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GAUGE THEORY for BEGINNERS

I – Gauge Invariance in Classical Field Theory:

basic object : Lagrangian density $\mathcal{L}\left(\phi_i(x), \partial_\mu \phi_i(x)\right)$
↳ fields $(i=1,\dots n)$

action : $S = \int_{-\infty}^{+\infty} dt \cdot L(t) = \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i)$

Hamilton Principle $\Longrightarrow$ Euler Equations:

$$\delta \int_{t_1}^{t_2} L(t)\, dt = 0 \qquad \Longrightarrow \qquad \frac{\delta \mathcal{L}}{\delta \phi_i} = \frac{\partial}{\partial x^\mu}\left(\frac{\delta \mathcal{L}}{\delta\left(\frac{\partial \phi_i}{\partial x^\mu}\right)}\right)$$

"as a proof": $\varepsilon(t)$: small perturb. of the $\phi_i(t)$'s such that: $\varepsilon(t_1) = \varepsilon(t_2) = 0$

$$\delta \mathcal{L} = \int_{t_1}^{t_2} \left[L(\phi_i + \varepsilon, \dot\phi_i + \dot\varepsilon) - L(\phi_i, \dot\phi_i)\right] dt = \int_{t_1}^{t_2} \left(\varepsilon \frac{\partial L}{\partial \phi_i} + \dot\varepsilon \frac{\partial L}{\partial \dot\phi_i}\right) dt$$

integ. by parts : $\delta \mathcal{L} = \underbrace{\left[\varepsilon \cdot \frac{\partial L}{\partial \dot\phi_i}\right]_{t_1}^{t_2}}_{=0 \text{ at boundaries}} + \int \varepsilon \underbrace{\left(\frac{\partial L}{\partial \phi_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot\phi_i}\right)}_{=0} dt$

$=0$ since $\delta S = 0$ for all possible perturb $\varepsilon(t)$

to be generalized to $\phi_i(\vec{x}, t)$ ...

Gauge transformation "Idea" comes from observing that, to each continuous symmetry of the Lagrangian under study corresponds a conservation law: Emmy NOETHER (1915–1918)

An example: let $\mathcal{L}$ indep^t of x^0

$$\delta \phi_i (x^0, \vec{x}) = \phi_i(x^0+\varepsilon, \vec{x}) - \phi_i(x) = \varepsilon \frac{\partial \phi_i}{\partial x^0}$$

$$\delta(\partial_\nu \phi_i) = \varepsilon\, \partial_\nu \left(\frac{\partial \phi_i}{\partial x^0}\right) \qquad \text{and} \quad \delta\mathcal{L} = \varepsilon \frac{\partial \mathcal{L}}{\partial x^0}$$

but
$$\delta \mathcal{L} = \sum_i \left[\frac{\delta \mathcal{L}}{\partial \phi_i}\, \delta\phi_i + \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi_i)} \cdot \delta(\partial_\nu \phi_i)\right]$$

$\downarrow$ Euler Eq

$$= \varepsilon \sum_i \left[\partial_\nu \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0} + \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi_i)}\, \partial_\nu \left(\frac{\partial \phi_i}{\partial x^0}\right)\right]$$

$$\frac{\partial \mathcal{L}}{\partial x^0} = \partial_\nu \left[\sum_i \left(\frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0}\right)\right]$$

$$\Rightarrow \quad \frac{\partial}{\partial x^0}\Big[\underbrace{\mathcal{L} - \sum_i \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi_i)} \cdot \frac{\partial \phi_i}{\partial x^0}}_{\mathcal{H}\ :\ \text{Hamilt. density}}\Big] = \vec{\nabla} \cdot \sum_i \frac{\delta \mathcal{L}}{\delta(\vec{\nabla}\phi_i)} \frac{\partial \phi_i}{\partial x^0}$$

Since ϕ_i's $\to 0$ as $|\vec{x}| \gg$
$$\boxed{\frac{\partial}{\partial x^0} \int \mathcal{H}\, d^3x = \frac{\partial}{\partial t} H = 0}$$

"constant of motion"

In the same way, in a theory which is Lorentz invariant energy and angular momentum are conserved. ($\mathcal{L}$ being a ~~scalar~~ Lorentz scalar density).

