

$B_s \rightarrow \mu^+ \mu^-$ Theory Status

Mikołaj Misiak

University of Warsaw

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B -meson or Kaon decays occur at low energies, at scales $\mu \ll M_W$.

We pass from the full theory of electroweak interactions to an **effective theory** by removing the high-energy degrees of freedom, i.e. integrating out the W -boson and all the other particles with $m \sim M_W$.

$$\mathcal{L}_{(\text{full EW} \times \text{QCD})} \longrightarrow \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED} \times \text{QCD}} \left(\begin{array}{c} \text{quarks } \neq t \\ \& \text{ leptons} \end{array} \right) + N \sum_n C_n(\mu) Q_n$$

Q_n – local interaction terms (operators), C_n – coupling constants (Wilson coefficients).

Operators (dim 6) that matter for $B_s \rightarrow \mu^+ \mu^-$ read:

$$Q_A = (\bar{b} \gamma^\alpha \gamma_5 s) (\bar{\mu} \gamma_\alpha \gamma_5 \mu) \quad \text{– the only relevant one in the SM at the LO in QED}$$

$$Q_{S(P)} = (\bar{b} \gamma_5 s) (\bar{\mu} (\gamma_5) \mu) = \frac{i(\bar{b} \gamma^\alpha \gamma_5 s) \partial_\alpha (\bar{\mu} (\gamma_5) \mu)}{m_b + m_s} + \underbrace{\boxed{E}}_{\text{vanishing by EOM}} + \underbrace{\boxed{T}}_{\text{total derivative}}$$

Necessary non-perturbative input: $\langle 0 | \bar{b} \gamma^\alpha \gamma_5 s | B_s(p) \rangle = ip^\alpha f_{B_s}$
decay constant

Such a matrix element vanishes for $(\bar{b} \gamma^\alpha s)$ and $(\bar{b} s)$ because B_s is a pseudoscalar.

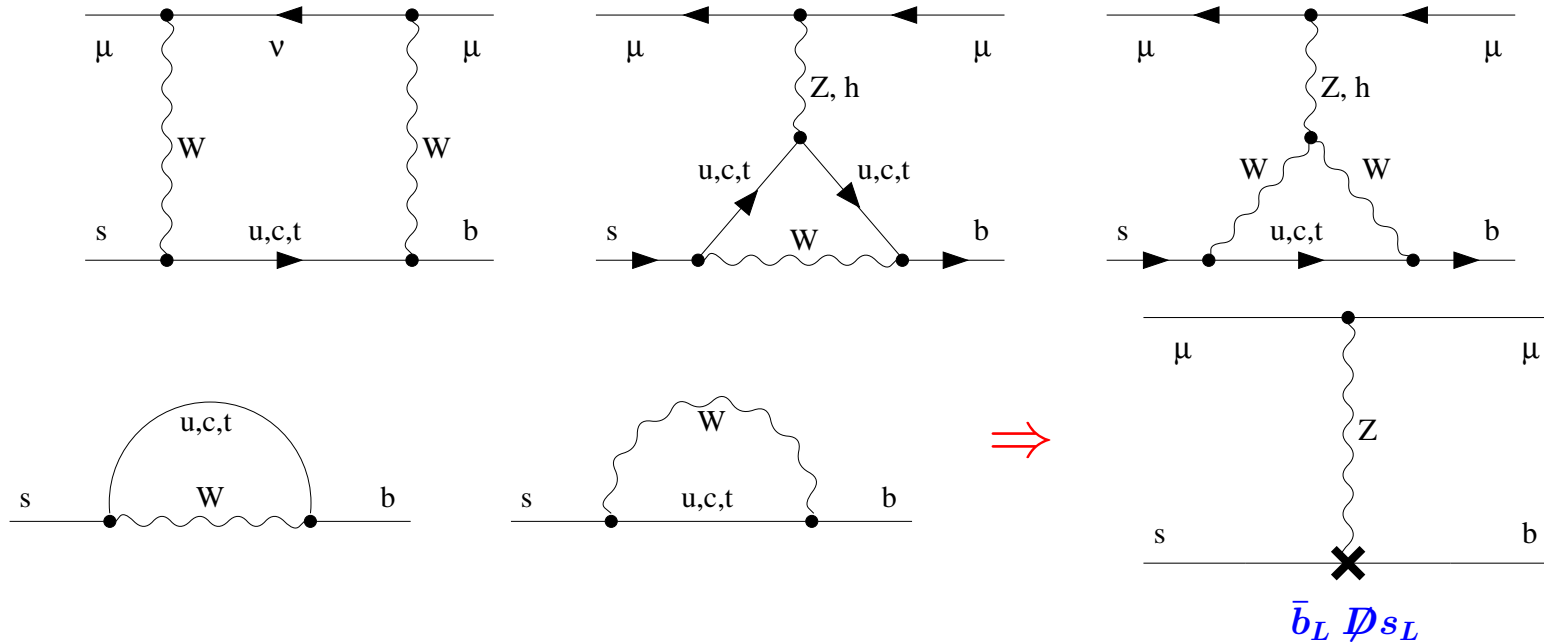
It also vanishes for $(\bar{b} \sigma^{\alpha\beta} s)$ because no antisymmetric tensor can be formed from p^α alone.

$$Q_V = (\bar{b} \gamma^\alpha s) (\bar{\mu} \gamma_\alpha \mu) \quad \text{gives no contribution at the LO in QED because}$$

$$p^\alpha (\bar{\mu} \gamma_\alpha \mu) = \bar{\mu} \not{p} \mu = \bar{\mu} (\not{p}_{\mu^+} + \not{p}_{\mu^-}) \mu = \bar{\mu} (-m_\mu + m_\mu) \mu = 0.$$

Q_S gets generated in the SM via the Higgs exchange, but... — see next page.

