PART I

(0)

Tentative Plan

- Introduction.

- Elements of Group Theory.
  (defin. & first properties).

- Lie groups. Lie algebras.

- Classification of semi-simple Lie algebras.
  Killing form. Cartan classif.
  Dynkin diagrams....

- Representations of simple Lie algebras.
  Weight diagrams.
  Represent. of $O(n)$ and $SU(n)$.
  Product of representations: Young tableaux.
  Clebsch-Gordan

- (Octonions and Exceptional groups)

Rk: - Applications : mainly in Gauge theory and GUT
and Spectroscopy of Elem. Pa

WHAT IS A GROUP ?

Def.: A group G is a set of elements together with a composition law $\cdot$ such that:

i) $\forall\, x, y \in G \quad x \cdot y \in G$ (internal law)

ii) $\forall\, x, y, z \in G \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativit

iii) $\exists\, e \in G$ s.t. $\forall\, x \in G : \; x \cdot e = x$ (e = identity

iv) $\forall\, x \in G \;\; \exists\, x^{-1} \in G$ s.t. : $x \cdot x^{-1} = e$ (x^{-1}: inverse

Examples: . $Z = \{n\}$ n integer ; R (with $+$) ; $R - \{0\}$ with .

. P_n : permutation group of n objects.

. Rotations in the plane around O.

Actually, in Physics, groups are never considered abstractly but by <u>an action</u> on some set S. (transform. group

Def.: Let G be a group, S a set. An <u>action</u> of G on S is an application : $G \times S \longrightarrow S$

such that: $\forall\, (g, s) \longrightarrow g(s)$ with the properties:

i) $g(g'(s)) = (g \cdot g')(s) \quad \forall\, g, g' \in G, \; \forall\, s \in S$

ii) $e(s) = s$

The group needs to be "represented".

Def.: A linear <u>representation</u> of a group G in a vector space V (on R or C) is an homomorphism D of G on the group o the linear and reversible operators on V:

$$\forall\, g \in G \longrightarrow \underset{\text{matrix}}{D(g)} \quad \text{s.t.} \quad D(g) \cdot D(g') = D(gg') \quad \forall\, g, g' \in G$$

Nature is full of symmetries
and symmetry breakdowns

SYMMETRY = Regularity and Harmony
(SUN METRON)
= with measure

Types of Symm. Groups in Physics
- with finite nber of el^{ts} (crystal. groups)
- with ∞ nber of el^{ts}:
  - discrete (1-1 correspondence with Z)
  - continuous (Lie groups "local" gauge groups; Superg...

"Geometrical Inv^{ce}"
- SPACE Groups [Rotations in R^3; Euclid. group; Cry... gro...
- SPACE·TIME " [Poincaré group; Galilean group ..]

"Dynamical Inv^{ce}"
- INV^{ce} Group of Hamiltonian ...
- SU(3) of colour and Confin^t.

"Gauge Groups"
- UNIFICATION Groups.
(Grand Unif.., SuperUnif...)

"Internal Sym"
.. CLASSIF. Groups (SU(3), SU(6).... in Elem. Part).

Sym. Breaking
- EXPLICIT ($H = H_0 + H_1$)
- SPONTANEOUS (Higgs)

Mechanics.

. Classical Mech.: $\mathcal{L}(q^\alpha, \dot{q}^\alpha, t)$ q^α: configuration of the system at t

Noether theorem: If $\mathcal{L}$ is invariant under a continuous group of transform. of dim. n, then there are n conserved quantities.

Inv^{ce} Group:	Conserved Quantity:
time translations	Energy H
space translat.	Impulsion $\vec{p}$
rotations around O	Kinetic mom^t /O : $\vec{L} = \vec{r} \times$

From $\mathcal{L} \longrightarrow H = \sum_\alpha p_\alpha \cdot \dot{q}^\alpha - \mathcal{L}$

Poisson braket: $\{f, g\} = \frac{\partial f}{\partial q^\alpha} \cdot \frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q^\alpha} \cdot \frac{\partial f}{\partial p_\alpha}$

Lie algebra structure.

Equ. of motion: $\begin{cases} \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} \\ \dot{q}^\alpha = -\frac{\partial H}{\partial p_\alpha} \end{cases} \Rightarrow \begin{aligned} \frac{dp_\alpha}{dt} &= -\{H, p_\alpha\} \\ \frac{dq^\alpha}{dt} &= -\{H, q^\alpha\} \end{aligned}$

If $\frac{d f(q^\alpha, p_\alpha)}{dt} = 0 \Rightarrow \{H, f\} = 0$ C^t of Motion

Example: $H = \frac{p^2}{2m} - \frac{1}{r}$ time transl. inv^{ce}: H

rotations in space: $\vec{L} = \vec{r} \times \vec{p}$

Runge-Lenz vector: $\vec{A} = \left(\vec{L} \times \vec{p} + \frac{\vec{r}}{r}\right) \frac{1}{\sqrt{2H}}$

So(4) group $\begin{cases} \{\vec{A}, \vec{A}\} = \varepsilon \vec{L} \\ \{\vec{L}, \vec{A}\} = \vec{A} \\ \{\vec{L}, \vec{L}\} = \vec{L} \end{cases}$

