Lectures on compact objects

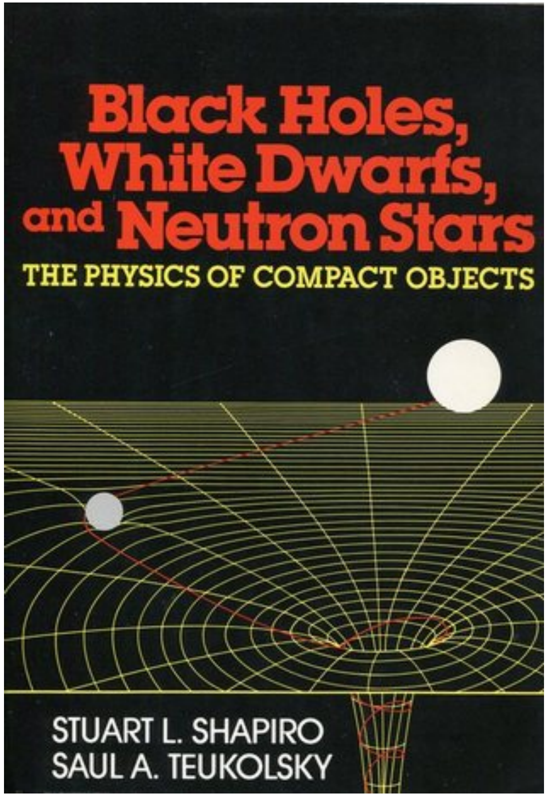
Sebastiano Bernuzzi (FSU Jena)

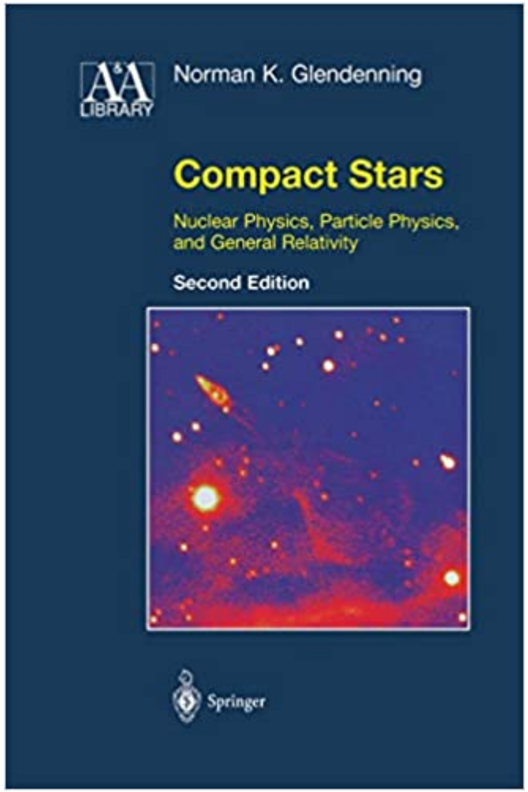
Contact: sebastiano.bernuzzi@uni-jena.de

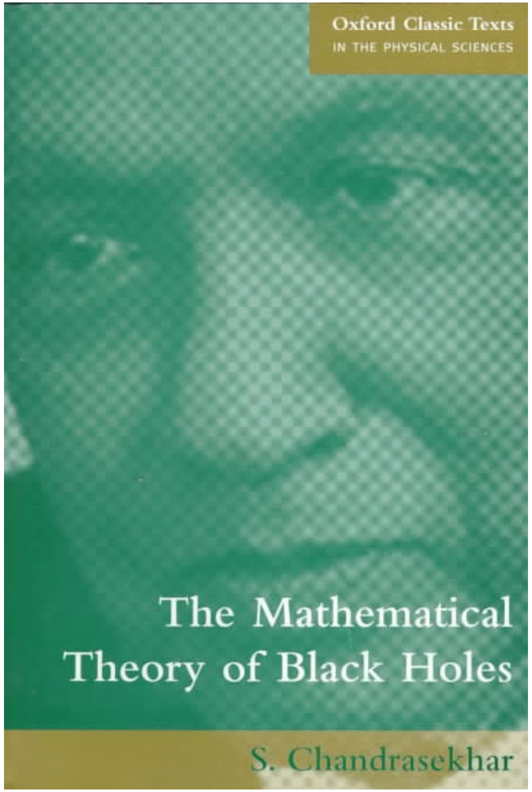
ISAPP Summer School on Gravitational Waves
June 2021