Evaluation of the LO Wilson coefficients in the SM:



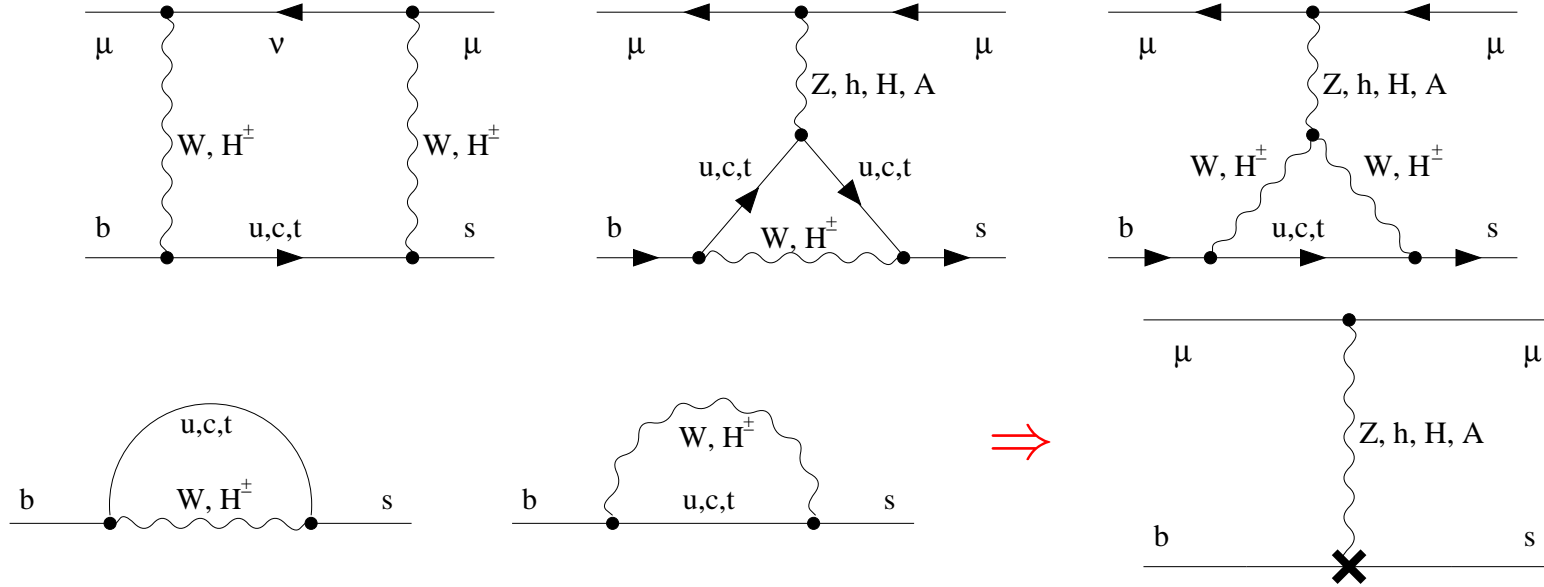
$$C_A^{(0)} = \frac{1}{2} Y_0 (m_t^2 / M_W^2), \quad Y_0(x) = \frac{3x^2}{8(x-1)^2} \ln x + \frac{x^2 - 4x}{8(x-1)},$$

$$C_S = \mathcal{O} \left(\frac{m_\mu}{M_W} \right), \quad C_P = 0.$$

Effects of C_S on the branching ratio are suppressed by $M_{B_s}^2 / M_W^2 \Rightarrow$ negligible.

Thus, only C_A matters in the SM.

Evaluation of the Wilson coefficients in the Two-Higgs-Doublet Model II



$$\tan \beta = v_2/v_1, \quad z = M_{H^\pm}^2/m_t^2,$$

$$C_S \simeq C_P \simeq \frac{m_\mu m_b \tan^2 \beta}{4M_W^2} \frac{\ln z}{z-1} > 0,$$

H.E. Logan and U. Nierste,
NPB 586 (2000) 39
($\mathcal{O}(\tan \beta)$ neglected)

$$\mathcal{B}(B_s \rightarrow \mu^+ \mu^-) \simeq (\text{const.}) \left[\left| \frac{2m_\mu}{M_{B_s}} C_A - C_P \right|^2 + |C_S|^2 \right]$$

$$C_A = \underbrace{C_A^{\text{SM}}}_{\text{positive}} + \underbrace{\Delta C_A}_{\text{small}} \Rightarrow \begin{cases} \text{suppression for moderate } C_{S,P} \\ \text{enhancement for huge } \tan \beta \text{ only} \end{cases}$$

