant under $SU(n-1)$.

broken generators : $(n^2 - 1) - [(n-1)^2 - 1] = 2n - 1$

real Higgs : $2n \Rightarrow$ only 1 "physical" Higgs

Rh: in (grand unification group) $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$
 ϕ in the (real) adjoint.

broken gener. : $24 - 12 = 12 \quad \dim(\text{adj } SU(5)) = 24 \Rightarrow$ ^{12 physical}
^{Higgs}

(17)

Summary :

- \mathcal{L} $V(\phi)$: ϕ "Higgs" scalars

(local) G-invar^t

$\langle \phi \rangle$ inv^t under S C G

$$\begin{cases} \dim(G) = N \\ \dim(S) = M \end{cases} \quad \xrightarrow{\text{(Goldstone + Higgs)}} \quad \begin{cases} (N-M) \text{ gauge bosons get a mass} \\ M \text{ gauge bosons without mass} \end{cases}$$

- ϕ in repes R of G

$$\dim(R) = r$$

if R complex \rightarrow we have $2r$ real fields.

$$\# \text{ "physical" Higgs} \quad || \quad \begin{array}{l} (r) \\ \text{or } (2r) \end{array} \quad - \quad \begin{array}{l} (N-M) \\ \# \text{ broken generators} \end{array}$$

Standard Model:

$$SU(2) \times U(1) \longrightarrow U(1)_{\text{em}}$$

ϕ in the 2 of $SU(2)$, i.e. 4 real fields

3 generators are broken $\Rightarrow 4 - 3 = 1$ "physical Higgs"

A (general) Math. pb for Higgs:

Given G symm. of \mathcal{L} and S s.t. $G \rightarrow S$

Determine G-reps. s.t. its repes-space R contains a

state ϕ_0 satisfying : $\text{stab}(\phi_0) = \{g(\phi_0) = \phi_0 \mid g \in G\} = S$