$\{H, \vec{A}\} = 0$

$\{H, \vec{L}\} = 0$

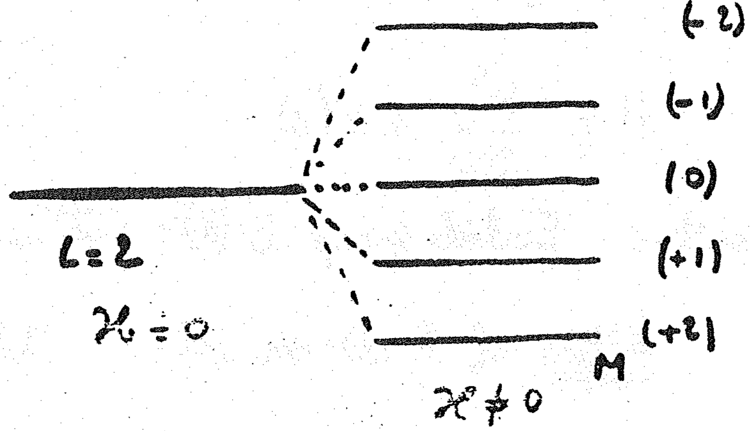
Breaking of symm: $H = H_0 + H_1$

inv under L_z only (O(

then: $$E_{nLM} = E^{0}_{nLM} - \frac{e}{2m}\mathcal{H} M .$$

$-L \leq M \leq +L$

ZEEMAN EFFECT



<u>Therefore</u> : From the knowledge of the irreduc. repr. of the invar^ce group G of H , one can have:

- a classification of eigenstates of H.
- indications on the d° of degeneracy of energy levels.
- transformation law of eigenstates under

• Quantum Mech. :

H Hamiltonian of a quantum system described by an Hilbert space $\mathcal{H}$. The set of unitary (and antiunitary) operators which commute with H is a group.

Example: $H = \frac{p^2}{2m} + V(x)$

Consider Euclid. group $E(3)$: $\vec{x} \to R\vec{x} + \vec{a}$

Action of $E(3)$ on $\mathcal{H}$: $U(\vec{a},R)\ \psi(x) = \psi\left(R^{-1}(\vec{x}-\vec{a})\right)$

on position : $U(\vec{a},R)^{-1}\,\vec{q}\,U(\vec{a},R) = R\vec{q} + \vec{a}$

impulsion: $U(a,R)^{-1}\,\vec{p}\,U(\vec{a},R) = R\vec{p}$

So: $U(\vec{a},R)^{-1} H\, U(\vec{a},R) = \frac{p^2}{2m} + V(R\vec{x}+\vec{a})$.

Therefore if $V(R\vec{x}+\vec{a}) = V(x) \Rightarrow [H, U(\vec{a},R)] = 0$.

symmetry group

⇕

csts of motion.

Take Schrödinger Equa. of an atom of Z electrons in a constant magnetic field //Oz

$$H = \underbrace{\sum_{i=1}^{Z}\left(\frac{p_i^2}{2m} - \frac{Ze^2}{r_i}\right) + \sum_{i<j}\frac{e^2}{|\vec{r}_i-\vec{r}_j|}}_{H_0 \text{ rotation invar}^t} - \underbrace{\frac{e}{2m}\mathcal{H}_1 L_z}_{H_1}$$ ← EXPLICIT BREAKIN OF SYMM

$[H_0, \vec{L}] = 0 \qquad \vec{L} = \sum_{i=1}^{Z} \vec{r}_i \times \vec{p}_i$

H_0 commuting with L_x, L_y, L_z operators, which generate the Lie algebra of the rotation group, the eigenstates of H_0 in a same irreduc. repres of $O(3)$ will have the same energy

irred repr of $O(3)$: $2L+1$ states $\begin{cases} L^2|LM\rangle = L(L+1)\,|LM\rangle \\ L_z\,|LM\rangle = M\,|LM\rangle \end{cases}$ $-L\le M\le L$

Wigner: Elementary particle $\Longleftrightarrow$ Irreducible representation of $\mathcal{P}$.

$M^{ij} = \varepsilon^{ijk} J_k$

$M^{0i} = K_i$

characterized by 2 quantities: <u>mass</u> m & <u>spin</u> j.

$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$

$W^2 = W_\mu W^\mu$
$P^2 = P_\mu P^\mu$
commute with lie alg. of $\mathcal{P}$: Invariants.

$$W^2 \,|\, p\, j\, j_3 \rangle = -m^2 j(j+1) \,|\, p\, j\, j_3 \rangle$$
$$P^2 \,|\, p\, j\, j_3 \rangle = m^2 \,|\, p\, j\, j_3 \rangle.$$

<u>Galilei Group</u> Non relativistic limit of $\mathcal{P}$ $(c \to \infty)$

Rk: boost in (x,t) plane:
$$x' \longrightarrow x + vt$$
$$t \to t$$

$$G : \begin{cases} \vec{x} \longrightarrow R\vec{x} + \vec{v}t + \vec{a} \\ t \longrightarrow t + b. \end{cases} \qquad (c \to \infty)$$

<u>Supersymmetry : Super Poincaré</u>:

add new generators which turn bosons into fermions and conversely.

$\Rightarrow$ to gather fermions and bosons in a same SUPER-multiplet.

Poincaré Groups.

Laws of nature identical in all reference frame of inertia.

c = velocity of light = c^t $\Rightarrow$ in any refer. frame:

$(\vec{x}_1, t_1)$
$(\vec{x}_2, t_2)$ events $\qquad c^2(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2 = \text{const}^t$.

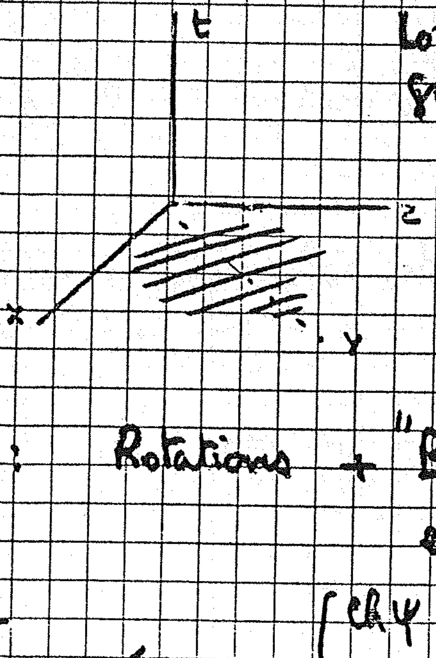
metric in Minkowsky space:

$$(x_0, \vec{\vec{x}}) \qquad |\vec{\vec{x}}|^2 = \overbrace{x_0^2 - (x_1^2}^{\text{complex rotations}} + \underbrace{x_2^2 + x_3^2)}_{\text{ordinary rotations}}$$

$\mathcal{P}$: Poincaré group : $\vec{\vec{x}} \to L\vec{\vec{x}} + \vec{\vec{a}}$

Lorentz group + Translations

6 + 4 = 10 parameters group.