Average time-integrated branching ratio:

$$\overline{\mathcal{B}}(B_s \rightarrow \mu^+ \mu^-) = \frac{|N|^2 M_{B_s}^3 f_{B_s}^2}{8\pi \Gamma_H^s} \beta \left(|r C_A - u C_P|^2 F_P + |u \beta C_S|^2 F_S \right) + \mathcal{O}(\alpha_{em}),$$

$$\text{where } N = \frac{V_{tb}^* V_{ts} G_F^2 M_W^2}{\pi^2}, \quad r = \frac{2m_\mu}{M_{B_s}}, \quad \beta = \sqrt{1-r^2}, \quad u = \frac{M_{B_s}}{m_b+m_s},$$

$$F_P = 1 - \frac{\Delta \Gamma_L^s}{\Gamma_L^s} \sin^2 \left[\frac{1}{2} \phi_s^{\text{NP}} + \arg(r C_A - u C_P) \right] \xrightarrow{\text{SM CP}} 1,$$

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In the limit of no CP-violation, mass eigenstates are CP eigenstates:

Heavier, CP-odd: $B_s^H = \frac{1}{\sqrt{2}}(B_s + \bar{B}_s)$, annihilated by $\bar{b}\gamma_5 s + \bar{s}\gamma_5 b$, ($\tau_H = 1.616(10)$ ps)

Lighter, CP-even: $B_s^L = \frac{1}{\sqrt{2}}(B_s - \bar{B}_s)$, annihilated by $\bar{b}\gamma_5 s - \bar{s}\gamma_5 b$, ($\tau_L = 1.519(4)$ ps)

Our interactions in this limit are all CP-even:

$$\left. \begin{aligned} Q_A + Q_A^\dagger &= [(\bar{b}\gamma^\alpha \gamma_5 s) + (\bar{s}\gamma^\alpha \gamma_5 b)] (\bar{\mu}\gamma_\alpha \gamma_5 \mu) \\ Q_P + Q_P^\dagger &= [(\bar{b}\gamma_5 s) + (\bar{s}\gamma_5 b)] (\bar{\mu}\gamma_5 \mu) \\ Q_S + Q_S^\dagger &= [(\bar{b}\gamma_5 s) - (\bar{s}\gamma_5 b)] (\bar{\mu}\mu) \end{aligned} \right\} \begin{aligned} &\text{annihilate } B_s^H, \text{ produce CP-odd dimuons} \\ &\text{annihilates } B_s^L, \text{ produces CP-even dimuons} \end{aligned}$$

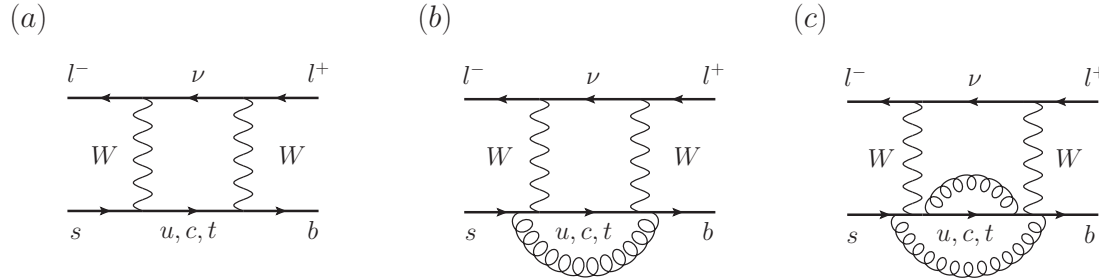
With SM-like CP-violation – still $Q_{A,P}$ annihilate B_s^H and Q_S annihilates B_s^L .

Beyond SM – interesting time-dependent observables, see arXiv:1303.3820, 1407.2771.

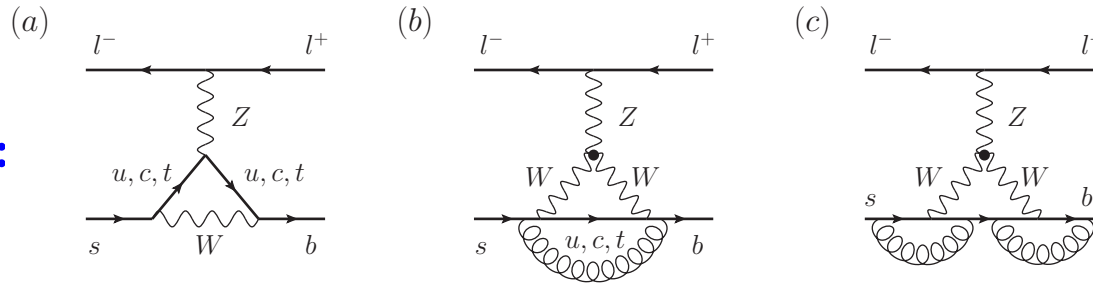
Evaluation of the NNLO QCD matching corrections in the SM

[T. Hermann, MM, M. Steinhauser, JHEP 1312 (2013) 097]

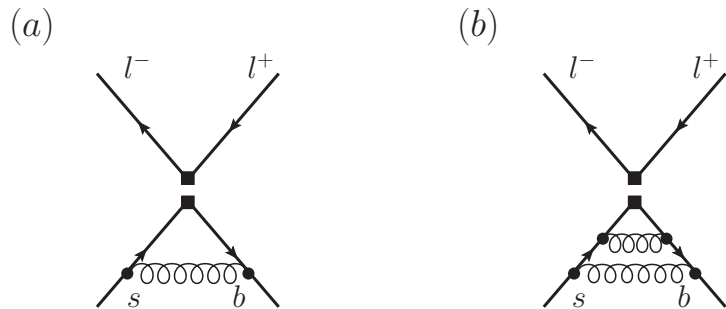
W-boxes:
(1LPI)