Topics

- Compact objects & compactness
- WD: Degenerate Fermi gas
- WD: Chandrasekhar mass
- NS: TOV equations & Buchdal limit
- NS: Maximum mass & Mass-radius diagram
- NS: Pulsations & stability
- NS: Equation of State
- BH: Schwarzschild solution and maximal extension
- BH: Birkhoff theorem & Schwarzschild's Orbits
- BH: the simpest GR 2-body problem
- BH: Gravitational collapse
- BH: Perturbation, Stability & Quasi-Normal-Modes
- BH (in binaries): Ringdown
- NS (in binaries): Love number, tidal polarizability & interactions


Black Holes,
White Dwarfs,
and Neutron Stars
THE PHYSICS OF COMPACT OBJECTS
STUART L. SHAPIRO
SAUL A. TEUKOLSKY


A&A LIBRARY
Norman K. Glendenning
Compact Stars
Nuclear Physics, Particle Physics, and General Relativity
Second Edition
Springer

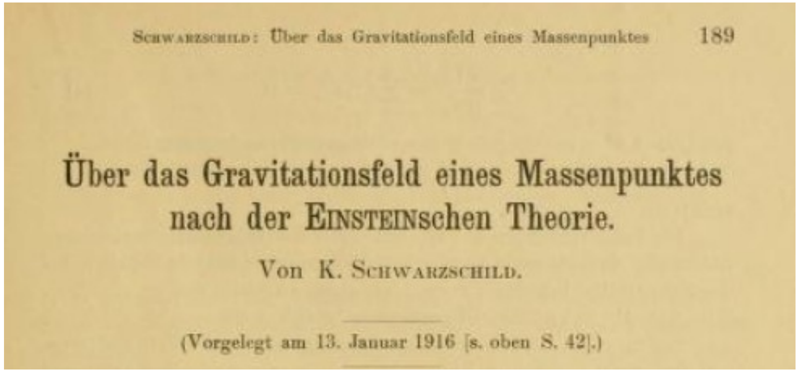

Oxford Classic Texts
IN THE PHYSICAL SCIENCES
The Mathematical
Theory of Black Holes
S. Chandrasekhar

Recap

	WD		NS		BH
• $C = \frac{GM}{c^2 R}$	10^{-4}	$\longrightarrow$	0.1	$\longrightarrow$	$1/2$
• $\max \rho \sim \frac{M}{R^3}$	$10^{8} - 10^{9}$	$\longrightarrow$	$10^{15} - 10^{16}$	$\longrightarrow$	"∞" $[g/cm^3]$

- "cold" & "small" ("quantum mechanically supported")
- DEGENERATE FERMION GAS $\underline{P = k\rho^{\Gamma}}$ $\Gamma = \begin{cases} 5/3 & \text{non rel. limit} \\ 4/3 & \text{relativistic limit} \end{cases}$
- $\Gamma \to 4/3$ unstable star : $R \sim M^{\frac{\Gamma-2}{3\Gamma-4}}$
- $M_{Ch} \sim 1.435 \, M_\odot$ (WD) <u>Chandrasekhar limit</u>
- TOV Eqs (GR stellar structure)
- Buchdahl inequality : $R > \frac{9}{4} M$ $(R > 2M)$
- $M_{max} \lesssim 3.2 \, (\rho_0/\rho_{nuc})^{1/2} \, M_\odot$ (independent on EoS $\rho_0 > \rho_{nuc}$)

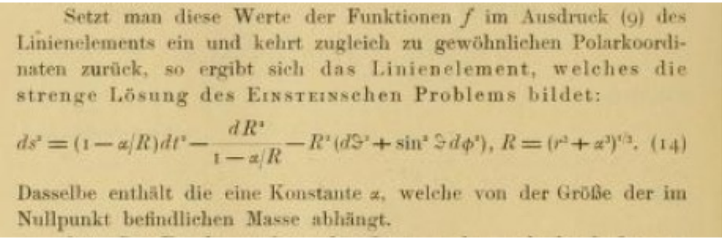
Black holes



Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie.