But one can look at conservat. laws which are not space-time symmetries

Ex: electric charge.

Suppose ϕ_i of charge q_i and $\mathcal{L}$ neutral

define: $\phi_i(x) \to \exp(-i q_i \theta)\, \phi_i(x)$

U(1) group.

(if $\phi_1(x) .. \phi_n(x)$ in $\mathcal{L} \Rightarrow q_1 + \dots + q_n = 0$)..

then $\partial_\mu \phi_i(x) \longrightarrow \exp(-i q_i \theta)\, \partial_\mu \phi_i(x)$

("global" gauge transf.)

infinit. $\delta \phi_i = -i \varepsilon q_i \phi_i$

and $\delta \mathcal{L} = 0 \Rightarrow \frac{\partial}{\partial x^\mu}\left[\underbrace{\frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, q_i \phi_i}_{J^\mu} \right] = 0$ (see p. 1)

$\Rightarrow J^\mu$ = conserved current

$\partial_\mu J^\mu = 0$.

Note:

- Actually, the q_i's are the eigenvalues of infin. oper. Q.

In Quantum theory: $\delta \phi_i = -i\varepsilon [Q, \phi_i] = -i \varepsilon q_i \phi_i$

and $\boxed{Q = \int d^3x\, J_0(\vec{x}, t)}$ with $\frac{\partial Q}{\partial x^0} = 0$.

- But, one can have more than one - not space-time - conserved quant.

ex: with SU(2) $\vec{\phi} \Rightarrow \exp(-i\, \vec{L}.\vec{\theta})\, \vec{\phi}$

↳ matricial repres. of SU(2)

$[L^i, L^j] = i \varepsilon_{ijk} L^k$

$\boxed{\text{"Local" gauge transformations}}$:

It is well known that electrodynamics has a symmetry <u>larger than a global gauge symmetry</u>, i.e:

$$\phi_i(x) \longrightarrow \phi_i'(x) = \exp\left[-i q_i\, \theta(x)\right] \phi_i(x)$$

x - dependent

we note : $\delta\phi_i = -i q_i \theta(x) \phi_i(x)$

<u>and</u> $\partial_\mu \phi_i \longrightarrow \partial_\mu \phi_i' = \exp(-i q_i \theta(x))\, \partial_\mu \phi_i(x) - i q_i \left(\partial_\mu \theta(x)\right) \exp(-i q_i \theta)\, \phi_i(x)$

pb!

and we know that "all works" (i.e. invariance restored) by introducing photon field following a "<u>minimal coupling</u>"

$$\boxed{\partial_\mu \phi_i \longrightarrow (\partial_\mu - i e q_i A_\mu)\phi_i}$$

A_μ spin 1 field

1st example of gauge boson.

and A_μ must transform s.t. $\mathcal{L}$ to be $U(1)$ invariant.

i.e: $A_\mu \to A'_\mu \qquad \phi_i \to \phi_i'$

with $(\partial_\mu - i e q_i A'_\mu)\phi_i'(x) = \exp(-i e q_i \theta(x))\, (\partial_\mu - i e q_i A_\mu)\phi_i(x)$

$$\Rightarrow \boxed{A'_\mu(x) = -\frac{1}{e}\partial_\mu \theta(x) + A_\mu(x)}$$

or: $\boxed{\delta A_\mu(x) = -\frac{1}{e}\partial_\mu \theta(x)}$

<u>Rk</u>: in addition to term coupling γ to charged particle there is the kinetic energy term coupling A_μ to itself

with: $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

with $\delta F_{\mu\nu} = 0$ (from above)