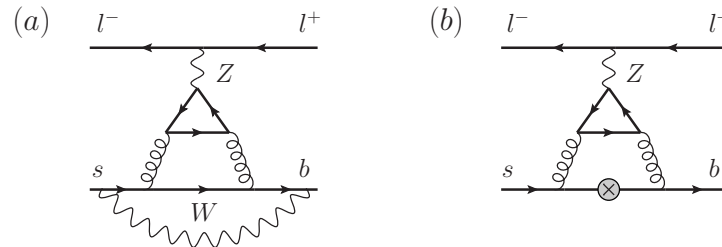
Z-penguins:
(1LPI)



Subtleties: (i) counterterms with finite parts $\sim \bar{b}_L \not{D} s_L$
(ii) evanescent operators: $E_B = (\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 s)(\bar{\mu}\gamma^\sigma\gamma^\rho\gamma^\nu\gamma_5\mu) - 4(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$
 $E_T = \text{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5)(\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma s)(\bar{\mu}\gamma_\alpha\gamma_5\mu) + 24(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$



Renormalization of E_B

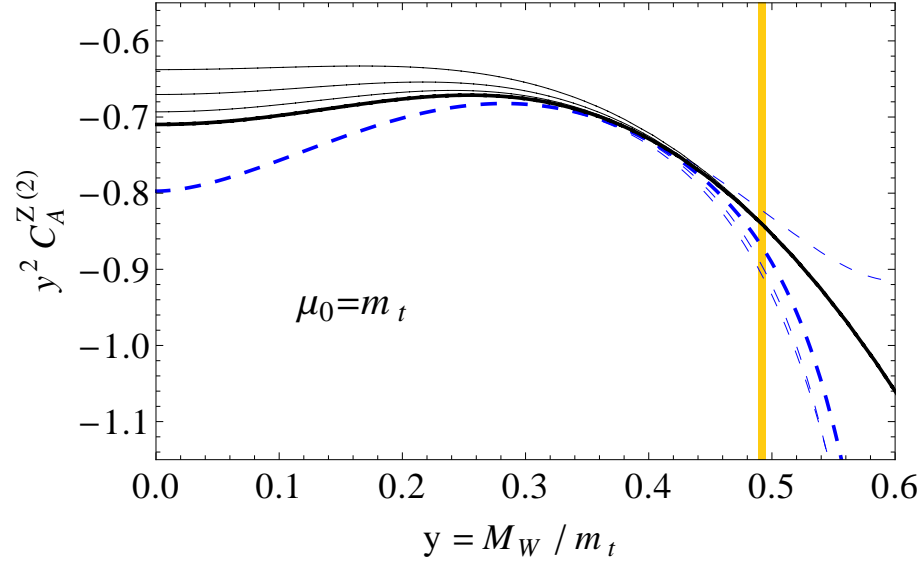


Diagrams generating E_T

Perturbative series for the Wilson coefficient at $\mu = \mu_0 \sim m_t, M_W$:

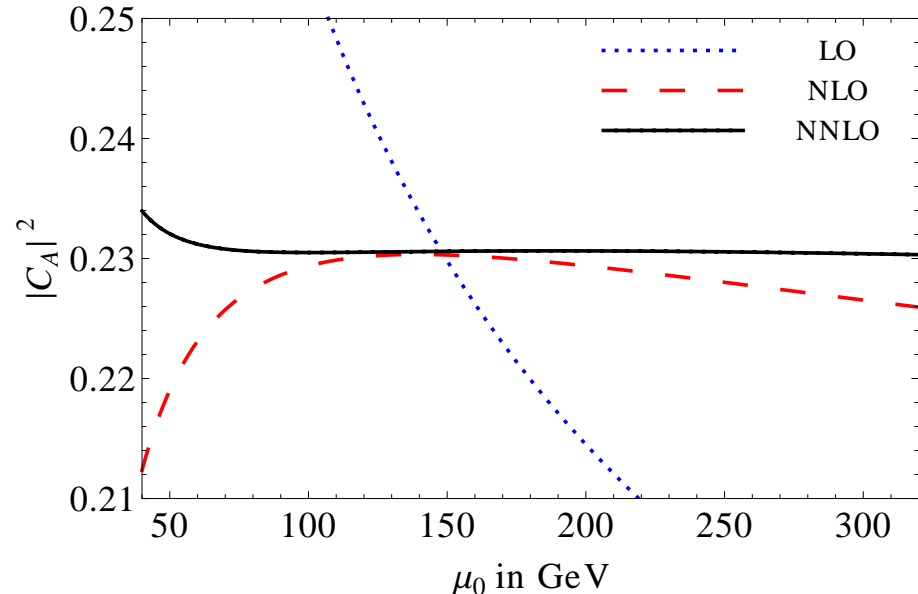
$$C_A(\mu_0) = C_A^{(0)}(\mu_0) + \frac{\alpha_s}{4\pi} C_A^{(1)}(\mu_0) + \left(\frac{\alpha_s}{4\pi}\right)^2 C_A^{(2)}(\mu_0) + \frac{\alpha_{em}}{4\pi} \Delta_{EW} C_A(\mu_0) + \dots$$

The top quark mass is $\overline{\text{MS}}$ -renormalized at μ_0 with respect to QCD, and on shell with respect to the EW interactions. Both α_s and α_{em} are $\overline{\text{MS}}$ -renormalized at μ_0 in the effective theory.



$$C_A^{(n)} = C_A^{W,(n)} + C_A^{Z,(n)}$$

To deal with single-scale tadpole integrals, we expand around $y = 1$ (solid lines) and around $y = 0$ (dashed lines), where $y = M_W/m_t$. The expansions reach $(1 - y^2)^{16}$ and y^{12} , respectively. The blue band indicates the physical region.