$\mathcal{L}$: Lorentz : Rotations + "Boosts" : group $\cong SO(3,1)$

ex.: in the plane (x, t)

$$x' = \frac{x + Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \longleftarrow \quad \begin{bmatrix} \mathrm{ch}\,\psi & \mathrm{sh}\,\psi \\ \mathrm{sh}\,\psi & \mathrm{ch}\,\psi \end{bmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

$$t' = \frac{t + \frac{V}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Any element of $\mathcal{P}$ $= e^{\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}} \, e^{\vec{\gamma}\vec{P} + \delta P_0} \longrightarrow$ Lie algebra of $\mathcal{P}$:

$\vec{J}$ rotations $\qquad [J_i, J_j] = \varepsilon_{ijk} J_k$

$\vec{K}$ boost $\qquad [J_i, K_j] = \varepsilon_{ijk} K_k$

P_i : space translat. $\qquad [K_i, K_j] = -\varepsilon_{ijk} J_k$

P_0 : time translat. $\qquad [K_i, P_j] = \delta_{ij} P^0 \quad [K_i, P^0] = P_i$

$$[M_{..}, M_{..}] = ..M_{..}$$ Lorentz

$$[M_{..}, P_{.}] = ..P_{.}$$ Translations

$$[P_{.}, P_{.}] = 0$$

anti-commutat. $\{Q_\alpha, \bar{Q}_\beta\}_+ = -2(\gamma^\mu)_{\alpha\beta} P_\mu$ Fermionic Generator

Q_α : Majorana spinor $[M^{\mu\nu}, Q_\alpha] = i\,\sigma^{\mu\nu}_{\alpha\beta} Q_\beta$ γ^μ real (Majorana representation)

$$[P_\mu, Q_\alpha] = 0$$

<u>Rk</u>: $P_\mu P^\mu = m^2 = C^{te}$ invariant.

<u>but</u>: $W_\mu W^\mu$ is <u>no more</u> an invariant.

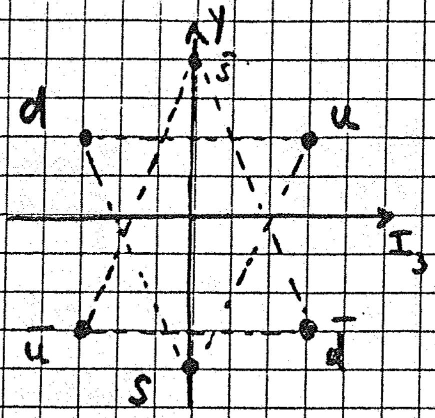
$m = 0 \longrightarrow$ supermultiplet $(s, s+1/2)$

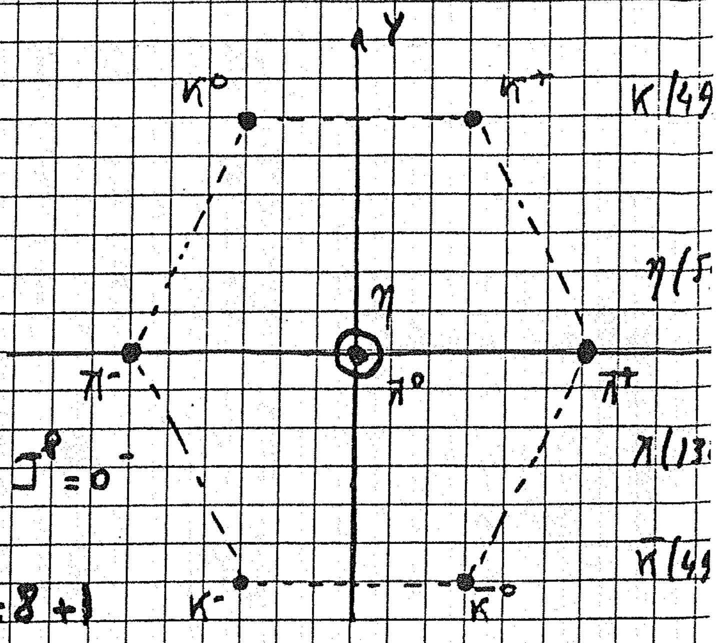
$m \neq 0$ " $(s-1/2, s, s, s+1/2)$.

Gravity + Supersymmetry $\longrightarrow$ Super Gravity.
(local Poincaré)

Unitary Internal Symmetries

- Quarks:





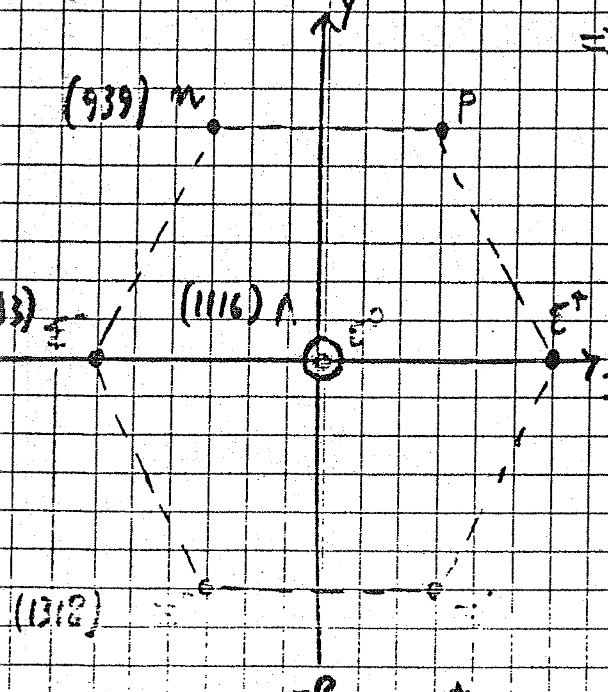
$\underline{3}$ of $SU(3)$ $\Rightarrow$ $3 \times \bar{3} = 8 + 1$

