

Classification of (semi) simple Lie Algebras.

Owing to converse Lie's theorem, all Lie groups with the same lie alg. are locally isomorphic (N.b.: $G \cong \bar{G}/D$ p.40)
 \Rightarrow Attempt to classify Lie algebras.

- \mathfrak{g} n-dim. Lie alg. characterized by n generators X_i ($i=1, \dots, n$) constituting a basis of a linear v. space, as well as by the Lie product $[X_i, X_j] = C_{ij}^k X_k$ (structure const. depending on the chosen basis).
- Solvable algebras : not very much known (classif. up to dim. 6).
- (Semi) simple algebras: all is known using the "standard" or "Cartan" basis (\Rightarrow classification of all simple alg. and of their representat.).

I Cartan Classification:

Problem: $A \in \mathfrak{g}$, look for $X \in \mathfrak{g}$ s.t.:

$$(\text{Ad } A)(X) = [A, X] = \rho X$$

$$A = a^i X_i \quad \Rightarrow \quad a^i x^j c_{ij}^k = \rho x^k$$

$X = x^j X_j$ system of linear equa.

(56)

non trivial solut iff: $\det \{ a^i C_{ij}^{lk} - p \delta_j^k \} = 0.$ *

equa. n^{th} order in $p \Rightarrow n \text{ solut. } \{x^i\}_p$

and then n operators $x_p.$

Are they linearly indept.?

N.b.: Necessary to "complexify" G i.e. to work with G defined on complex field, in order to solve equa. (since the field \mathbb{R} is not algebraically closed).



Cartan's theorem: Let G be semi-simple, If A is chosen such that equa. * has a maximal nber of \neq roots, then:

i) the root $p=0$ is 2 times degenerate, and
 \exists 2 linearly indept elts H_1, \dots, H_2 such that:

$$[A, H_i] = 0$$

$$[H_i, H_j] = 0 \quad i, j = 1, \dots, 2.$$

ii) the other roots are not degenerate: their nba is $(n-2)$. Denoting E_α the eigenvectors with eigenvalues α : $[A, E_\alpha] = \alpha E_\alpha$

The elts H_i ($i=1 \dots 2$) and E_α ($j=1 \dots, n-2$) are linearly indept and therefore constitute a basis of G .

- Def.: . the number r is called the rank of \mathfrak{g} -
 • the H_i ($i=1, \dots r$) constitute the Cartan subalg.
 i.e. maximal Abelian subalg. of \mathfrak{g} . (= $\text{Cart}(\mathfrak{g})$)

Property: $\text{Cart}(\mathfrak{g})$ is unique up to a conjugation of \mathfrak{g} -

Rk: $\text{Cart}(\mathfrak{g})$ "maximal" as Abel. subalg. i.e. any Abelian subalg of \mathfrak{g} is in $\text{Cart}(\mathfrak{g})$.

Theorem: Any el^t. of a semi-simple alg. \mathfrak{g} is conjugate to an el^t. of $\text{Cart}(\mathfrak{g})$, once given a basis for \mathfrak{g} and $\text{Cart}(\mathfrak{g})$.

Solv differently: $\forall X \in \mathfrak{g}$ one can choose $\text{Cart}(\mathfrak{g})$ such that $X \in \text{Cart}(\mathfrak{g})$.

Example: $SU(2)$: $[J_i, J_j] = i \epsilon_{ijk} J_k$. complex field.

$$\text{Choose } J_3 = H_1, \quad J_{\pm} = J_1 \pm i J_2 = E_{\pm}$$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = 2 J_3$$

First remarks:

$$\begin{aligned} [A, [H_i, E_\alpha]] &= [[\underbrace{A, H_i}_0], E_\alpha] + [H_i, [A, E_\alpha]] \\ &= \alpha [H_i, E_\alpha] \end{aligned}$$

α eigenvalue & non degenerate \Rightarrow

$$[H_i, E_\alpha] = \alpha_i E_\alpha.$$

α_i : coeff^t of proportionality.

Now: $[A, [E_\alpha, E_\beta]] = [[A, E_\alpha], E_\beta] + [E_\alpha, [A, E_\beta]]$

$$= (\alpha + \beta) [E_\alpha, E_\beta].$$

So 3 possibilities:

- i) $(\alpha + \beta)$ is not a root: $[E_\alpha, E_\beta] = 0.$
- ii) $(\alpha + \beta) \neq 0$: $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$
- iii) $\alpha + \beta = 0$
 $E_{-\alpha}$ exists : $[E_\alpha, E_{-\alpha}] = \sum_{i=1}^2 \lambda_i H_i$

Theorem: If α is a root, $-\alpha$ is also a root.

Proof: Consider Killing form: $g_{\alpha\gamma} = C_{\alpha i}^m C_{\gamma m}^i$

the non-zero structure const. are at most:

$$C_{i\alpha}^{\alpha} = \alpha_i \quad C_{\alpha\beta}^{\alpha+\beta} = N_{\alpha\beta} \quad C_{\alpha-\alpha}^i = \lambda_i^i$$

$$\Rightarrow g_{\alpha\gamma} = C_{\alpha i}^{\alpha} C_{\gamma\alpha}^i + C_{\alpha\beta}^{\alpha+\beta} \underbrace{C_{\gamma, \alpha+\beta}^{\beta}}_{\text{if } \neq 0} + C_{\alpha, -\alpha}^i \cdot C_{\gamma i}^{-\alpha}$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\gamma = -\alpha \qquad \qquad \qquad \gamma = -\alpha$$

So: $g_{\alpha\gamma} = 0 \neq 0$ if $E_{-\alpha}$ does not belong to the basis.
 impossible since \mathfrak{g} semi-simple

$$(\det[g_{ij}] \neq 0).$$

Remark: E_α root $\rightarrow E_{-\alpha}$ root \Rightarrow number of root is even
 $n - r = 2p.$

Property: $\text{Ad } H_i (= [H_i, .])$ can be chosen hermitian,
 and therefore the eigenvalues α_i real -

Simple but tedious calculations lead to:

Summary of results:

$$[H_i, H_j] = 0 \quad i, j = 1, \dots, r.$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i$$

$$[E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha + \beta} \quad \text{if } \alpha + \beta \text{ root} \neq 0.$$

with : $\forall \alpha, \beta \text{ roots } \exists 2 \text{ non negative integers } j, k \text{ such that}$

$$N_{\alpha, \beta}^2 = \frac{1}{2} j(k+1) \vec{\alpha} \cdot \vec{\alpha}$$

$$\text{with: } \vec{\alpha} = (\alpha_1, \dots, \alpha_r)$$

and:

$$\begin{cases} \vec{\beta} + j \vec{\alpha} \\ \vec{\beta} - k \vec{\alpha} \end{cases} \quad \text{roots}$$

the sign of $N_{\alpha, \beta}$ submitted to restriction:

$$N_{\alpha, \beta} = -N_{\beta, \alpha} = -N_{-\alpha, -\beta}$$

. The roots are normalized s.t.:

$$\delta_{ij} = \sum_{\vec{\alpha} \neq 0} \alpha_i \alpha_j \quad \sum_{\vec{\alpha} \neq 0} \vec{\alpha} \cdot \vec{\alpha} = 1$$

with the Killing form:

$$g = \left[\begin{array}{cc|c|c|c} & & & & \\ & & \overbrace{\delta_{ij}}^{n} & & 0 \\ \hline & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & \\ \hline & 0 & & & \\ & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ & & & & \end{array} \right] \quad \left\{ n-r = 2p \right.$$

Def.: the term root will be used for α_i or for $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$
 the root diagram of G will be constituted by the set
 of vectors $\vec{\alpha}$ (corresponding to the $\pm E_\alpha$).

Properties of the roots:

(60)

i) $\vec{\alpha}$ root $\rightarrow k\vec{\alpha}$ root iff $k = 1, 0, -1$.

ii) $\vec{\alpha}$ & $\vec{\beta}$ roots \Rightarrow

①
$$l \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} \text{ integer}$$

②
$$\vec{\gamma} = \vec{\beta} - l \left(\frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} \right) \vec{\alpha} \text{ is a root}$$

in R^2 : $\vec{\gamma}$ is the symmetric of $\vec{\beta}$ with respect to the hyperplane $\perp \vec{\alpha}$

(= Weyl reflection. All these Weyl reflections and their products constitute the Weyl group):

Weyl hyperplane

Consequences of these properties:

$$\begin{aligned} \textcircled{a} \Rightarrow \quad \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|^2} &= \frac{1}{2} m \Rightarrow \vec{\alpha} \cdot \vec{\beta} = \frac{m}{2} \cdot |\vec{\alpha}|^2 \\ \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\beta}|^2} &= \frac{1}{2} n \Rightarrow \vec{\alpha} \cdot \vec{\beta} = \frac{n}{2} \cdot |\vec{\beta}|^2 \end{aligned} \quad \left\{ \begin{array}{l} \cos^2 \varphi = \frac{(\vec{\alpha} \cdot \vec{\beta})^2}{(|\vec{\alpha}|^2)(|\vec{\beta}|^2)} = \frac{mn}{4} \\ \frac{|\vec{\alpha}|^2}{|\vec{\beta}|^2} = \frac{n}{m} \end{array} \right.$$

$0 \leq \varphi \leq 90^\circ$ (for $\varphi > 90^\circ$ change α (or β) in $-\alpha$ (or $-\beta$))

$\cos^2 \varphi$	1	$3/4$	$1/2$	$1/4$	0
φ	0°	30°	45°	60°	90°
$\frac{\vec{\alpha} \cdot \vec{\beta}}{ \vec{\beta} ^2} = \frac{ \vec{\alpha} ^2}{ \vec{\beta} ^2}$	1	3	2	1	undeter- mined