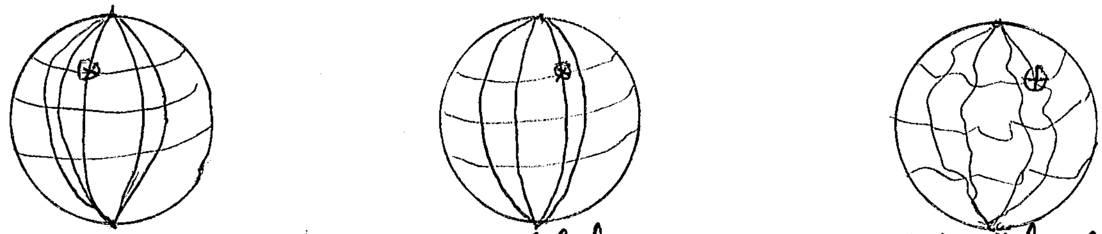
that is: $$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(factor $-\frac{1}{4}$ to get a good normal. for e in Euler eqns.)

Note : a mass term $-\frac{1}{2} m^2 A_\mu A^\mu$ would violate local gauge inv^{ce}.

Of course, photon has not been discovered by imposing local gauge inv^{ce} ! Gauge transform. came as a property of Maxwell eqns. (gauge inv^{ce} in electrodynamics allow to derive Ward identities proving in part renormalazibility).

<u>Remark on local gauge transf.</u>:



(1) (2) global rotation (3) "local" rotation

While rotating the sphere does not change, but surface is modified. The transformations <u>induce tensions</u>, <u>forces</u> between pts on the sphere. One can think that forces at the basis of matter behaviour are of this kind.

Yang & Mills (1954) generalize the $U(1)$ case to $SU(2)$ "isotopic spin".

$$[L_i, L_j] = i\,\varepsilon_{ijk} L_k$$

$$\phi(x) \to \phi'(x) = \exp\left(-i\,\vec{L}\cdot\vec{\theta}(x)\right)\phi(x)$$
$$= U(\theta(x))\,\phi(x).$$

$$\partial_\mu \phi \to U(\theta)\,\partial_\mu\phi + \left(\partial_\mu U(\theta)\right)\phi$$

Introducing the "<u>covariant derivative</u>"

$$\boxed{D_\mu \phi = \left(\partial_\mu - i\underbrace{g}_{\text{cst } g}\ \vec{L}\cdot\vec{A}_\mu\right)\phi}$$

analogous to "e"

now 3 "gauge bosons"

such that $D_\mu\phi \to U(\theta)\,D_\mu\phi$

How A_μ will be transformed ?

$$D'_\mu \phi' = \partial_\mu\phi' - ig\,\vec{A}'_\mu\cdot\vec{L}\,\phi' = U(\theta)\,D_\mu\phi$$

i.e. $$U\left[\partial_\mu - ig\,\vec{A}_\mu\cdot\vec{L}\right]\phi = (\partial_\mu U)\phi + U(\partial_\mu\phi) - ig\,\vec{A}'_\mu\cdot\vec{L}\,U\phi$$

$$\Rightarrow \boxed{\vec{A}'_\mu\cdot\vec{L} = U\,\vec{A}_\mu\cdot\vec{L}\,U^{-1} - \frac{i}{g}(\partial_\mu U)\,U^{-1}}$$

<u>Exercise</u> : prove these transf. satisfy group action.

$$\vec{A}_\mu\vec{L} \xrightarrow{U} \vec{A}'_\mu L \xrightarrow{V} \vec{A}''_\mu \vec{L}$$

with $$\vec{A}''_\mu\cdot\vec{L} = (VU)\,\vec{A}_\mu\vec{L}\,(VU)^{-1} - \frac{i}{g}\left[\partial_\mu(VU)\right](VU)^{-1}$$