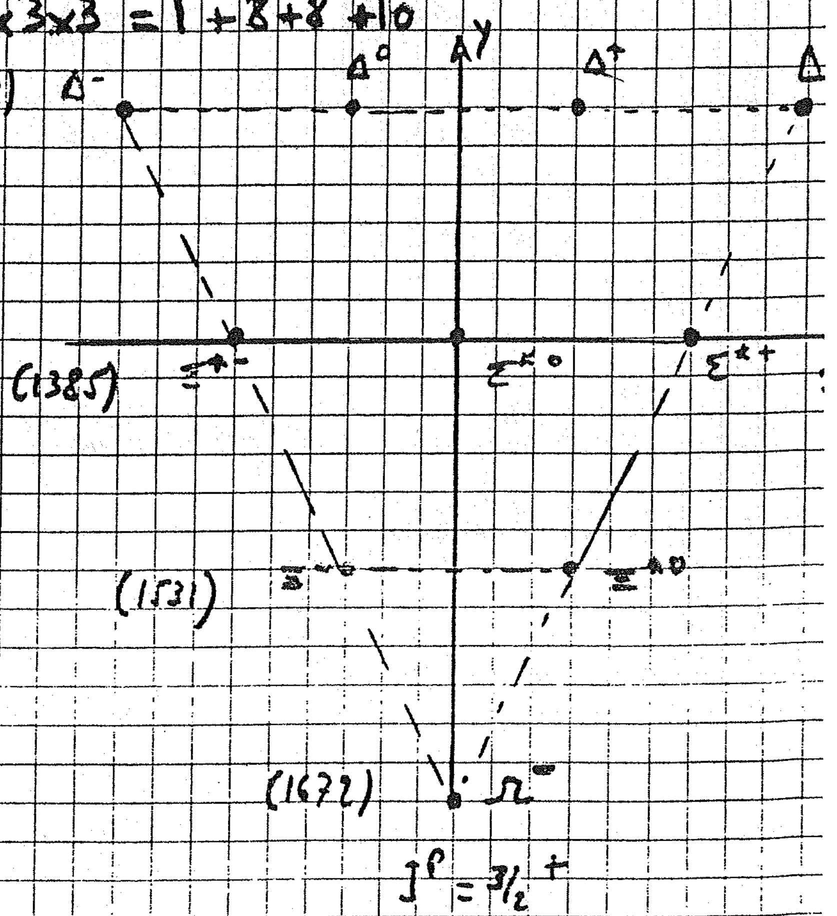
mesons octet: $8 = 3 + 1 + 2 +$

$SU(3) \supset SU(2) \times U(1)$

baryons

$\Rightarrow 3 \times 3 \times 3 = 1 + 8 + 8 + 10$





- Selection Rules
- Gell-Mann – Okubo mass formula $M = M_0 + M_8$

$= 8^{\text{th}}$ compon of an octet –

Generalizations :

- $\underline{\underline{SU(6)}} \supset SU(3)_{fl.} \times SU(2)_{spin}$ Non Relat.!

$6 = (3,2)$ $\quad \begin{pmatrix} u \\ d \\ s \end{pmatrix} \times \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$ i.e $\quad 6 = \begin{pmatrix} u\uparrow \\ d\uparrow \\ s\uparrow \\ u\downarrow \\ d\downarrow \\ s\downarrow \end{pmatrix}$

Classif. of hadrons:

$$6 \times \bar{6} = 1 + 35$$

$$6 \times 6 \times 6 = 56 + 70 + 70 + 20$$

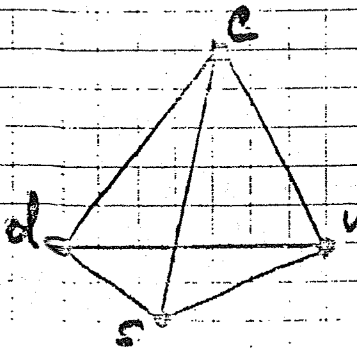
For high spin $SU(6) \times O(3)_L \leftarrow$ orbit. ang. mom!

Pauli principle for baryons: $56, L=0$; $70, L=1$.

- Charm Quark c (and b... and t...

GIM mechanism: add a $q^{\frac{2}{3}}q$ to avoid neutral current with $\Delta S \neq 0$.

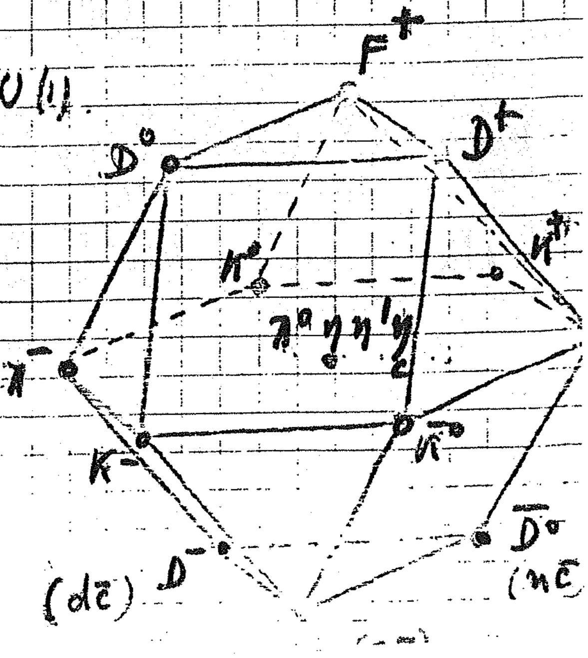
Weak $SU(2)_L \times U(1)$: $\underset{q}{\begin{pmatrix} u \\ d \end{pmatrix}_L}$ $\underset{\ell}{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L}$ $\underset{q}{\begin{pmatrix} \ \\ s \end{pmatrix}_L}$ $\underset{\ell}{\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L}$ doublet only.