Matching scale dependence of $|C_A|^2$ gets significantly reduced. The plot corresponds to $\Delta_{EW} C_A(\mu_0) = 0$. However, with our conventions for m_t and the global normalization, μ_0 -dependence is due to QCD only.

NNLO fit (with $\Delta_{EW} C_A(\mu_0) = 0$):

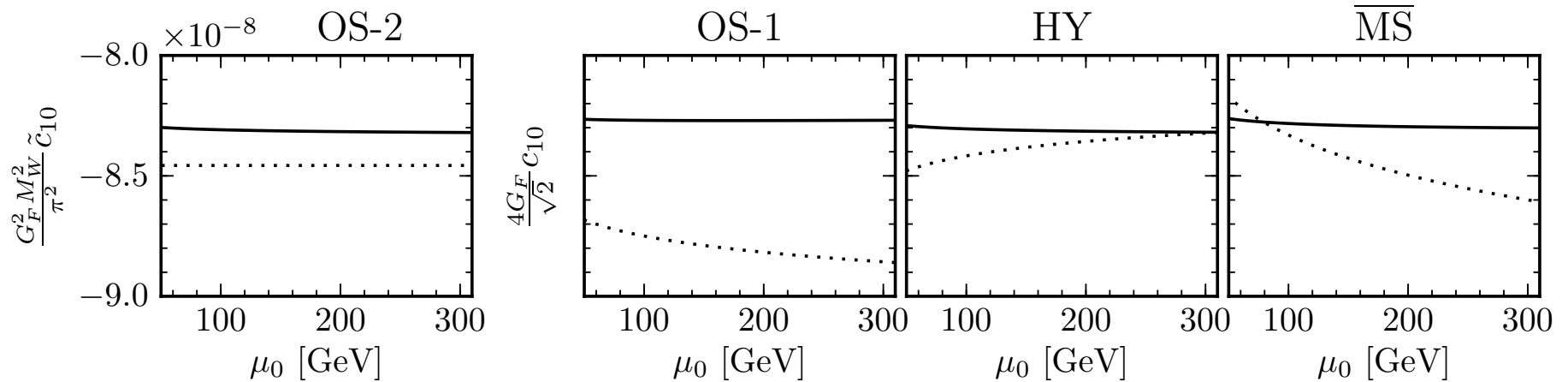
$$C_A = 0.4802 \left(\frac{M_t}{173.1}\right)^{1.52} \left(\frac{\alpha_s(M_Z)}{0.1184}\right)^{-0.09} + \mathcal{O}(\alpha_{em})$$

Evaluation of the NLO EW matching corrections in the SM

[C. Bobeth, M. Gorbahn, E. Stamou, Phys. Rev. D 89 (2014) 034023]

Method: similar to the NNLO QCD case. Two-loop integrals with three mass scales are present.

Dependence of the final result on μ_0 in various renormalization schemes (dotted – LO, solid – NLO):



In all the four plots: no QCD corrections to C_A included, $m_t(m_t)$ w.r.t. QCD used.

OS-2 scheme: Global normalization factor in \mathcal{L}_{eff} set to $N = V_{tb}^* V_{ts} G_F^2 M_W^2 / \pi^2$
 Masses at the LO renormalized on-shell w.r.t. EW interactions (including M_W in N)

Plotted quantity: $-2C_A G_F^2 M_W^2 / \pi^2$ in GeV^{-2}

NLO EW matching correction to the BR: -3.7%

other schemes: Global normalization factor in \mathcal{L}_{eff} set to $4V_{tb}^* V_{ts} G_F / \sqrt{2}$