7/ Infin^t with $U(\theta) = \exp(-i\vec{\theta}.\vec{L})$

$$\boxed{U(\theta)\ \vec{A}_\mu.\vec{L}\ U(\theta)^{-1}} = (1 - i\vec{\theta}.\vec{L})(\vec{A}_\mu.\vec{L})(1 + i\vec{\theta}.\vec{L})$$

$$= \vec{A}_\mu \vec{L} + \varepsilon_{ijk}\,\theta^j A^i_\mu L^k + \dots$$

$$\boxed{[\partial_\mu U(\theta)]\ U(\theta)^{-1}} = -i\,\partial_\mu \theta^i . L^i$$ <u>in Lie algebra!</u>

$$\Rightarrow \quad \delta A^i_\mu . L^i = -\frac{1}{g}\,\partial_\mu \theta^j . L^j - \varepsilon_{ijk} A^i_\mu \theta^j L^k$$

that is $\boxed{\delta A^i_\mu} = -\frac{1}{g}\,\partial_\mu \theta^i + \varepsilon_{ijk}\,\theta^j A^k_\mu$

Note: the transform. of A^i_μ do not depend on the repres. but only on the commut. relations (struct. cst)

<u>Rk:</u> we recover the e.m. case $\delta A_\mu = -\frac{1}{g}\partial_\mu\theta$ with $U(1)$ group!

What about $F_{\mu\nu}$?

Define now $$\boxed{\vec{F}_{\mu\nu}.\vec{L} = \partial_\mu \vec{A}_\nu.\vec{L} - \partial_\nu \vec{A}_\mu.\vec{L} - ig\,[\vec{A}_\mu\vec{L}, \vec{A}_\nu\vec{L}]}$$

or $$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\,\varepsilon_{ijk} A^j_\mu A^k_\nu$$

Group action: $$\delta F^i_{\mu\nu} = \varepsilon_{ijk}\,\theta^j F^k_{\mu\nu}$$

that is also: $\boxed{\vec{F}_{\mu\nu}.\vec{L}} \rightarrow U(\theta)\,\vec{F}_{\mu\nu}.\vec{L}\,U(\theta)^{-1}$

and $\vec{F}_{\mu\nu}.\vec{F}_{\mu\nu}$ <u>scalar</u> under $SU(2)$.

Summary: $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ invariant under G simple with $[T_i, T_j] = i\, c_{ijk} T_k$

$$\phi \to \exp(-i\vec{\theta}.\vec{L})\, \phi \qquad \theta^i(x) \hookrightarrow \text{func. of } x$$

- add to each T_i a "bosonic field" A_μ^i
- Write $\mathcal{L}$ as $* \ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}(\phi_i, (\partial_\mu - ig\vec{A}_\mu.\vec{L})\phi_i)$

$$* \ \mathcal{L}_0 = -\frac{1}{4} \vec{F}_{\mu\nu} . \vec{F}_{\mu\nu}$$

with $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\, c_{ijk} A^j_\mu A^k_\nu$

the gauge bosons transforming under G as:

$$** \left\| \begin{array}{l} \vec{L}.\vec{A}_\mu \to U(\theta)\, \vec{L}.\vec{A}_\mu\, U(\theta)^{-1} - \frac{i}{g}[\partial_\mu U(\theta)].\, U(\theta)^{-1} \\ \\ \vec{F}_{\mu\nu}.\vec{L} \to U(\theta)\, \vec{F}_{\mu\nu}.\vec{L}\, U(\theta)^{-1} \quad (\Rightarrow \vec{F}_{\mu\nu} \vec{F}_{\mu\nu} \ G\text{-scalar}) \end{array} \right.$$

and:

Note: If $G = G_1 \times G_2$

one can have g_1 and g_2 $(\neq)$ for G_1 and G_2 respectively

(i.e. only 1 coupling constant for each simple group or $U(1)$ group)