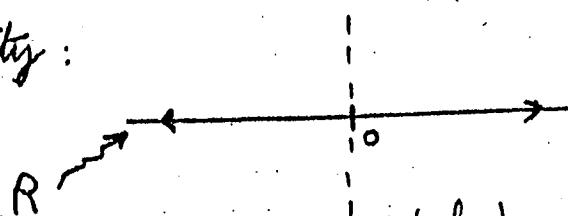
at first

Construction of the root diagrams:

Case $r=1$:

only 1 possibility:

$$(\text{SU}(2) \cong) \boxed{A_1}$$



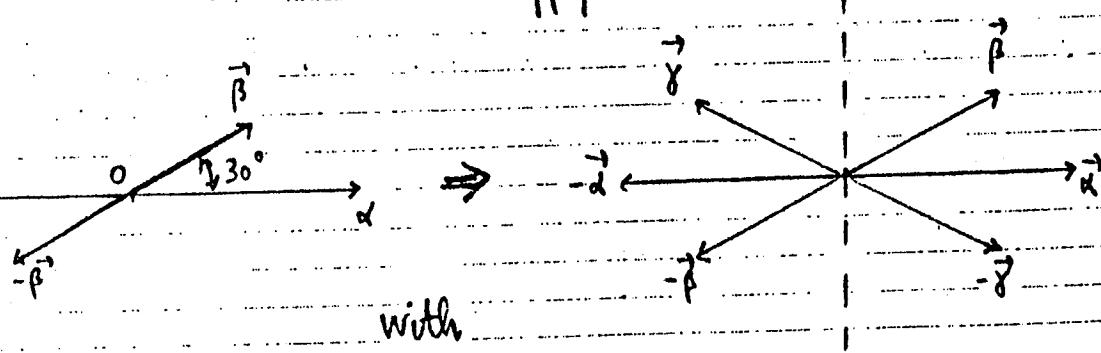
Weyl hyperplane

$$\text{length of } \vec{\alpha} = 1/\sqrt{2} \quad (\delta_{ij} = \sum_k \alpha_i \alpha_j)$$

Case $r=2$:

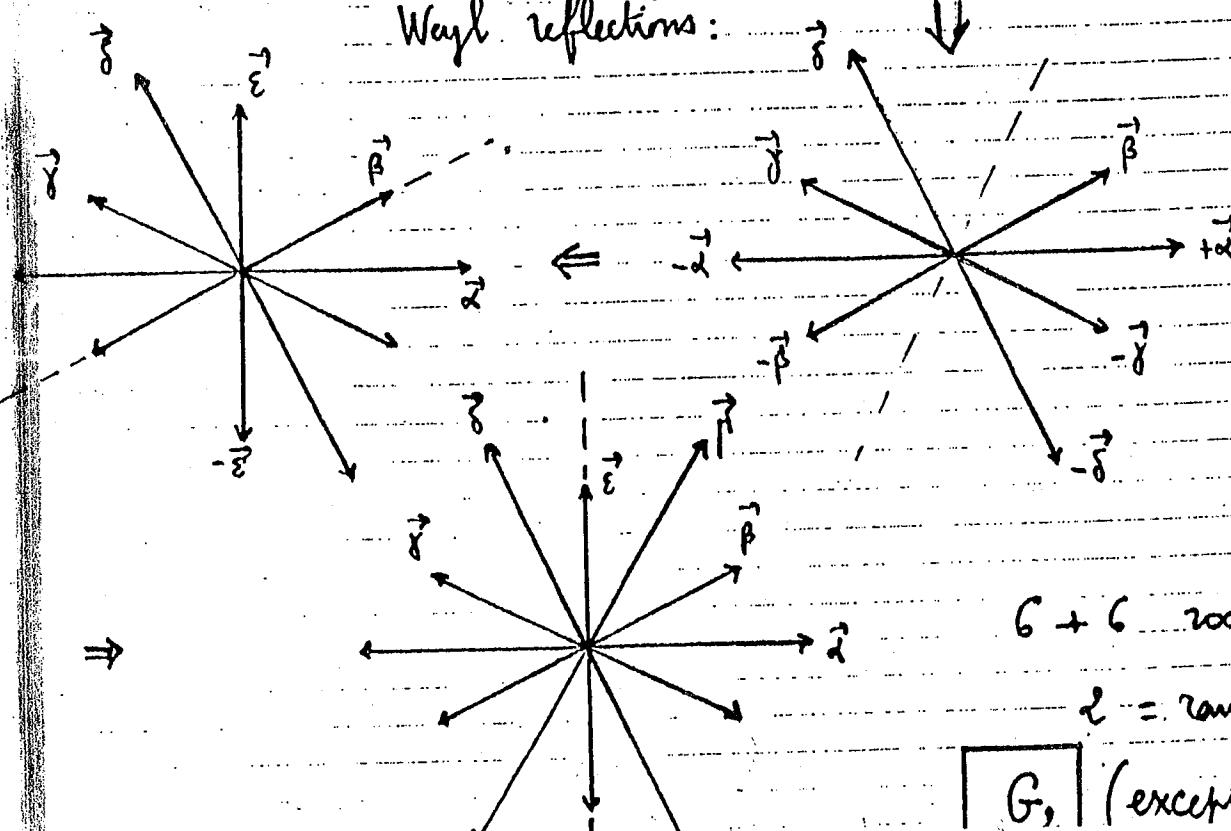
$\vec{\alpha} \& \vec{\beta}$ roots

$$\text{a) } q = 30^\circ \Rightarrow \frac{|\vec{\alpha}|}{|\vec{\beta}|} = \sqrt{3}$$



with

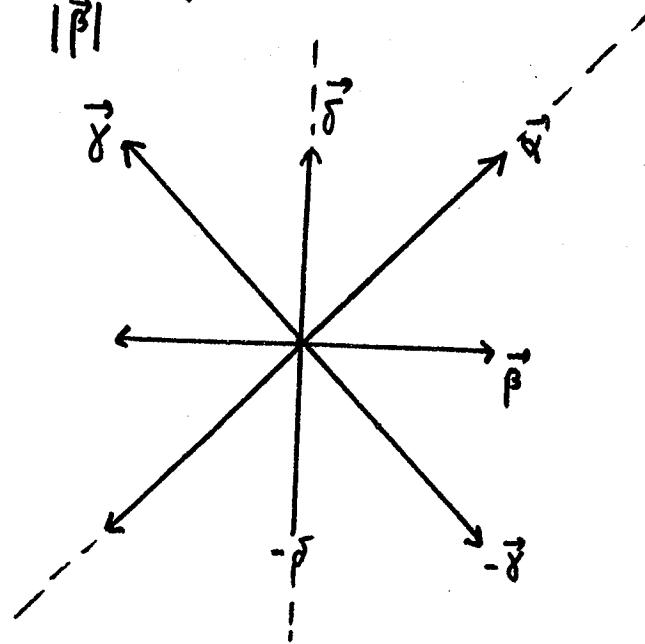
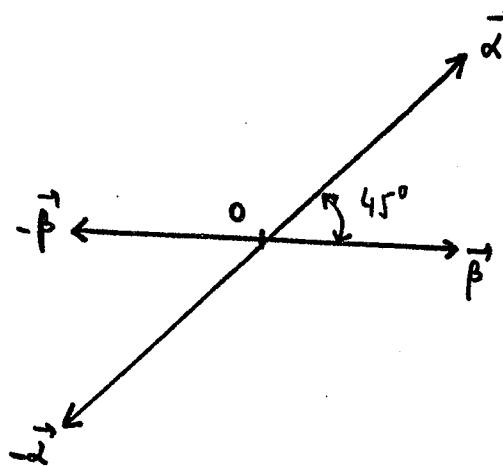
Weyl reflections:



$6 + 6 \text{ roots } \left\{ \begin{array}{l} \text{dim.} \\ \text{rank} \end{array} \right. = 14$

$\boxed{G_2}$ (exceptional Lie alg.)

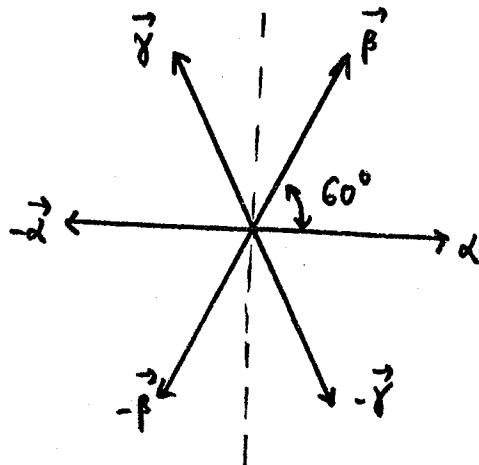
b) $\varphi = 45^\circ \rightarrow \frac{|\vec{\alpha}|}{|\vec{\beta}|} = \sqrt{2}$



$4+4$ roots
 $2 = \text{rank}$ } dim. 10

$$SO(10) = [B_2 \cong C_2] (= Sp(4))$$

c) $\varphi = 60^\circ \rightarrow |\vec{\alpha}| = |\vec{\beta}|$



$3+3$ roots
 $2 = \text{rank}$ } dim. 8

$$A_2 \cong SU(3)$$

d) $\varphi = 90^\circ$

Special case

$$[A_1 \oplus A_1]$$

$$= SU(2) \oplus SU(2)$$

e) $\varphi = 0$.

Special case

excluded

$\vec{\alpha}$ & $\vec{\beta}$ proportional
 $(= \text{degenerate})$

not a s. single alg.