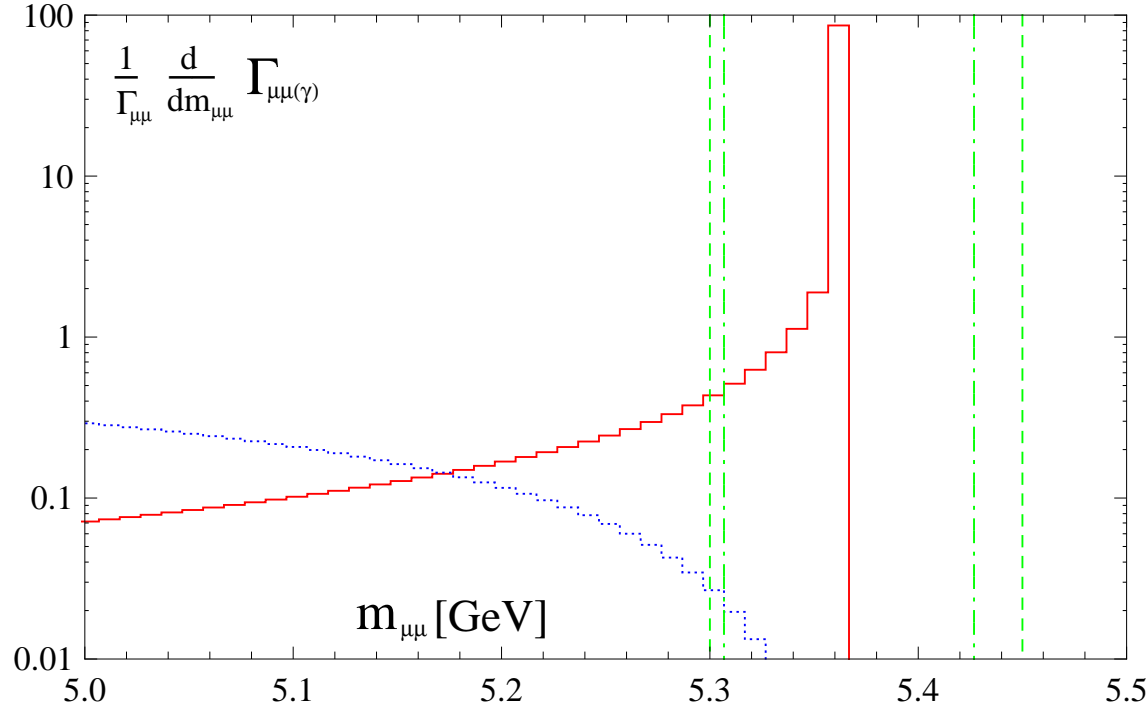
At the LO, $\alpha_{em}(\mu_0)$ used

$\overline{\text{MS}}$: Masses and $\sin^2 \theta_W$ renormalized at μ_0

OS-1: Masses as in OS-2, $\sin^2 \theta_W$ on-shell

HY (hybrid): Masses as in OS-2, $\sin^2 \theta_W$ as in $\overline{\text{MS}}$.

Radiative tail in the dimuon invariant mass spectrum



Green vertical lines – experimental “blinded” windows [CMS and LHCb, Nature 522 (2015) 68]

Red line – no real photon and/or radiation only from the muons. It vanishes when $m_\mu \rightarrow 0$.

[A.J. Buras, J. Girrbach, D. Guadagnoli, G. Isidori, Eur.Phys.J. C72 (2012) 2172]

[S. Jadach, B.F.L. Ward, Z. Was, Phys.Rev. D63 (2001) 113009], Eq. (204) as in PHOTOS

Blue line – remainder due to radiation from the quarks. IR-safe because B_s is neutral.

Phase-space suppressed but survives in the $m_\mu \rightarrow 0$ limit.

[Y.G. Aditya, K.J. Healey, A.A. Petrov, Phys.Rev. D87 (2013) 074028]

[D. Melikhov, N. Nikitin, Phys.Rev. D70 (2004) 114028]

Interference between the two contributions is negligible – suppressed both by phase-space and $m_\mu^2/M_{B_s}^2$.

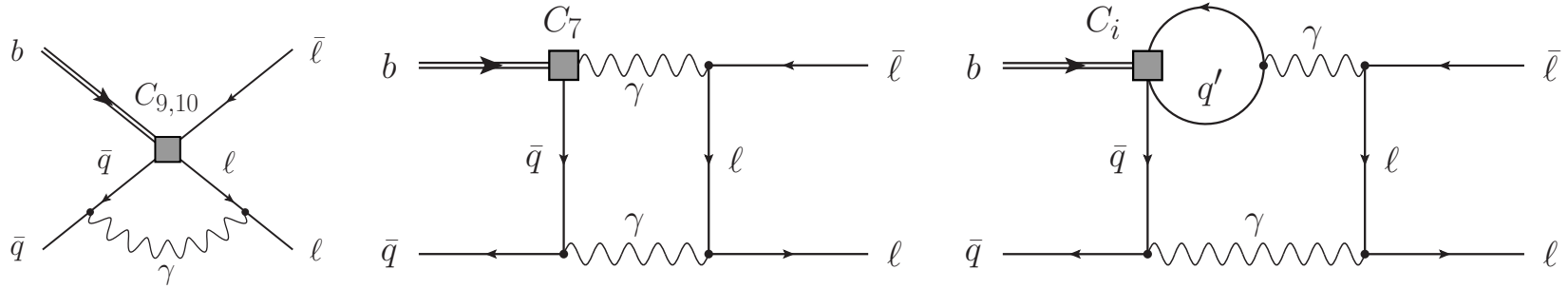
Enhanced QED effects in $B_q \rightarrow \ell^+ \ell^-$

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some of the QED corrections receive suppression by $\frac{m_\ell^2}{\Lambda M_{B_q}}$ only:



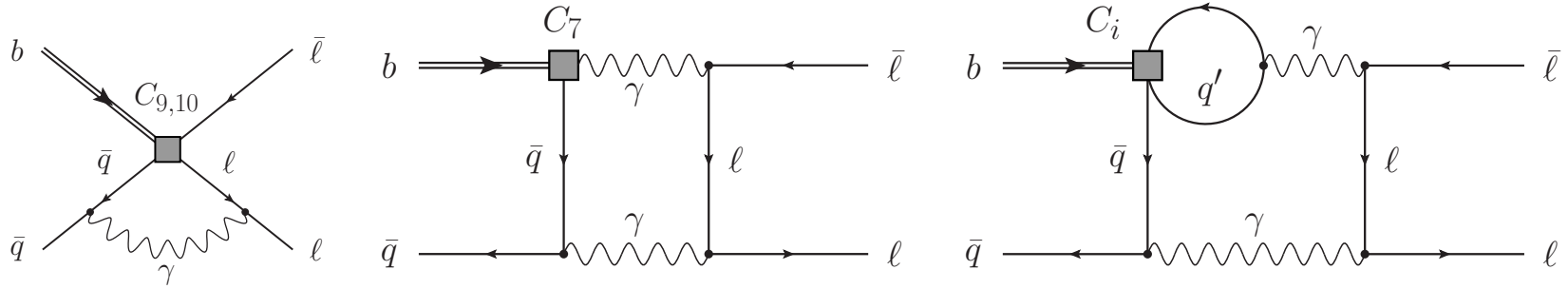
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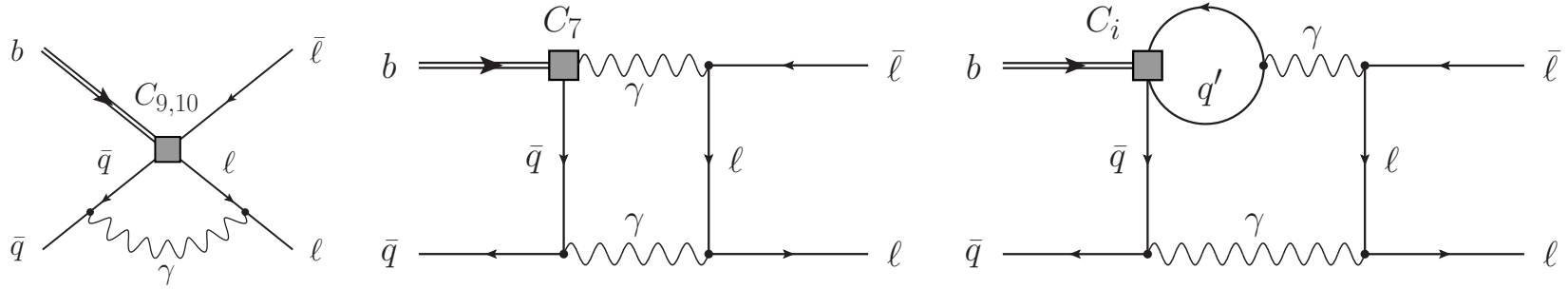
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Consequently, the relative QED correction scales like $\frac{\alpha_{em}}{\pi} \frac{M_{Bq}}{\Lambda}$.

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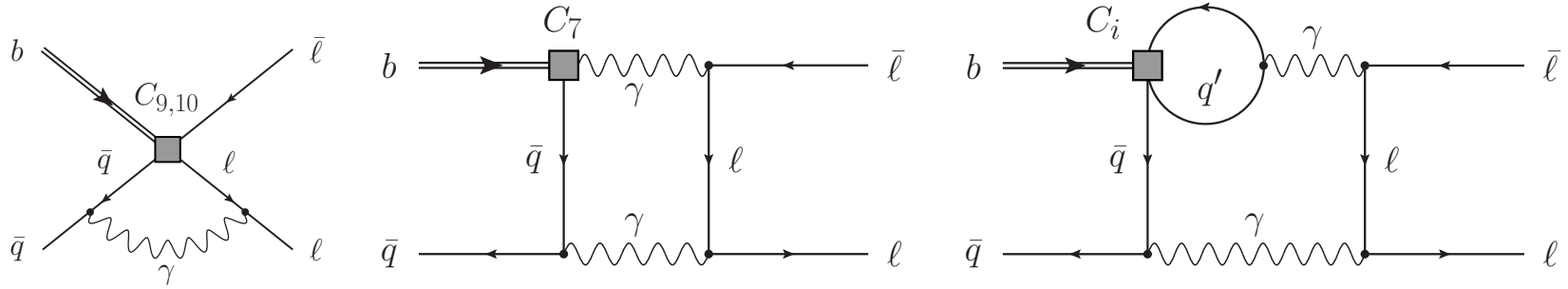
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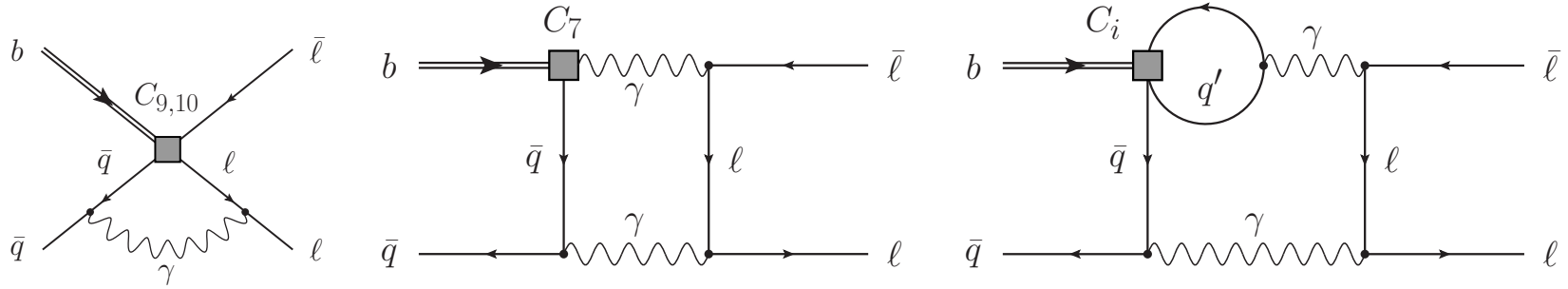
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Enhanced QED effects in $B_q \rightarrow \ell^+ \ell^-$

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However, it is larger than $\pm 0.3\%$ due to scale-variation of the Wilson coefficient $C_A(\mu_b)$.