$SU(4) \supset SU(3) \times U(1)$.

$4 \times \bar{4} = 1 + 15$

$15 = 8 + 3 + \bar{3} + 1$



GUTS

Unification of $\begin{cases}\text{Strong} \\ \text{Electro Magnetic} \\ \text{Weak}\end{cases}$ Interactions by Gauge Theory.

$$G \supset SU(3)_c \times SU(2) \times U(1).$$

Candidates: $SU(5)$ $SO(10)$ $E(6)$ ...

Let us consider $\underline{SU(5)}$: Each family of fermions in: $\bar{5} \oplus 10$. $\begin{cases}(u, d, \nu_e, e \\ (c, s, \nu_\mu, \mu \\ (\cdot, b, \nu_\tau, \tau\end{cases}$

$$\left(\begin{array}{c|c} SU(3)_c & 0 \\ \hline 0 & SU(2) \end{array}\right) \qquad \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ e^- \\ \nu_e \end{pmatrix}_L \qquad \qquad \begin{pmatrix} 0 & \bar{u}_3 & -\bar{u}_2 & u_1 & d_1 \\ \cdot & 0 & \bar{u}_1 & u_2 & d_2 \\ \cdot & \cdot & 0 & u_3 & d_3 \\ \cdot & (-) & \cdot & 0 & e^+ \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}_L$$

$$5 = (3,1) + (1,2) \qquad\qquad 10 = (\bar{3},1) + (3,2) + (1,1).$$

unific with good q.nb

Recall in G.W.S: $SU(2) \times U(1)$.

$$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \quad e^-_R \quad \begin{pmatrix} u \\ d \end{pmatrix}_L \quad u_R \quad d_R$$

Rk: $(\bar{5} + 10)_L \longrightarrow (5 + \overline{10})_R$

No room for $(\nu_e)_R$: take $SO(10) \supset SU(5) \supset \ldots$

$$16 = 10 + \bar{5} + 1_{\text{for } (\nu_e)_R}$$

Gauge Bosons:

$$\mathcal{L}_{eff} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{\Psi}_{L/R}\, \gamma_\mu \left(\partial^\mu - ig\, \bar{T}_i \bar{V}_i^\mu\right) \Psi$$

fermions — generators of $SU(5)$ — gauge bosons

Spontaneous Breakdown (Higgs fld.).

Ex: $SU(2) \times U(1)$ W.S.

$\{T_1\, T_2\, T_3\} \times \{Y\}$

$\downarrow$ $W^\pm, Z$

$U(1)$

$\{T_3 + \frac{1}{2} Y\}$ $\gamma_{//}$

Ex: $SU(5)$

$\downarrow$ $X, Y, \bar{X}, \bar{Y}$

$SU(3) \times SU(2) \times U(1)$.

$\downarrow$ $W^\pm, Z$

$SU(3) \times U(1)$.

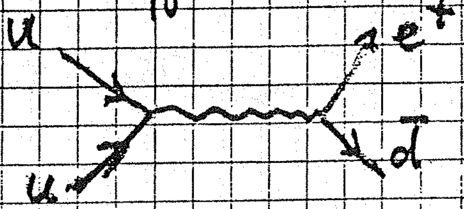
gluons + γ

$$24 = \underset{\text{gluons}}{(8,1)} + \underbrace{(1,3) + (1,1)}_{\substack{W^\pm, Z, \gamma \\ \text{mixing}}} + \underset{(\bar{X},\bar{Y})}{(\bar{3},2)} + \underset{(X,Y)}{(3,2)}$$

charges $\pm\frac{4}{3}, \pm$

q & ℓ in same multiplet $\Rightarrow$ in $\mathcal{L}_{eff}$ terms such that:

PROTON DECAY:



The Higgs problem:

ϕ being scalar fields in a Representat. of G $(= SU(5))$.

$G \xrightarrow[S.S.B.]{} S$

Write a 4^{th} order polynom. in ϕ, invar^t under G s.t. its "smallest" minimum is invariant under (and only under) a chosen subgroup S.

in partic: $\frac{\partial V}{\partial \phi} (\phi_0) = 0$ $\quad S = \{g \in G \mid g(\phi_0) = \phi_0\}$

Mass relations for fermions: terms $\bar{\Psi}_L\, \phi\, \Psi_R$ in $\mathcal{L}$.

Group theoretical pbs with GUTS: (13)

- Choose a group $G \supset SU(3)\times SU(2)\times U(1)$

with:

— for each family of q & l:

. a repres. R of G (if possible irreducible) without Triangular anomaly

. R complex (part and antipart in complex conjugate repres.).

— solve the Higgs pb.

— write a Higgs potential V (invariants) with represent. of G containing a vector ϕ_0 stabilized by $S \subset G$ ($S = SU(3)\times SU(2)\times U(1)$ or $SU(3)\times U(1)$), and such that the minim. of V is along ϕ_0.

— the repres. for q and l and for Higgs must give good mass relations for fermions: $\bar{L}\,\phi\, R$.

- Try to solve the family pb:

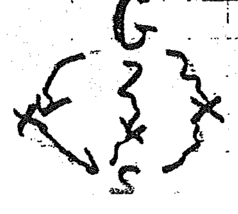
$$G \supset SU(3)\times SU(2)\times U(1)\times Fa$$

- replace the Higgs pb. by Technicolor.

Remarks: EXPLICIT realization of Represent. of G.

Knowledge of the subgroups of G, and the $\neq$ choices of breakings.

(1)



(2) it can exist several S which are not conjugate.