Generalisation:

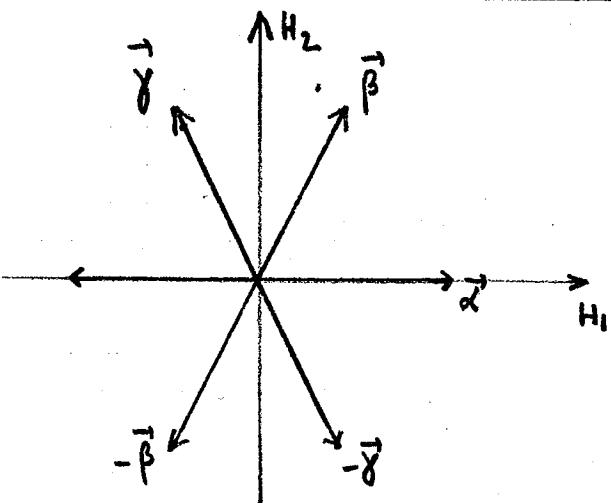
In the same way one can construct the root space E_2 from E_1 , one can build E_{n+1} from E_n by - adding to E_n an additional vector such that.

- i) it coincides with none of the vectors in E_n
- ii) the condition : $2 \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}}$ is satisfied.

- completing then the space by reflection.

If all vectors resulting from the completion still obey $2 \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} = \text{integer}$, then the completion is a root space; if not, it is not

$SU(3) = A_2$ root space again:



$$[H_1, E_{\pm\alpha}] = \pm \alpha_1 E_{\pm\alpha}$$

$$(\text{Ad } H_1) E_{\pm\alpha} = \pm \alpha_1 E_{\pm\alpha}$$

take:

$$\left\{ \begin{array}{l} H_1 = \lambda_3 \\ H_2 = \lambda_8 \\ E_{\pm\alpha} = \lambda_1 \pm i\lambda_2 \\ E_{\pm\beta} = \lambda_4 \pm i\lambda_5 \\ E_{\pm\gamma} = \lambda_6 \pm i\lambda_7 \end{array} \right.$$

Note: $\left. \begin{array}{l} \lambda_3 \\ \lambda_1 \pm i\lambda_2 \end{array} \right\} A_1 (\equiv SU(2))$.

$$\left. \begin{array}{l} \lambda_3 + i\sqrt{3}\lambda_8 \\ \lambda_4 \pm i\lambda_5 \end{array} \right\} A_1$$

On the axes ($\vec{\alpha}$) ($\vec{\beta}$) ($\vec{\gamma}$) one recognizes an $SU(2)$ alg

$$\left. \begin{array}{l} \lambda - i\sqrt{3}\lambda_8 \\ \lambda_6 \pm i\lambda_7 \end{array} \right\} A_1 \quad \begin{array}{l} (\text{Cl: I, U and V spin algebra}) \\ \text{spin algebra} \end{array}$$

Root Space for (semi-) simple Lie algebras.

Root space.	Non-zero roots	Nbr of non-zero roots	Rank	Nbr of generators	Normalizing $\vec{e}_i $
A_l $= SU(l+1)$	$e_i - e_j \quad 1 \leq i \neq j \leq l+1$ (n-dimens. hyperplane perp vector: $\vec{v} = [\vec{e}_1 + \dots + \vec{e}_{l+1}]$)	$l(l+1)$	l	$(l+1)^{\frac{l}{2}-1}$	$\frac{1}{\sqrt{2(l+1)}}$
B_l $= SO(2l+1)$	$\pm e_i \pm e_j \quad 1 \leq i \neq j \leq l$ $\pm e_i$	$2l(l+1)$ $2l$	l	$\frac{2l(2l+1)}{2}$	$\frac{1}{\sqrt{2(2l+1)}}$
C_l $= Sp(2l)$	$\pm e_i \pm e_j \quad 1 \leq i \neq j \leq l$ $\pm 2e_i$	$2l(l-1)$ $2l$	l	$\frac{2l(2l+1)}{2}$	$\frac{1}{\sqrt{2(2l+2)}}$
D_l $= SO(2l)$	$\pm e_i \pm e_j \quad 1 \leq i+j \leq l$	$2l(l-1)$	l	$\frac{2l(2l-1)}{2}$	$\frac{1}{\sqrt{2(2l-2)}}$
G_2	$e_i - e_j \quad 1 \leq i+j+l \leq 3$ $\pm (e_i + e_j) \mp 2e_k$	6 6	2	14	$\frac{1}{\sqrt{24}}$
F_4	$\pm e_i \pm e_j \quad 1 \leq i \neq j \leq 4$ $\pm 2e_j$ $\pm e_1 \pm e_2 \pm e_3 \pm e_4$	24 8 16	4	52	$\frac{1}{6}$
E_6	$\pm e_i \pm e_j \quad 1 \leq i \neq j \leq 5$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm \dots \pm e_5) \pm \sqrt{2-\frac{5}{4}}e_6$ even number of + signs.	40 32	6	78	$\frac{1}{\sqrt{24}}$
E_7	$\pm e_i \pm e_j \quad 1 \leq i \neq j \leq 6$ $\pm \sqrt{2}e_7$ $\frac{1}{2}(\pm e_1 \pm \dots \pm e_6) \pm \sqrt{2-\frac{5}{4}}e_7$ even number of + signs	60 24	7	133	$\frac{1}{6}$
E_8	$(\pm e_i \pm e_j \pm \dots \pm e_8) ; \pm e_i \pm e_i$	128; 112	8	256	$\frac{1}{\sqrt{24}}$

Remarks: In the above tableau:

- + all algebras simple except D_2 or $SO(4)$ semi-simple
(as groups: $SO(4) \cong [SU(2) \times SU(2)]/\mathbb{Z}_2$
see p. 41 & 43).
- + $A_1 \cong B_1 \cong C_1$ (as groups: $SO(3) \cong SU(2)/\mathbb{Z}_2$
 $SU(2) \cong Sp(2)$)
- + $B_2 \cong C_2$ (as groups: $SO(5) \cong Sp(4)/\mathbb{Z}_2$)
- + $A_3 \cong D_3$ (as groups: $SO(6) \cong SU(4)/\mathbb{Z}_2$)
- + G_2, F_4, E_6, E_7, E_8 are the five "exceptional" Lie algebras
- + the description of root space can be simplified
 \Rightarrow Dynkin diagrams.

II Dynkin Diagrams.

Defn: A root $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ is defined to be positive if its first non-vanishing component is positive.

Rk: Since $\vec{\alpha}$ root $\Rightarrow -\vec{\alpha}$ root, the number of positive roots is $\underline{n-2/2}$.

Ex.: $B_2 (\cong SO(5))$ roots: $\pm e_0 \pm e_j; \pm e_i \quad 1 \leq i \neq j \leq 2$
positive roots: $(1,1); (1,-1); (1,0); (0,1)$

Def 2: A root is said simple if it is positive and cannot be decomposed as the sum of two other positive roots. (66)

Theorem: In a simple Lie algebra \mathfrak{g} of rank 2, there are 2 simple roots which are linearly independent, and fully characterize the algebra.

Consequence: Since the root space Σ is 2-dim., these 2 simple roots constitute a basis for Σ .

Ex.: B_2 : simple roots: $(1, -1)$ and $(0, 1)$

$$\text{and: } (1, 0) = (1, -1) + (0, 1)$$

$$(1, 1) = (1, -1) + 2(0, 1)$$

only positive coeffs.

We can classify all simple algebra owing to:

Theorem: Let Π be the Σ_2 -basis constructed with the 2 simple roots. Then:

$$\begin{aligned} \textcircled{a} \quad & \nexists \alpha, \beta \in \Pi \Rightarrow \alpha - \beta \notin \Pi \\ & \Rightarrow 2 \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} = -p \quad p \text{ integer} \end{aligned}$$

$$\begin{aligned} \textcircled{b} \quad & \nexists \alpha, \beta \in \Pi \Rightarrow \begin{cases} = 1 \Rightarrow \theta_{\alpha\beta} = 120^\circ \\ \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} = 2 \Rightarrow \theta_{\alpha\beta} = 135^\circ \\ = 3 \Rightarrow \theta_{\alpha\beta} = 150^\circ \\ = -2 \Rightarrow \theta_{\alpha\beta} = 90^\circ \end{cases} \end{aligned}$$

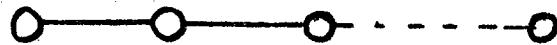
$$\text{with } \cos \theta = \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}| |\vec{\beta}|}$$

Dynkin Diagram:

- A simple root is represented by a circle
- for the longer } only 2 possible lengths
- for the shorter } in a simple algebra!
- Two simple roots are connected by
- $\begin{array}{c} \overset{0}{\textcircled{1}} \\ | \\ 2 \\ 3 \end{array} \left. \right\}$ lines if their angle $\theta_{\alpha\beta}$ is $\begin{cases} 90^\circ \\ 120^\circ \\ 135^\circ \\ 150^\circ \end{cases}$.

Then the correspondence (semi-) simple Lie alg. \leftrightarrow Dynkin Diagram is unique!

A_1



(use p. 64).

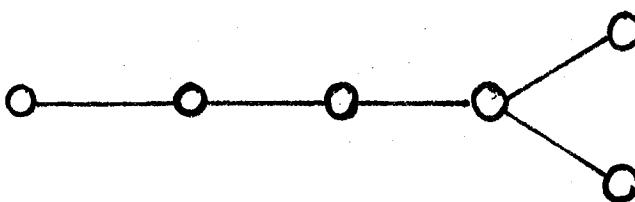
B_2



C_2

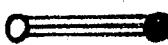


D_4

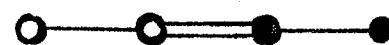


$(l \geq 3)$

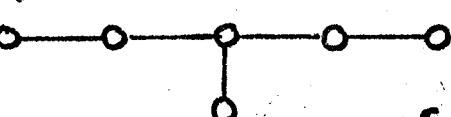
G_2 :



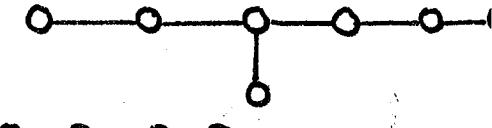
F_4 :



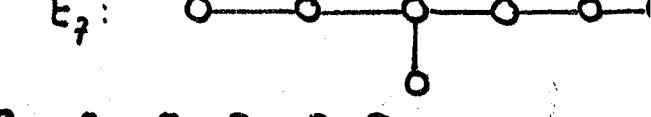
E_6



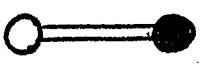
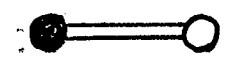
E_7 :

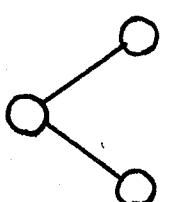


E_8 :



First Remarks on D.D.:

a) B_2 :  \cong  : C_2 (\cong as Lie alg
no group).