SM predictions for all the branching ratios $\overline{\mathcal{B}}_{q\ell} \equiv \overline{\mathcal{B}}(B_q^0 \rightarrow \ell^+ \ell^-)$ including 2-loop electroweak and 3-loop QCD matching at $\mu_0 \sim m_t$
 [C. Bobeth, M. Gorbahn, T. Hermann, MM, E. Stamou, M. Steinhauser, PRL 112 (2014) 101801]

$$\begin{aligned}\overline{\mathcal{B}}_{se} \times 10^{14} &= \eta_{\text{QED}} (8.54 \pm 0.13) R_{t\alpha} R_s, \\ \overline{\mathcal{B}}_{s\mu} \times 10^9 &= \eta_{\text{QED}} (3.65 \pm 0.06) R_{t\alpha} R_s, \\ \overline{\mathcal{B}}_{s\tau} \times 10^7 &= \eta_{\text{QED}} (7.73 \pm 0.12) R_{t\alpha} R_s, \\ \overline{\mathcal{B}}_{de} \times 10^{15} &= \eta_{\text{QED}} (2.48 \pm 0.04) R_{t\alpha} R_d, \\ \overline{\mathcal{B}}_{d\mu} \times 10^{10} &= \eta_{\text{QED}} (1.06 \pm 0.02) R_{t\alpha} R_d, \\ \overline{\mathcal{B}}_{d\tau} \times 10^8 &= \eta_{\text{QED}} (2.22 \pm 0.04) R_{t\alpha} R_d,\end{aligned}$$

where

$$\begin{aligned}R_{t\alpha} &= \left(\frac{M_t}{173.1 \text{ GeV}} \right)^{3.06} \left(\frac{\alpha_s(M_Z)}{0.1184} \right)^{-0.18}, \\ R_s &= \left(\frac{f_{B_s} [\text{MeV}]}{227.7} \right)^2 \left(\frac{|V_{cb}|}{0.0424} \right)^2 \left(\frac{|V_{tb}^* V_{ts} / V_{cb}|}{0.980} \right)^2 \frac{\tau_H^s [\text{ps}]}{1.615}, \\ R_d &= \left(\frac{f_{B_d} [\text{MeV}]}{190.5} \right)^2 \left(\frac{|V_{tb}^* V_{td}|}{0.0088} \right)^2 \frac{\tau_d^{\text{av}} [\text{ps}]}{1.519}.\end{aligned}$$

Update of the input parameters

	2014 paper	this talk	source
M_t [GeV]	173.1(9)	172.69(30)	PDG 2022, http://pdglive.lbl.gov
$\alpha_s(M_Z)$	0.1184(7)	0.1179(9)	arXiv:2203.08271, Eq.(9.2)
f_{B_s} [GeV]	0.2277(45)	0.2303(13)	FLAG, arXiv:2111.09849
f_{B_d} [GeV]	0.1905(42)	0.1900(13)	FLAG, arXiv:2111.09849
$ V_{cb} \times 10^3$	42.40(90)	42.16(50)	inclusive, arXiv:2107.00604
$ V_{tb}^* V_{ts} / V_{cb} $	0.9800(10)	0.9819(5)	derived from CKMfitter 2019, http://ckmfitter.in2p3.fr
$ V_{tb}^* V_{td} \times 10^4$	88(3)	$87.1^{+0.86}_{-2.46}$	CKMfitter 2019, http://ckmfitter.in2p3.fr
τ_H^s [ps]	1.615(21)	1.616(10)	HFLAV 2022, https://hflav.web.cern.ch
τ_H^d [ps]	1.519(7)	1.519(4)	HFLAV 2022, https://hflav.web.cern.ch
$\overline{\mathcal{B}}_{s\mu} \times 10^9$	3.65(23)	3.66(12)	
$\overline{\mathcal{B}}_{d\mu} \times 10^{10}$	1.06(9)	$1.02^{+0.03}_{-0.06}$	

Sources of uncertainties	f_{B_q}	CKM	τ_H^q	M_t	α_s	other parametric	non-parametric	Σ
$\overline{\mathcal{B}}_{s\ell}$	1.1%	2.4%	0.6%	0.5%	0.2%	$< 0.1\%$	1.5%	3.2%
$\overline{\mathcal{B}}_{d\ell}$	1.4%	$(^{+2.0}_{-5.6})\%$	0.3%	0.5%	0.2%	$< 0.1\%$	1.5%	$(^{+3.1}_{-6.2})\%$

Summary

- Uncertainties in the SM predictions for $\overline{\mathcal{B}}_{q\ell}$ are dominated by the parametric ones, mainly due to the decay constants and CKM factors.
- In the $\overline{\mathcal{B}}_{s\ell}$ case, resolving the inclusive-exclusive tension in $|V_{cb}|$ would help a lot.
- The central values of the SM predictions for $\overline{\mathcal{B}}_{s\mu}$ and $\overline{\mathcal{B}}_{d\mu}$ are in good agreement with the data from LHCb, CMS and ATLAS.
- Some of the QED corrections involve non-perturbative physics beyond what is contained in the decay constants. Despite the powerlike enhancement factors in such corrections, the non-parametric uncertainty can be retained at the $\pm 1.5\%$ level.