Spontaneous Symmetry Breaking:

$\mathcal{L}$ invariant under a continuous group of transform. G

If only one ground state (vacuum) inv^t under G: no

If several ground states which transform into each other under G, and if one of them is singled out for some physical reason as the physical ground state of the system : the relevant theory is Spontan. Broken.

Importance: In a gauge theory, when local invariance is spontan. broken, we get massive gauge bosons.

Example:
$$\mathcal{L} = \underbrace{\frac{\partial \varphi^*}{\partial x_\mu}\frac{\partial \varphi}{\partial x^\mu}}_{T} - \underbrace{\left(m^2 \varphi^*\varphi + h(\varphi^*\varphi)^2\right)}_{V}$$

$\varphi = \varphi_1 + i\varphi_2$ real scalar fields

m^2, h cst

$h > 0$

$\mathcal{L}$ is $U(1)$ invar^t.

$$\varphi(x) \to e^{i\Lambda}\varphi(x)$$

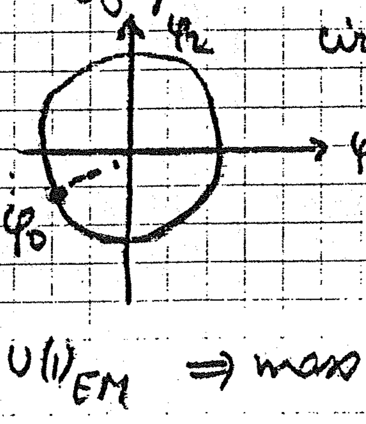
Λ real, cst

ground states $\Rightarrow$ minimizing the potential: $\frac{\partial V}{\partial \varphi} = -2\mu^2|\varphi| + 4h|\varphi|^3 = 0$

- if $m^2 > 0$ $\quad V \geq 0 \quad$ min (V) : $|\varphi| = 0$
- if $m^2 < 0$ $\quad m^2 = -\mu^2 \quad \Rightarrow |\varphi| = \frac{\mu^2}{h} = v^2$

∞ of ground states on the circle $|\varphi| = v$

choose one ground state to be the good one.



Symm. Spont. Broken $\Leftarrow$ it is not $U(1)$ inv^t.

Rk: In $SU(2) \times U(1)$ G.W.S. : breaking $\to U(1)_{EM}$ $\Rightarrow$ mass to $W^\pm, Z$

ELEMENTS OF GROUP THEORY.

(I) Defin.:

Def 1: A group G is a set of elem^ts together with a composition law $\circ$ such that:

(1) $\forall x, y \in G \quad x \circ y \in G$ — internal law

(2) $\forall x, y, z \in G \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$ — associat.

(3) $\exists e \in G$ s.t. $\forall x \in G: \ x \cdot e = x$ — identity

(4) $\forall x \in G \quad \exists x^{-1} \in G$ s.t.: $x \cdot x^{-1} = e$ — inverse

A group is said finite if it contains a finite nber of elem^ts (this nber is the order of the group).

A group is Abelian or Commutative if $\forall x, y \in G: x \cdot y = y \cdot x$

Theor 1: Let G be a group:

(1) $\forall x, x^{-1} \in G: \quad x^{-1} \cdot x = e$

(2) $\forall x \in G \quad : \quad e \cdot x = x$

(3) e is unique; $\forall x \in G \quad x^{-1}$ is unique.

Def 2: A subgroup of a group G is a (non empty) part $H \subset G$ which is a group with the conjugation law induced by G.
H is a proper subgroup if $H \neq G$ and $H \neq \{e\}$.

Theor2: $H \subset G$ is a subgroup of G iff $\forall x, y \in H: x \cdot y^{-1} \in H$
(H non empty)

Property: Let G be a group, H_i, $i \in J$, a family of G subgroups. Then $\bigcap_{i \in I} H_i$ is a ~~grab~~-group.

Def: Let $A \subset G$, G group. The subgroup generated by A is the smallest subgroup of G which contains A.

$$\Big(= \bigcap_{\substack{K \text{ subgroup of } G \\ K \supset A}} K \Big).$$

Example: G a group.

$$C(G) = \{ c \in G \mid c.g = g.c \ \ \forall g \in G \}$$
$$= \text{center of } G \quad \{ \text{Abelian subgroup} \}$$

(II) Cosets of a group with respect to a subgroup; Conjugation; Invariant subgroup:

a) H subgroup of G

$\exists$ equivalence relation: $\quad x \sim y \iff x^{-1}.y \in H$

i.e.: $\quad x \sim y \iff \exists\, h \in H$ s.t. $y = xh$

Def: left coset (mod. H) of x:

$$\dot{x} = \{ y \mid x \sim y \} = \{ xh \mid h \in H \}$$
$$\equiv xH$$

Rk: $x \sim y \Rightarrow \dot{x} = \dot{y}$

The set of left cosets mod H, denoted G/H is a partition of G.

By the same way, set of right cosets, mod H:

another equivalence relation : $\quad H\backslash G$

$$x \approx y \iff x.y^{-1} \in H$$

$$\dot{x} = \{ y \mid x \approx y \} = \{ hx \mid h \in H \}$$
$$\equiv Hx$$

<u>Question</u> : When do the partitions constituted by G/H and $H\backslash G$ coincide ?

one must then have: $xH = Hx \quad \forall x \in G$

<u>Def</u>: A subgroup H of G is said normal (or invariant) if $\forall x \in G$: $\quad x H x^{-1} = H$.

Then G/H has (canonically) a structure of group
G/H is called the <u>Quotient group</u> of G by H.

$$(xH).(yH) = x.y H$$

identity : $eH = H$

inverse : $(xH)^{-1} = x^{-1}H$.

Canonical homomorphism: $\pi : G \longrightarrow G/H$
$\qquad x \longrightarrow xH$.

$\forall x, y \in G \quad \pi(x).\pi(y) = \pi(xy)$.