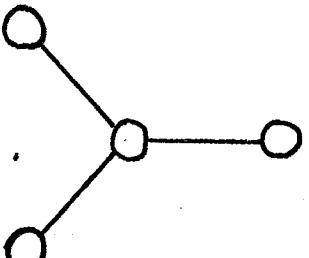
b) A_3 :  \cong  : D_3

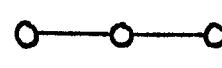
c)  \cong  $\Rightarrow A_1 \cong B_1 \cong C_1$

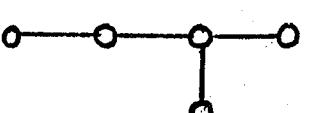
but not D_1 ($SO(2)$)

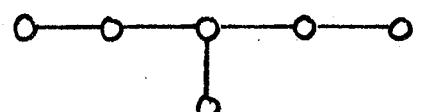
in the same way D_2 ($\cong SO(4)$) \neq  ---

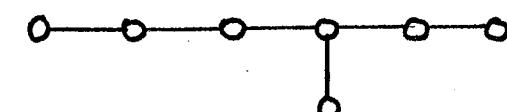
$\Rightarrow D_r \quad r \geq 3$.

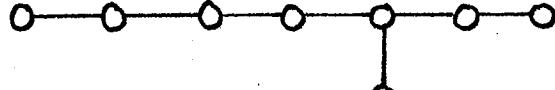
d) D_4 :  special symmetry \Rightarrow Triality
in partic.: 3 \neq represent. of $\dim 8$,
satisfying: $\begin{cases} 8 \times 8' = 8'' + \dots \\ 8' \times 8'' = 8 + \dots \\ 8'' \times 8 = 8' + \dots \end{cases}$

e) A_4 (or $SO(5)$):  = E_4 .

D_5 (or $SO(10)$):  = E_5

E_6 : 

E_7 : 

E_8 : 

Because the two famous candidates for GUTs are $SU(5)$ and $SO(10)$, is it a reason to believe that the truth is in one of the three others E_6, E_7 or E_8 groups?

(69)

f.) G simple group: $\frac{\text{Aut}(G)}{\text{Int}(G)}$ = factor group is

exactly on the corresponding Lie algebra \mathfrak{G} , the finite group of mappings of the Dynkin diagram into itself, which preserves all inner products. These mappings are described below:

		Group
$A_n :$		$\alpha_i \leftrightarrow \alpha_{n+1-i}$ P_2
$D_n :$		$\alpha_{n-1} \leftrightarrow \alpha_n$ P_2
$D_4 :$		$(e) (12) (23) (34)$ $(123) (321)$ P_3
$E_6 :$		$\alpha_i \leftrightarrow \alpha_{6-i}$ $\alpha_6 \leftrightarrow \alpha_5$ P_2
$B_n, C_n, G_2, F_4, E_7, E_8$		$P_1 (=e)$

Dynkin Diagrams and Subalgebras of a Simple Algebra:

I.D. can be used to classify all the (s.) simple subalgebras of a (s.) simple algebra. This is not an easy job!

E.B. DYNKIN.

American Math. Society Transl.

Series 2, Vol. 5. (p. 111 - 378). (1957)

(70)

(general theorems, general methods, classif. of
(8) simple algebras of exceptional algebras).

M. Lorente & B. Gruber

J.M.P. Vol. 13, n° 10, (1972) p. 1639.

classif. of (8) simple algebras of classical algebras
up to rank 6.

But we can have some subalgebras easily from D.D.,
by suppressing one or more lines circles:

Example:

$$A_{\ell}: \quad \begin{array}{ccccccccc} \alpha_1 & & \alpha_2 & & \alpha_3 & \cdots & \alpha_k & | & \alpha_{k+1} & | & \alpha_{k+2} & | & \alpha_{k+3} & \cdots & \alpha_{n-1} & \alpha_n \\ \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ & - & \circ \end{array} \Rightarrow A_k \oplus A_{\ell-k-1}$$

$$D_{\ell}: \quad \begin{array}{ccccccccc} \alpha_1 & & \alpha_2 & & \alpha_3 & \cdots & \alpha_k & | & \alpha_{k+1} & | & \alpha_{k+2} & | & \alpha_{k+3} & \cdots & \alpha_{n-1} & \alpha_n \\ \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ & - & \circ \end{array} \Rightarrow A_k \oplus D_{\ell-k-1}$$

$$E_7: \quad \begin{array}{ccccccccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array}$$

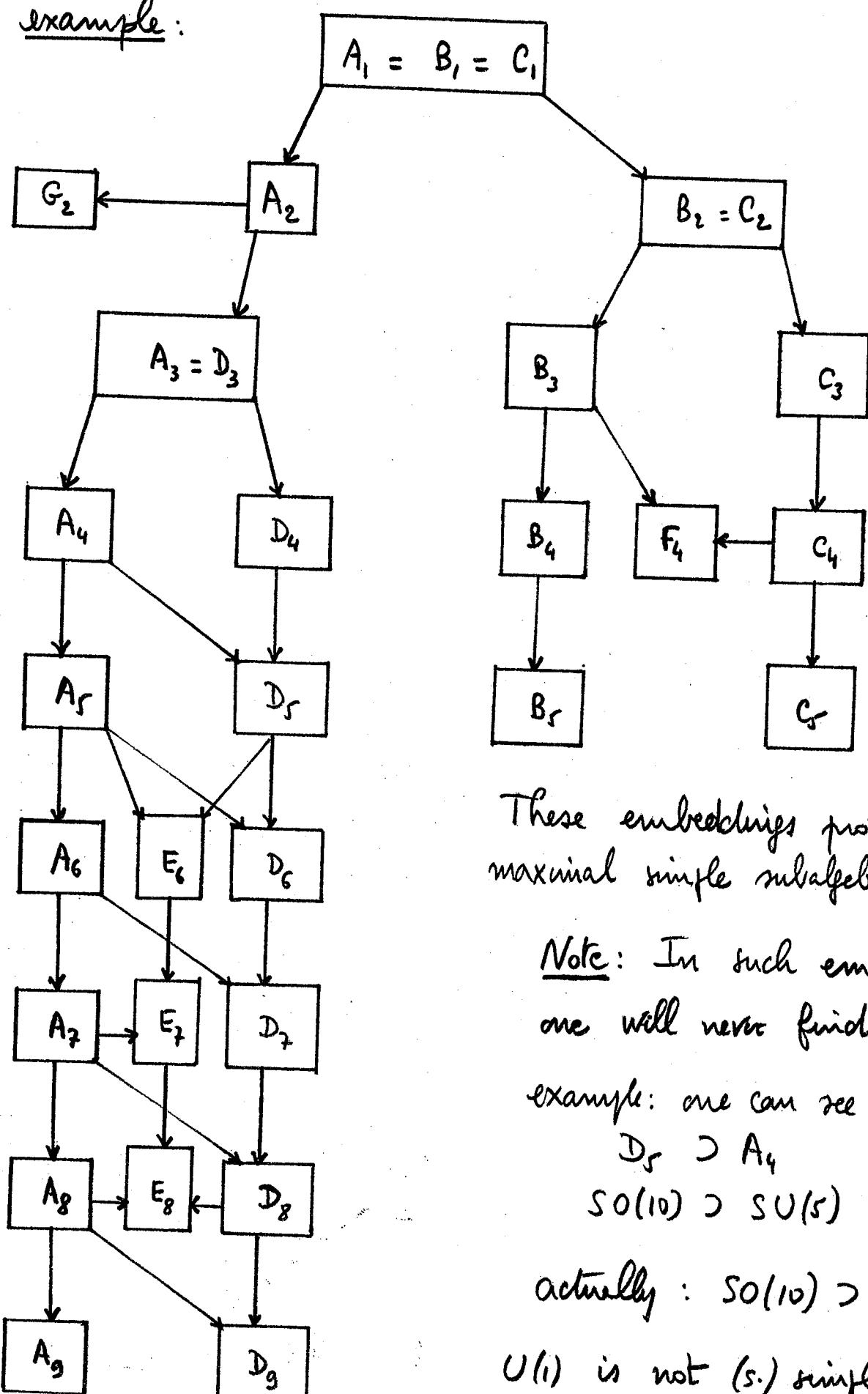
$$\bullet \oplus \begin{array}{ccccccccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array} = A_1 \oplus D_5$$

$$\bullet \oplus \begin{array}{ccccccccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array} = A_2 \oplus A_1 \oplus A_3$$

$$\bullet \bullet \bullet \bullet \bullet \bullet \bullet \oplus \begin{array}{ccccccccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array} = A_3 \oplus A_1 \oplus A_3$$

Inversely, increasing correctly a D.D. allows to see the inclusions among simple algebras: +1

example:



These embeddings provide some maximal simple subalgebras.

Note: In such embeddings one will never find "U(1)" part

example: one can see:

$$D_5 \supset A_4 \\ SO(10) \supset SU(5)$$

actually: $SO(10) \supset SU(5) \times U(1)$

$U(1)$ is not (s.) simple!

Linear Representations of Groups

and Lie Algebras.

(I) Definitions and first Properties:

Def 1: Let V be a linear vector space on \mathbb{R} or \mathbb{C} .
A linear representation of a group G in V is
an homomorphism D of G in the group of
the linear and invertible operators of V
 $\forall g, g' \in G : D(g) \cdot D(g') = D(g \cdot g')$

From Chapter II : $D(e) = \mathbb{1}$

$$D(g^{-1}) = D(g)^{-1} \quad \forall g \in G.$$

Trivial representation: $\forall g \in G : D(g) = \mathbb{1}$

Faithful representation: $D(g) \neq \mathbb{1} \quad \forall g \neq e$

(ex: adjoint represent. for an Abelian group is not faithful)

Def 2: Let \mathcal{H} be an Hilbert space.

An unitary representation of a group G in \mathcal{H}
is an homomorphism U of G in the group of
the unitary operators of \mathcal{H} .

Rk: $U(e) = \mathbb{1}$ and $U(g^{-1}) = U(g)^{-1} = U(g)^+ \quad \forall g \in G$

Theorem 1: Let D be a linear represent. of a finite group G in a vector space V , and \langle , \rangle a scalar product on V , then the hermitian form

$$\forall u, v \in V \quad h(u, v) = \sum_{g \in G} \langle D(g)u, D(g)v \rangle$$

defines a new scalar product on V invariant with respect to D .

(since: $h(D(g)u, D(g)v) = h(u, v)$).

Consequence: For any linear represent. D of a finite group G in a vector space V , \exists a scalar product making this representation unitary.

Now, it appears that most of the properties on represent. of finite groups are valid for compact lie groups.

Theorem 2: Let D be a linear represent. of a compact Lie group in a vector space V , and \langle , \rangle a scalar product in V . Then the hermitian form

$$\forall u, v \in V \quad h(u, v) = \int_G \langle D(g)u, D(g)v \rangle d\mu(g)$$

where μ is the Haar measure, defines a new scalar product on V , invariant with respect to D .

Consequence: For any linear represent. D of a compact lie group G on V , \exists a scalar product making repes. unitary.

(14)

Haar measure for lie groups (summary):

- Start with G : finite group

$$\Delta \subset G \rightarrow \mu(\Delta) = \text{num of elmts in } \Delta$$

e.g.: $g\Delta = \{gx \mid x \in \Delta\}$ we have: $\mu(g\Delta) = \mu(\Delta) = \mu(\Delta g)$
 the measure μ is left inv. and right invariant.

- Consider now G lie group:

$$\Delta \subset G \rightarrow \Xi(\Delta) \subset \mathcal{A} \text{ space of the parameters.}$$

Let $\rho : \mathcal{A} \rightarrow \mathbb{R}_+$ we define: $\mu(\Delta) = \int_{\Xi(\Delta)} \rho(a) d^v a$
 (density)

if $v = \text{num of parameters } a$:

One can prove: \exists an unique choice of ρ such that μ is left invariant for a Lie group, i.e.:

μ and $\tilde{\mu}$ are \Leftarrow $\# g$ and Δ : $\mu(g\Delta) = \mu(\Delta)$
 called Haar measures. \exists also an unique choice $\tilde{\rho}$ such that $\tilde{\mu}$ is right invariant. $\tilde{\mu}(\Delta g) = \tilde{\mu}(\Delta)$.

Theorem: If G is a compact lie group: $\mu = \tilde{\mu}$
 and $\mu(G) = \tilde{\mu}(G) = \text{finite}$

Now: $f : G \rightarrow \mathbb{R}^n$ (or \mathbb{C}^n)

$f \circ \Phi^{-1} : \mathcal{A} \rightarrow \mathbb{R}^n$

$$\int_{\Xi(\Delta)} (f \circ \Phi^{-1}(a)) \rho(a) d^v a = \int_G f(g) d\mu(g)$$

formally

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g). \quad \text{The} \Leftarrow \text{with: } \int_{\Delta} d\mu(g) = \mu(\Delta)$$

Def. 3: Let $D: V \rightarrow V$, $D': V' \rightarrow V'$ two linear represent. of G in the linear v. space V and V' resp. D and D' are called equivalent represent. if \exists a one-to-one linear mapping ($\text{Ker } A = \{0\}$), $A: V \rightarrow V'$ such that:

$$\forall g \in G : \boxed{A D(g) A^{-1} = D'(g)}.$$

Rk.: If V and V' are Hilbert spaces, and A unitary, then D and D' are called unitarily equivalent.

- Sum and Product of Representations:

Direct Sum of two lin. v. spaces: V (dim. n), V' (dim n')

$$\rightarrow V \oplus V' \ni \vec{x} = \sum_{i=1}^{n+n'} \alpha^i \vec{e}_i \quad \text{if } (\vec{e}_1, \dots, \vec{e}_{n'}) \text{ basis of } V \\ \dim = n + n' \quad (\vec{e}_{n+1}, \dots, \vec{e}_{n+n'}) \text{ basis of } V'$$

- Direct Product of two lin. v. spaces:

$$\rightarrow V \otimes V' \ni \vec{x} = \sum_{i=1..n} \sum_{j=1..n'} \alpha^{ij} \vec{e}_i \otimes \vec{e}_{n+j} \\ \dim = n \cdot n'.$$

- Direct sum of Representations: $D: V \rightarrow V$; $D': V' \rightarrow V'$

Def. 4: the represent. $D \oplus D'$ defined by:

$$\forall g \in G \rightarrow D \cdot (g) \oplus D' \cdot (g) \quad \text{such that:}$$

$$\begin{pmatrix} \vec{v} \\ \vdots \\ \vec{v}' \end{pmatrix} \rightarrow \begin{pmatrix} D(g) & & \\ & \ddots & \\ & & D'(g) \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vdots \\ \vec{v}' \end{pmatrix} = \begin{pmatrix} D(g)\vec{v} \\ \vdots \\ D'(g)\vec{v}' \end{pmatrix}$$

is called the direct sum of the represent D and D' of G in $V \oplus V'$.
It is obviously a represent!

- Tensorial product of representations:

Def 5: the represent. $D \otimes D'$ defined by:

$\forall g \in G \rightarrow D(g) \otimes D'(g)$ such that:

$$\vec{v} \otimes \vec{v}' \rightarrow D(g)\vec{v} \otimes D'(g)\vec{v}'$$

is called the tensorial - or Kronecker - product of the represent. D and D' of G in $V \otimes V'$.

Note: $D(g) = e^X \Rightarrow \vec{v} \otimes \vec{v}' \rightarrow (e^X \vec{v}) \otimes (e^X \vec{v}')$

$$D'(g) = e^{X'}$$

infinit⁺: $[(1+X)\vec{v}] \otimes [(1+X')\vec{v}']$

$X \otimes X'$ same generator
in \neq represent.

$$= \vec{v} \otimes \vec{v}' + (X \otimes 1)\vec{v} \otimes (0 \otimes X')\vec{v}' + \dots$$

so: $[D \otimes D'](g) = e^{(X \otimes 1 + 1 \otimes X')}$

More generally: Kronecker product:

Def 6: $G = H \times K$ direct product of the groups H and K .
 $D^{(H)}$ and $D^{(K)}$ represent. of H and K in V_H and V_K

$D^{(H)} \otimes D^{(K)} : (h, k) \in H \times K \rightarrow D^{(H)}(h) \otimes D^{(K)}(k)$.

Example: $G = SU(3) \times SU(2)$ on $\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$

$\begin{matrix} \uparrow & \downarrow \\ 3 \text{ dim. space} & 2 \text{ dim. space} \end{matrix} \quad \downarrow$
6 dimens. space $\left\{ \begin{matrix} u^\dagger d^\dagger s^\dagger \\ u^\dagger d^\dagger s^\dagger \end{matrix} \right.$

$$g \in SU(3) \times SU(2)$$

$$\{[D \otimes D'](g)\} (q \otimes s) = (e^{i\alpha^i \lambda_i} q) \otimes (e^{i\beta^j \sigma_j} s)$$

$\begin{matrix} \uparrow & \downarrow \\ \text{quark} & \text{spin} \end{matrix} \quad = e^{i(\alpha^i \lambda_i \otimes \mathbb{I} + \beta^j \sigma_j)} (q \otimes s)$

actually 6.dim. represent. space of $SU(6)$

which reduces / $SU(3) \times SU(2)$ as: $6 = (3, 2) + (\bar{1}, 2)$

- Reducibility of Representations:

Invariant subspace: Let D be a represent. of G in V .

$E \subset V$ is an invariant subspace of V under D if

$$[D(g)](E) \subset E \quad \forall g \in G$$

i.e.: $\begin{bmatrix} D^{(1)}(g) & | & D^{(11)}(g) \\ \hline - & + & - \\ 0 & | & D^{(1)}(g) \end{bmatrix} \begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix} \}_{\mathcal{E}} \Rightarrow$ the restriction $D^{(1)}$ of D to E
 $\begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix} \}_{\mathcal{F}} \Rightarrow$ and the restriction $D^{(11)}$ of D to a complementary subspace
of E in V are themselves repres.

Defn. D of G in V is irreducible if there is no invariant subspaces, except trivial ones. If not D is said reducible

(70)

Def 8: D of G in V is said completely reducible if for any invariant subspace, \exists a complementary subspace which is invariant.

Theorem: Any unitary represent. U of G in an Hilbert space λ is completely reducible.

Proof: Let E be a V subspace invariant by U unitary:

$$\forall x \in E, y \in E^\perp : (x, U(g)y) = 0 \quad \text{since}$$

$$\begin{aligned} \forall g \in G & \quad (U(g)^{-1}x, y) = (U(g^{-1})x, y) = 0 \\ & \quad \downarrow \\ & \quad U(g)E^\perp \subset E^\perp \quad \forall g \in G. \end{aligned}$$

Corollary: Any representation D of a compact group G (in particular finite) in a linear v. space V is irreducible completely reducible.