<u>Example</u> : Euclidean group in 3 dim. space: = rotations + translat.

$$\vec{x} \xrightarrow[(\vec{a}, R)]{} R\vec{x} + \vec{a}$$

$$(\vec{a}', R')(\vec{a}, R) = (\vec{a}' + R'\vec{a}, RR')$$

$H = \{ (\vec{a}, \mathbb{1}) \}$ is an invariant subgroup -

Def: G is a <u>simple</u> group if $G \not\supset$ proper invariant subgroups.

G is a <u>semi-simple</u> group if $G \not\supset$ proper abelian invariant subgroups.

b) <u>Conjugation</u> (classes):

<u>**Def**</u>: In a group G, the applic.

$$g \in G : \quad ad_g : G \longrightarrow G$$
$$\qquad\qquad\qquad x \longrightarrow g x g^{-1}$$

are called conjugation (by g).

. Two elts. x & $y \in G$ are conjugate if $\exists\ g \in G$ with: $ad_g\, x = y$

or: $g x g^{-1} = y$

. Conjugation is an equivalence relation.

. Conjugation class of x: $\quad C_x = \{ g x g^{-1} \mid g \in G \}$.

<u>Examples</u>: . In $SO(3)$ acting on R^3: $\quad R R'(\vec{n}, \alpha) R^{-1} = R'(R\vec{n}, \alpha)$

. Permutation group of 3 elts: S_3:

3 classes:

$$3! = 6 \text{ elts} \begin{cases} (a) & e \\ (b) & (12) \quad (13) \quad (23) \\ (c) & (123) \quad (132). \end{cases}$$

in particular: $\quad (12) = (23)(13)(23)$.

<u>Conjugate subgroups</u>: H & K subgroups of G are conjugate if $\exists\, g \in G$ s.t.: $g H g^{-1} = K$.

<u>Rk:</u> $SO(3)$ & $SU(2)$ in $SU(3)$

III Group Morphisms

Homomorphism: of a group G in (on) a group G'

applic.: $\mu : G \to G'$

s.t.: $\mu(x).\mu(y) = \mu(x.y)\ \forall x,y \in G$

Isomorphism: bijective homomorphism.

Endomorphism: homomorphism of G in (on) itself.

ex.: conjugation by g_0: $g \to g_0 g g_0^{-1}$

one obtains: $g.g' \to g_0 g g' g_0^{-1}$

$= (g_0 g g_0^{-1})(g_0 g' g_0^{-1})$

Automorphism: bijective endomorphism.

Theor. 3: let $\mu : G \to G'$ a group homom. Then:

a) $\mu(e) = e'$; $\mu(x^{-1}) = \mu(x)^{-1}$

b) $\mathrm{Ker}\, \mu = \{ x \in G \mid \mu(x) = e' \}$ kernel of μ is a normal G subgroup.

c) H subgroup of $G \Rightarrow \mu(H)$ subgroup of G'.

d) $G/\mathrm{Ker}\,\mu$ is isomorphic to $\mu(G)$.

Automorphism group of G = $\boxed{\mathrm{Aut}\, G}$

group with internal law: composit. of applic.

identity: $e = \mathbb{1}$

inverse: μ^{-1} = inverse applic.

<u>Internal autom. group</u>: $\boxed{\text{Int } G}$ $\Big(\cong G/C(G)\Big)$ $\boxed{= \text{Ad } G}$
(Voir dessous plus bas)

with $\text{ad}_g : g' \to g g' g^{-1}$

these are particular examples of autom. of G.

<u>Rk</u>: External autom. of G = non internal autom. of G.
outer (generally not a group).

<u>Example</u>: Eucl. group $E(3) \ni (\vec{a}, R)$

"dilatation" $D_\alpha(\vec{b}, R) = (\alpha \vec{b}, R)$
is an outer autom.

<u>Theor 4</u>: The applic.: $\text{Ad} : g \to \text{Ad}g$ is an homom. $G \to \text{Aut } G$
$\text{Int}G$ is a normal subgroup of $\text{Aut}G$. ($\text{et} = \text{Ad } G$).

<u>Rk</u>: If G is simple, the quotient group $\dfrac{\text{Aut } G}{\text{Int } G}$ is finite.
(theorem)
(no proof now!).

<u>Property</u>: $\text{Ker}(\text{Ad}) = C(G)$ center of G.
d'où: $\text{Int } G = \text{Ad } G \simeq G/\text{Ker Ad} = G/C(G)$

<u>Example</u>: $E(3) \ni (\vec{a}, R)$

$C(E(3)) = \{(0, 1)\}$ trivial

(IV) Direct product; Semi-direct product of groups

a) <u>Def.</u>: A group G is the <u>direct product</u> of its subgroups H and K (we note: $G = H \times K$) if:

(1) $\forall h \in H$, $\forall k \in K$: $h.k = k.h$

(2) $\forall g \in G$ $\exists$ unique decomp.: $g = h.k$ with $h \in H$, $k \in K$

Def $\equiv$ $\Rightarrow$

<u>Theorem 5</u>: A group G is the <u>direct product</u> of its subgroups H and K iff:

(1) $G = H.K$

(2) $H \cap K = \{e\}$

(3) H & K normal.

<u>Construction</u>: Let A & B be two groups - Construct group G s.t. $\begin{cases} H \text{ \& } K \text{ subgroups of } G \\ H\ (K) \text{ isomorphic to } A\ (B) \\ G = H \times K. \end{cases}$

$\Downarrow$

isom. $\begin{cases} \mu : A \to H \\ \nu : B \to K \end{cases}$ G can be identified with the set $A \times B$ of couples (a,b) $a \in A$, $b \in B$ with the law:

$$(a,b)\ (a',b') = (aa', bb')$$

and μ and ν correspond to isomorphisms:

$$\mu : a \in A \to (a, e) \in H$$
$$\nu : b \in B \to (e', b) \in K.$$

<u>Examples</u>: • Translations : $\mathbb{Z}_x \times \mathbb{Z}_y$ in two dimens.