Ex.:

$$\left[\begin{array}{c|cc|c} D^{(1)} & 0 & 0 & | \\ \hline \hline 0 & D^{(2)} & 0 & | \\ 0 & 0 & D^{(3)} & | \\ \hline \hline & & & | \end{array} \right] \left(\begin{array}{c} E_1 \\ \vdots \\ E_2 \\ \vdots \\ E_3 \end{array} \right)$$

Theorem (Peter-Weyl): Any unitary irreducible represent. of a compact group is finite dimensional.

(T)

Theorem (Schur): Let D be an irreducible representation
of a group G in a complex lin. v. space V .
A linear applic.: $V \rightarrow V$ - Then:
 $[D(g), A] = 0 \quad \forall g \in G \Rightarrow A$ multiple of \mathbb{I}

Proof: λ an eigenvalue of A .

$$A_\lambda = A - \lambda \mathbb{I} \quad \Rightarrow [D(g), A_\lambda] = 0 \quad \forall g \in G.$$

now: $\text{Ker } A_\lambda = \{x \in V \mid A_\lambda x = 0\}$ is invariant as V subspace
under D , since: $x \in \text{Ker } A_\lambda : A_\lambda D(g) x = D(g) A_\lambda x = 0$.
 $\Rightarrow D(g)x \in \text{Ker } A_\lambda$.

But D irreducible $\Rightarrow \text{Ker } A_\lambda = \{0\}$ or V .

Since λ eigenvalue of $A \quad \exists v \in V \quad (A - \lambda \mathbb{I})v = 0 \Rightarrow \text{Ker } A \neq \{0\}$

The only possibility is then $A_\lambda = 0$
i.e.: $A = \lambda \mathbb{I}$.

Consequence:

① Any irreducible complex represent. of an Abelian group
G is unidimensional.

proof: $g_0 \in G \quad \forall g \in G : [D(g), D(g_0)] = 0 \Rightarrow D(g_0) = \lambda_{g_0} \in \mathbb{C}$

thus in order D to be irreducible, must be a 1 dim. \mathbb{C}

(2) Let D be an irreducible complex repres. of a Lie group G . Then all the fundamental invariants (Casimir) are multiple of identity (or even zero). (see just below)

Converse of Schur's theorem:

Let U be an unitary repres. of G in a complex v. space V - If for any operator A such that $[U(g), A] = 0 \forall g \in G$, A is λI , then U is irreducible.

(Remember: G compact \exists scalar product making D unitary)

- Fundamental Invariants - Casimir operators:

Let \mathfrak{g} be a Lie algebra with generators:

$$[X_\alpha, X_\beta] = i C_{\alpha\beta}^\gamma X_\gamma$$

Casimir has shown that the operator:

$$C = g^{\alpha\beta} X_\alpha X_\beta = X_\alpha X^\alpha$$

(With $g^{\alpha\beta}$ inverse of Killing form $g_{\alpha\beta} = C_{\alpha\mu}^\nu C_{\beta\nu}^\mu$)

is an invariant of \mathfrak{g} :

$$[C, X_\gamma] = 0.$$

C is a fundamental invariant since it cannot be written as a function of other invariants. C is called a Casimir.

Actually $\mathfrak{f} = r$ (= rank of \mathfrak{g}) fundamental invariants
in \mathfrak{g} semi-simple. (81)

Example : $\mathfrak{su}(2)$:

$$[J_i, J_j] = i \epsilon_{ijk} J_k \quad (i, j, k = 1, 2, 3)$$

Casimir operator:

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

$$= J_2^2 + \frac{1}{2} (J_+ J_- + J_- J_+)$$

$$[J^2, J_i] = 0.$$

$\mathfrak{su}(3)$:

$$[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$$

2nd d⁰:

$$C_{(2)} = \sum_{i=1}^8 \lambda_i^2$$

3rd d⁰:

$$C_{(3)} = d_{ijk} \lambda_i \lambda_j \lambda_k$$

with d_{ijk} completely symmetric in i, j, k

and defined by:

$$[\lambda_i, \lambda_j]_+ = \frac{4}{3} \delta_{ij} \mathbf{1} + 2 d_{ijk} \lambda_k$$

Note: non-zero structure const. f_{ijk} (completely antisym.).

$$\begin{cases} f_{123} = 1 \\ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ f_{458} = f_{678} = \sqrt{\frac{3}{2}} \end{cases}$$

non zero symmetric coeff's: d_{ijk} :

$$\begin{cases} d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3} \\ d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2 \\ d_{448} = d_{558} = d_{668} = d_{778} = -1/2\sqrt{3} \end{cases}$$

Remarks: - The fundamental weights are of degree:

2, 3, ..., n for $Sp(n)$.

2, 4, ..., 2l for $SO(2l)$ or $SO(2l+1)$

2 and 6 for G_2 .

⋮

- The eigenvalues of the l Casimirs on an unitary and irreducible representation D of G completely characterize this represent.

(II)

Classification of irreducible and finite dimens.

Representations of G (semi-) simple:

Def: Weight: In G , the Cartan H_i ($i=1, 2, \dots, r$) are Hermitian and commute. Let D be a represent. of G , one can choose a system of common eigenvectors for the H_i , which will provide a basis of the repres. space V . Let $|m\rangle$ be such a vector:

$$H_i |m\rangle = m_i |m\rangle$$

now H_i

is a $k \times k$ matrix

if $\dim V = k$

The r -dim. vector $\vec{m} = (m_1, \dots, m_r)$ associated to $|m\rangle$ is called the weight of $|m\rangle$

Example of Weights: adjoint represent.

$[H_i, E_\alpha] = \alpha_i E_\alpha \Rightarrow$ root diag. = weight diag. of the regular repres.

The knowledge of the weight diagram completely characterizes the represent, i.e. allows to obtain explicitly the matrices H_i and E_α .

Properties of the weights:

a) If $D: V \rightarrow V$ with $\dim V = N$, \exists at most N weights.

(Indeed eigenvectors corresponding to \neq weights are linearly indep^t, and \exists at most N indep^t vect. in V .)

b) $|m\rangle$ eigenvector
with weight \vec{m} } \Rightarrow $\left\{ \begin{array}{l} E_\alpha |m\rangle \text{ or } = 0 \\ \text{or eigenvector with} \\ \text{weight } \vec{m} + \vec{\alpha} \end{array} \right.$

$$\begin{aligned} \text{Indeed: } H_i E_\alpha |m\rangle &= (E_\alpha H_i + \alpha_i E_\alpha) |m\rangle \\ &= (m_i + \alpha_i) E_\alpha |m\rangle. \end{aligned}$$

c) $\begin{cases} \vec{m} \text{ weight} \\ + \vec{\alpha} \text{ root} \end{cases} \Rightarrow \begin{cases} 2 \frac{\vec{m} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \text{ integer} \\ \vec{m}' = \vec{m} - 2 \frac{\vec{m} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \vec{\alpha} \text{ is a weight} \end{cases}$

Weyl reflection with same multiplicity as \vec{m} .

Def: - The number v of eigenvectors $|m\rangle$ with the same weight \vec{m} is called the multiplicity of \vec{m} .

- A weight is said simple if $v = 1$.

- Two weights \vec{m} and \vec{m}' are said equivalent if they are related by Weyl reflection group.

- A weight \vec{m} is said higher than \vec{m}' if the vector $\vec{m} - \vec{m}'$ is positive, i.e. its first non-zero component is >0.

- The highest weight of a set of equivalent weights is said to be dominant.

Theorem: In any irreducible representation, there is an highest weight. This highest weight is simple.

Theorem: Two irreducible representations of G are equivalent iff they have the same highest weight.

Theorem (Cartan): For every simple algebra of rank l

there are l dominant weights called fundamental dominant weights $\vec{M}^{(i)} (i=1, \dots, l)$ such that any other

dominant weight \vec{M} is:

$$\vec{M} = \sum_{i=1}^l \lambda_i \vec{M}^{(i)} = \vec{M}(\lambda_1, \dots, \lambda_l)$$

where λ_i are non-negative integers.

- Furthermore, there exist l so-called fundamental irreducible repres. which have the l fundamental dominant weights as their highest weights.
- Finally to any dominant weight $\vec{M}(\lambda_1, \dots, \lambda_l)$ it corresponds one and only one (up to an equivalence) irreducible represent. with $\vec{M}(\lambda_1, \dots, \lambda_l)$ as its highest weight. This represent. D can be denoted $D(\lambda_1, \lambda_2, \dots, \lambda_l)$.

therefore: fundamental repres. $D^{(i)} = D(0, 0, \dots, \underset{i}{1}, 0, \dots, 0)$

trivial represent.: $D(0, 0, \dots, 0)$.

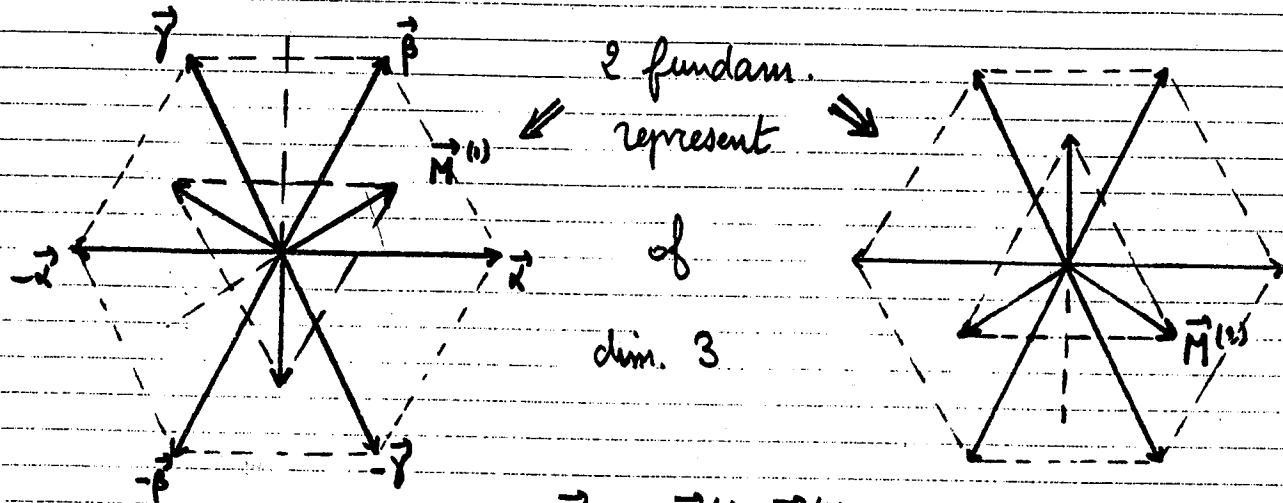
Determination of the fundamental weights:

Let $\vec{\alpha}_1, \dots, \vec{\alpha}_r$ the set of simple roots of G , one can characterize the $\vec{M}^{(i)}$ as follows:

$$2 \frac{\vec{M}^{(i)} \cdot \vec{\alpha}_j}{\vec{\alpha}_j \cdot \vec{\alpha}_j} = \delta_j^i.$$

example: A_2 :

simple roots: $\vec{\beta}$ and $-\vec{\beta}$



$D(1,0)$

$$\vec{M} = \vec{M}^{(1)} + \vec{M}^{(2)} = \vec{\alpha}$$

$D(0,1)$

with Weyl reflection: root diagram!

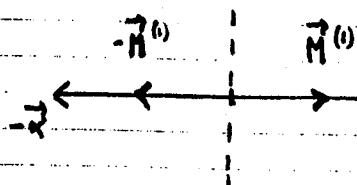
$D(1,1)$

A_1 :

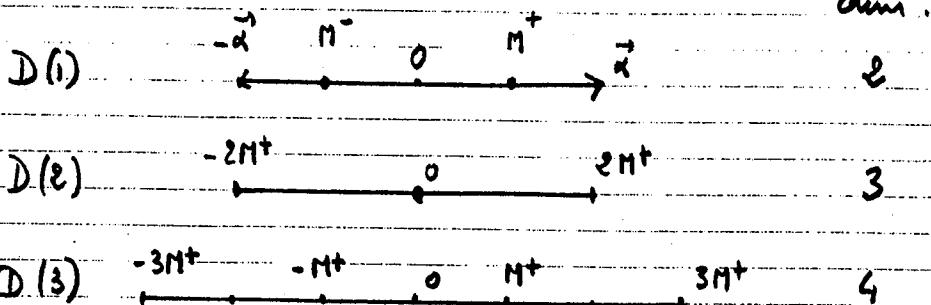
1 fundam. repres.

of dim. 2

$D(1)$



A₁ (again): $\dim D(\lambda) = \lambda + 1$.



using properties

p. 83

Usually: $\lambda = 2j$ j integer or half integer > 0 .

One denotes $D(\lambda)$ as $D^{(j)}$.

Remark: Characterization of irreducible repres. by highest weight
≡ eigenvalues of the 2 fundamental invariants

in A_1 :

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+)$$

Let $D^{(j)}$ be a represent., $|jjm\rangle$ eigenvector of J_z
corresponding to the weight m :

$$J_z |jjm\rangle = m |jjm\rangle$$

j is the highest weight: $J^+ |jjj\rangle = 0$.

$$J^2 \text{ multiple of } 0 \text{ in } D^{(j)} \Rightarrow J^2 |jjj\rangle = J_z(J_z + 1) |jjj\rangle = j(j+1) |jjj\rangle$$

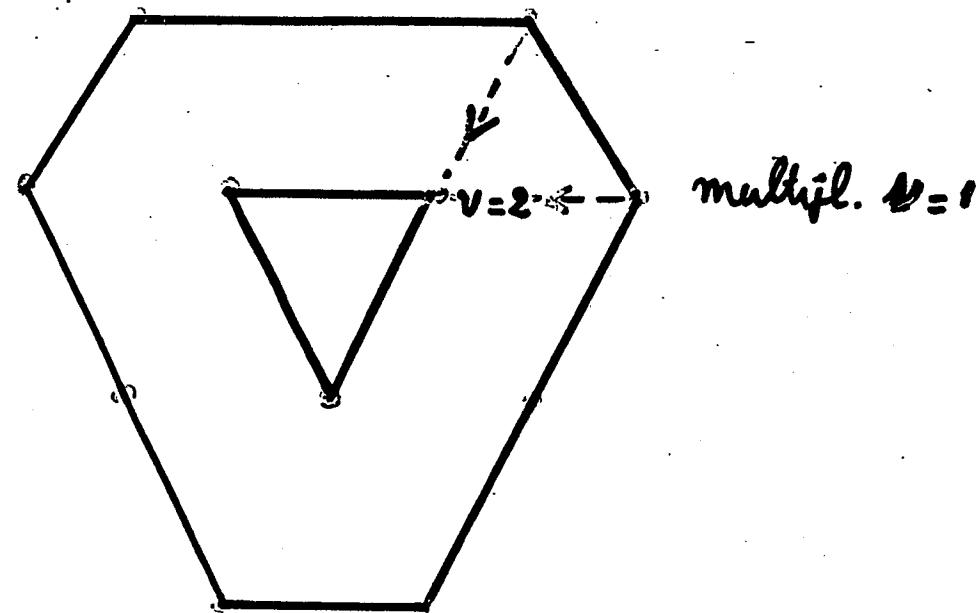
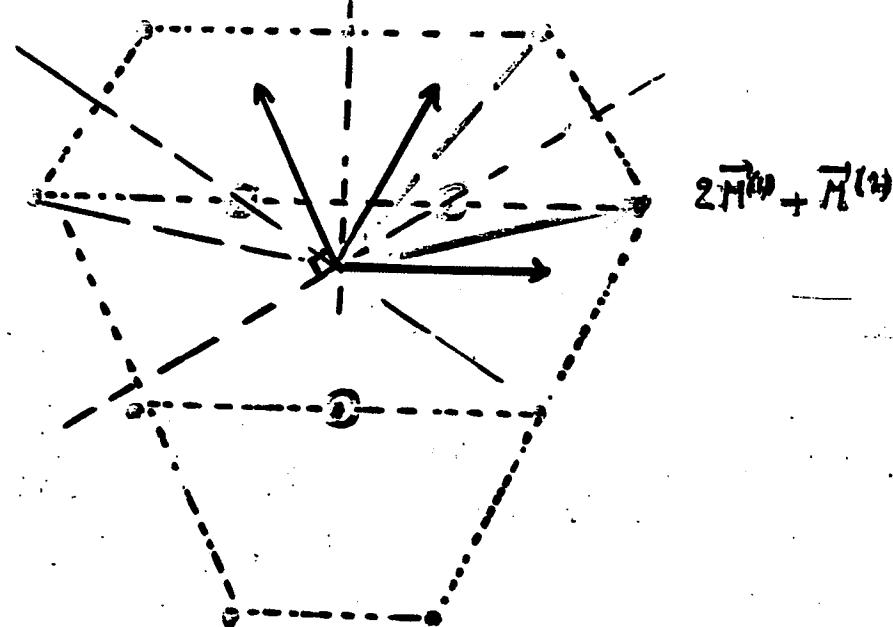
J^2 and J_z constitute a complete set of observables.

A₂ (again):

Same game: knowing the highest weight
 $\lambda_1 \tilde{M}^{(1)} + \lambda_2 \tilde{M}^{(2)}$ by Weyl reflections and
action of "ladder" operators $E_{\pm\alpha}$, one can
construct the representation.

Lec: Construction of $D(2,1)$:

(55)



For $SU(3)$, in general:

- boundary is (at most) a 6-sided figure, symmetric with respect to γ -axis.
- There is one state at each side of the boundary, two states at each side of next layer, three states at the sites on next layer, etc., until a triangular layer is reached: then multpl. ceases to increase.

$$\dim D(\lambda_1, \lambda_2) = \frac{1}{2} (\lambda_1 + 1) (\lambda_2 + 1) (\lambda_1 + \lambda_2 + 2)$$

Complete set of commuting observables for $SU(3)$:

for $D(\lambda_1, \lambda_2)$ eigenvalue of:

$$C_2 = \frac{1}{12} \left[(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) + 3(\lambda_1 + \lambda_2) \right]$$

$$C_3 = 16 (\lambda_1 - \lambda_2) \left[\frac{2}{9} (\lambda_1 + \lambda_2)^2 + \frac{1}{9} \lambda_1 \lambda_2 + \lambda_1 + \lambda_2 + 1 \right]$$

knowledge of $(\lambda_1, \lambda_2) \Leftrightarrow$ knowledge of C_2 & C_3 eigenvalues.

$SU(3) \supset SU(2) \times U(1)$ as sub-algebras.

$$\{\lambda_1, \lambda_2, \lambda_3\} \times \{\lambda_8\}$$

note that $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ commutes among them -
 λ_3 - selves and (of course)
 λ_8 with $C_{(2)}$ and $C_{(3)}$.

\Rightarrow These 5 commuting operators constitute a complete set of commuting observables: the knowledge of their eigenvalues fixes completely the \neq weights in $D(\lambda_1, \lambda_2)$!

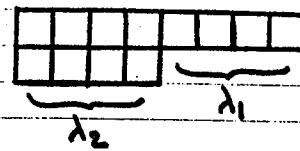
i.e. for any 5-plet of eigenvalues of these 5 operators in $D(\lambda_1, \lambda_2)$ corresponds one and only 1 point in the weight diagram.)