• $SU(2) \times SU(2)$ Wigner supermultiplet

$\left(SU(3) \times SU(2) \subset SU(6)_{fl.-spin} \right)$

• $SU(3)_c \times SU(2)_L \times U(1)_Y$ (gauge theory).

b.) <u>Def.</u>: A group G is the <u>semi-direct product</u> of a subgroup H by a subgroup K if:

(1) $G = H\,K$

(2) $H \cap K = \{e\}$

(3) H is normal.

Then $\forall\, g \in G$ $\exists$ unique decomp. : $g = h.k$ $\quad h \in H$, $k \in K$.

<u>Example</u>: $E(3)$; Poincaré group.

<u>Construction</u>:

Let A & B two groups. Construct group G

s.t. $\begin{cases} H \text{ \& } K \text{ subgroups of } G \\ H\ (K) \text{ isomorphic to } A\ (B) \\ G \text{ semi-direct product of } H \text{ by } K. \end{cases}$

$\mu : A \to H$

$\nu : B \to K$

$$\begin{tikzcd} A \ni a \arrow[r, "\mu"] \arrow[d, "(\psi_b)"'] & \mu(a) \in H \arrow[d, "(b)"] \\ A \ni \psi_b(a) \arrow[r, "\mu"'] & \nu(b)\,\mu(a)\,\nu(b)^{-1} \in H \end{tikzcd}$$

- $\forall g \in G \dashrightarrow (a,b)$ s.t.

$$g = \mu(a)\,\nu(b)$$

- H normal : $h \in H \xrightarrow[\text{of } H]{} \nu(b) \cdot h\, \nu(b)^{-1} \in H$

  auton of H

- ψ_b : autom. of A with : $\mu \circ \psi_b(a) = \nu(b)\,\mu(a)\,\nu(b^{-1})$

- $\psi : b \in B \to \psi_b \in \mathrm{Aut}\, A$ is an homom.

$$\mu \circ \psi_b \cdot \psi_{b'}(a) = \mu \circ \psi_{b.b'}(a)$$

- $gg' = \mu(a)\,\nu(b)\,\mu(a')\,\nu(b') = \mu(a)\,\underbrace{\nu(b)\,\mu(a')\,\nu(b)^{-1}}\,\nu(bb')$

$$\mu(a) \cdot \mu \circ \psi_b(a') \cdot \nu(bb')$$

$$= \mu(a \cdot \psi_b(a')) \cdot \nu(bb')$$

$\Rightarrow$ G can be identified with the set $A \times B$ of couples (a,b) with the group law : $(a,b)\,(a',b') = (a \cdot \psi_b(a'), bb')$

then : $\mu : a \to (a,e)$ $\Rightarrow$ Semi-Direct Product of A by B

$\nu : b \to (e',b)$ relative to aut. ψ : $\boxed{A \mathbin{\square}_{\psi} B}$

Example: Eucl. group $E(3)$:

$$(\vec{a}, R)\,(\vec{a}', R') = (\vec{a} + \underbrace{R\vec{a}'}_{\psi_R}, RR').$$

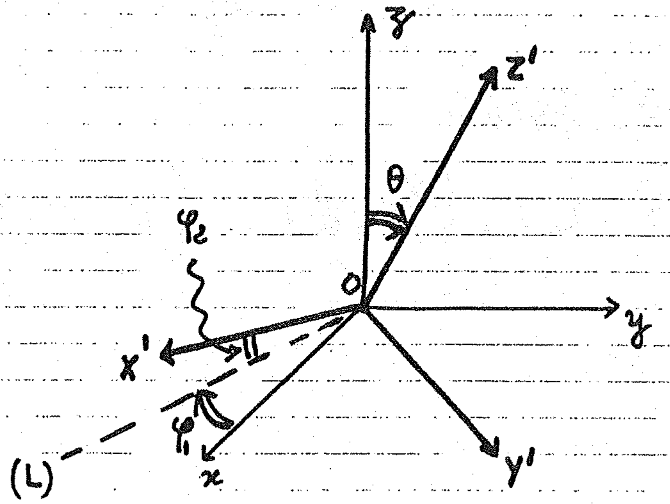
Poincaré group:

$$(\vec{a}, L).(\vec{a}', L') = (\vec{a} + L\vec{a}', L.L').$$

Lie Groups

① Defin.:

Example of Rotation group in 3 dim.:



3 parameters

$g : (Ox, Oy, Oz) \longrightarrow (Ox', Oy', Oz')$

(L) : intersection of the planes xOy, $x'Oy'$.

$$g(\varphi_1, \theta, \varphi_2) = \begin{bmatrix} \cos\varphi_1\cos\varphi_2 - \cos\theta\sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \cos\theta\sin\varphi_1\cos\varphi_2 & \sin\varphi_1\sin\theta \\ \sin\varphi_1\cos\varphi_2 + \cos\theta\cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\theta\cos\varphi_1\cos\varphi_2 & -\cos\varphi_1\sin\theta \\ \sin\varphi_2\sin\theta & \cos\varphi_2\sin\theta & \cos\theta \end{bmatrix}$$

$0 \leq \begin{matrix}\varphi_1 \\ \varphi_2\end{matrix} \leq 2\pi$

$0 \leq \theta \leq \pi$

$\Rightarrow$ The knowledge of $(\varphi_1, \theta, \varphi_2)$ completely determines the rotation.

<u>Rk</u>: • $\left[g(\varphi_1, \theta, \varphi_2)\right]^{-1} = g(\pi - \varphi_2, \theta, \pi - \varphi_1)$.

• $g(\varphi_1, \theta, \varphi_2) = \underbrace{g_{\varphi_2}}_{Oz'} \cdot \underbrace{g_\theta}_{(L)} \cdot \underbrace{g_{\varphi_1}}_{Oz}$

rotat. around : Oz' (L) Oz