Generalisation: $SU(n) \supset SU(n-1) \times U(1) \supset \dots \supset \overbrace{U(1) \times \dots \times U(1)}^{n-1}$

$$\dim D(\lambda_1, \dots, \lambda_{n-1}) = \frac{(\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_{n-1} + 1)(\lambda_1 + \lambda_2 + 2) \dots (\lambda_{n-2} + \lambda_{n-1} + 2) \dots (\lambda_1 + \dots + \lambda_{n-1} + n-1)}{2! 3! \dots (n-1)!}$$

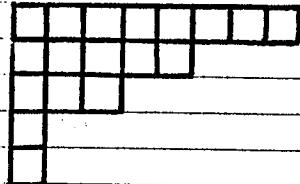
Representation of $D(\lambda_1, \dots, \lambda_p)$ by Young Tableaux (case of $SU(n)$):

$SU(3) \quad D(\lambda_1, \lambda_2) :$



$\lambda_1 + \lambda_2 = p$ boxes
 $\lambda_2 = q$ boxes

$SU(n) \quad D(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) :$



$m_{1,n}$ boxes
 $m_{2,n}$ "
 $m_{3,n}$ "
 \vdots
 $m_{n-1,n}$

$$\begin{aligned} \text{with : } m_{1,n} - m_{2,n} &= \lambda_1 & \text{and } m_{1,n} > m_{2,n} \\ m_{2,n} - m_{3,n} &= \lambda_2 & \geq \dots \geq m_{n-1,n} \\ \vdots \\ m_{n-2,n} - m_{n-1,n} &= \lambda_{n-2} \\ m_{n-1,n} &= \lambda_{n-1} \end{aligned}$$

Contragredient representation: $D \rightarrow \bar{D}$ such that

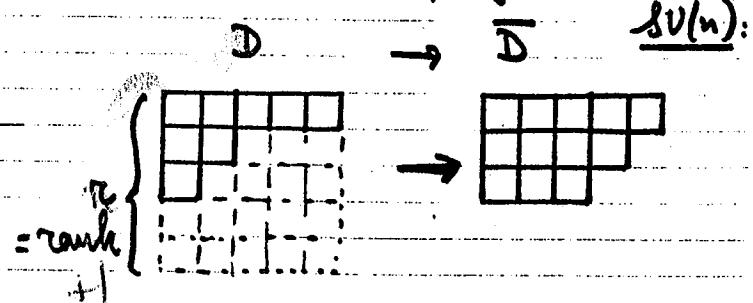
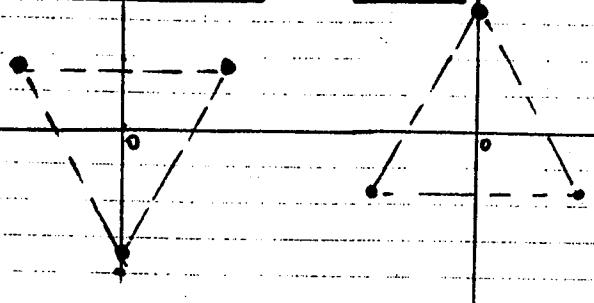
$$\forall g \in G: \bar{D}(g) = D(g')^t$$

$$\text{if } D \text{ unitary: } D = U: \quad \bar{U}(g) = U(g)^* \quad \bar{U}(g) = (U^g)^* = U_g^*$$

One can show that 2 contragredient representations D and \bar{D} have weight diagrams symmetric with respect to 0.

Ex: in $SU(3)$: $D(0,1)$ and $D(1,0)$:

As Young Tableaux in $SU(n)$:



Reduction of a representation into a sum of irreducible ones

(case of $Sp(2n)$):

2 cases : (a) $G : D$ irreducible repres. on V

$G \supset S$: with respect to $S \Rightarrow D$ reducible: $D = \sum_{i=1}^r \eta_i D^{[i]}$
 subgroups

(b) $G : D \text{ and } D'$ irreducible \Rightarrow

$$D \otimes D' = \sum_{i=1}^3 \eta_i D^{[i]}$$

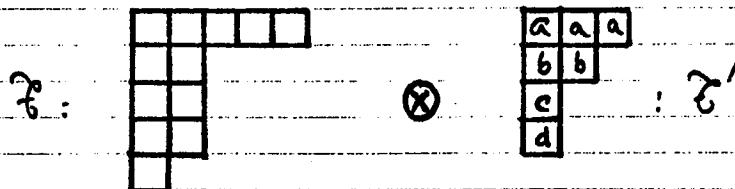
reducible

(b) Case: Let \mathfrak{T}_i and \mathfrak{T}' the Young tableaux associated with D and D' :

$$D \otimes D'$$

Method :

c) choose the simpler tableau, ex. \mathfrak{T}' : call a the boxes in the first line, b those of 2nd line, etc...



(ii) Add to \mathfrak{T} one box a of \mathfrak{T}' using all \neq ways such that one gets always a tableau. Then, add to this obtained tableau a second a with the prescription:

- 2 boxes a must not be in the same column

(iii) When all the a are used, add the b , then the c , etc.., in the same way, but such that:

Ex:



only one



two



Examples:

$$\square \otimes \square = \square \oplus \square$$

$$\square \otimes \square = \square \oplus \square$$

in $Sp(3)$ $3 \otimes 3 = 1 \oplus 8$

$$\square \otimes \square \otimes \square = \square \square \oplus 2 \square \oplus \square$$

in $SU(3)$ $3 \times 3 \times 3 = 10 \quad 8 \times 8 \quad 1$

(a) Case: Reduction of an irreducible $SU(n)$ representation
with respect to $SU(n-1), SU(n-2), \dots$

let $D(\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_i = m_{1,n} - m_{2,n}$

$$\lambda_{n-1} = m_{n-1,n}$$

with $m_{1,n}$ non negative integer

Consider:

$$\begin{matrix} m_{1,n} & m_{2,n} & m_{3,n} & \dots & m_{n-2,n} & m_{n-1,n} & 0 \\ \geq & \geq & \geq & \geq & \geq & \geq & \\ m_{1,n-1} & m_{2,n-1} & & & m_{n-2,n} & m_{n-1,n} & \end{matrix}$$

- two boxes with the same label must not be in the same column. (91)
- denoting $n_i(a)$ the number of a in the i^{th} first columns starting from the right, $n'_i(a)$ the number of a in the i^{th} first lines starting from the top, one must have (using identical definitions for b, c, d, \dots):

	n_i	\leftarrow
	$b b$	$a a$
c		
d		

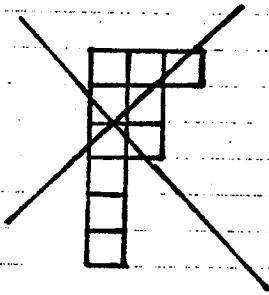
$$n_i(a) \geq n_i(b) \geq n_i(c) \geq \dots$$

$$n'_i(a) \geq n'_i(b) \geq n'_i(c) \geq \dots$$

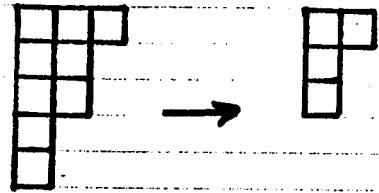
v) For $G = SU(n)$, any tableau with more than n lines will be suppressed.

And any tableau with n lines will be replaced by the corresponding tableau in which the columns with n boxes are suppressed.

ex: $SU(5)$:



and



v) The same "dummy" (without letters a, b, \dots) tableau may appear several times. Suppose it appears twice:

- if the distribution of a, b, \dots in the tableaux is the same, then one of the tableaux must be suppressed.
- if the distribution of a, b, \dots is different, then the irred. repres. associated with the dummy tabl. appears twice.

and more generally the pyramid:

$SU(n)$

$$m_{1n} \geq m_{2n} \geq m_{3n} \geq \dots \geq m_{n-2,n} \geq m_{n-1,n} = 0$$

$SU(n-1)$

$$\geq m_{1,n-1} \geq m_{2,n-1} \geq m_{3,n-1}$$

$SO(n-2)$

$$m_{1,n-2} \geq m_{2,n-2}$$

$$m_{n-3,n-2} \geq m_{n-2,n-2}$$

$$m_{r,i}$$

$$m_{i,i}$$

$$m_{12} \geq m_{22}$$

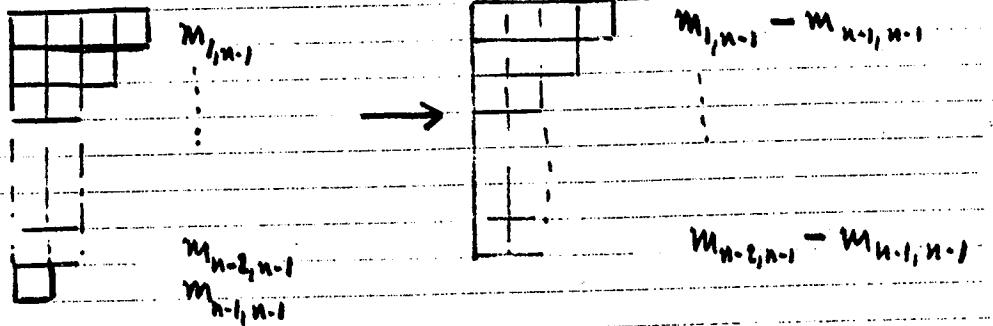
$$m_{11}$$

$U(1)$

Rule: The \neq possible choices of $m_{1,n-1}, \dots, m_{n-1,n-1}$ will give the \neq Young tableaux associated with the $SU(n-1)$ represent.

included in the $SU(n)$ represent. ($m_{1n}, \dots, m_{n-1,n}$), and so on
for $SO(n-2) \dots$ up to $U(1)$.

Rule: in $SU(n-1)$



Example: octet of $SU(3)$: $D(1,1)$:



2 1 0 $SU(3)$

2 1

2 0

1 1

1 0

$$8 = 2 + 3 + 1 + 2$$



$SU(3)$

$SU(2)$