Von K. SCHWARZSCHILD.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)



Setzt man diese Werte der Funktionen f im Ausdruck (9) des Linienelements ein und kehrt zugleich zu gewöhnlichen Polarkoordinaten zurück, so ergibt sich das Linienelement, welches die strenge Lösung des EINSTEINschen Problems bildet:

$$ds^2 = (1-\alpha/R)dt^2 - \frac{dR^2}{1-\alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2),\quad R = (r^3+\alpha^3)^{1/3}. \quad (14)$$

Dasselbe enthält die eine Konstante α, welche von der Größe der im Nullpunkt befindlichen Masse abhängt.

"Black Holes" in Space

The heavy densely packed dying stars that speckle space may help determine how matter behaves when enclosed in its own gravitational field—By Ann Ewing

➤ SPACE may be peppered with "black holes."

This was suggested at the American Association for the Advancement of Science meeting in Cleveland by astronomers and physicists who are experts on what are called degenerate stars.

Degenerate stars are not Hollywood types with low morals. They are dying stars, or white dwarfs, and make up about 10% of all stars in the sky.

The faint light they emit comes from the little heat left in their last stages of life. It is not known how a star quietly declines to become a white dwarf.

Degenerate stars are made of densely packed electrons and nuclei, or cores of atoms. They are so dense that a thimbleful of their matter weighs a ton.

Some such stars are predicted in theory to have a density of one million tons per thimbleful. When this happens, the star is essentially made of neutrons and strange particles.

Because a degenerate star is so dense, its gravitational field is very strong. According to Einstein's general theory of relativity, as mass is added to a degenerate star a sudden collapse will take place and the intense gravitational field of the star will close in on itself.

Such a star then forms a "black hole" in the universe.

Modern tools, such as telescopes on an orbiting space platform, may be used to detect such black holes and to help determine how matter behaves when it is enclosed by its own gravitational field.

The light from the most famous white dwarf star, Sirius B, a companion to Sirius—which is the brightest star in the heavens visible from earth—has been captured using the 200-inch telescope atop Mt. Palomar. This was done as part of a program to study at least 20 white dwarfs.

Preliminary analysis of the light from Sirius B indicates that it has an effective temperature of 16,800 degrees Kelvin, or 30,000 degrees Fahrenheit. Its radius can be calculated from the temperature, and is only nine-thousandths that of the sun.

The star must therefore consist mainly of helium or heavier elements.

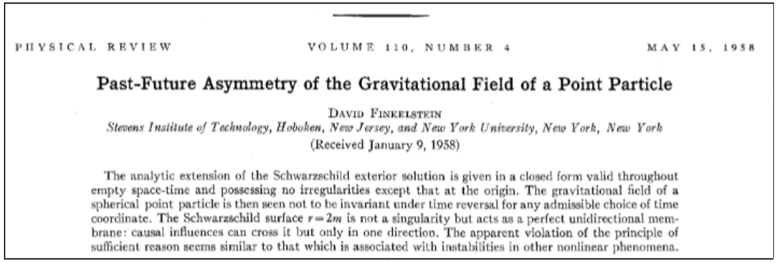
The speakers at the symposium were Drs. A. G. W. Cameron of the National Aeronautics and Space Administration's Goddard Institute for Space Studies, New York; Charles Misner of the University of Maryland; Volker Weidemann, Physikalisch-Technische Bundesanstalt, Braunschweig, Germany, and J. B. Oke of California Institute of Technology. The symposium was arranged by Dr. Hong-yee Chiu of the Goddard Institute for Space Studies.

• Science News Letter, 85:39 Jan. 18, 1964

SPACE

Original paper: https://archive.org/details/sitzungsberichte1916deutsch/page/188/mode/2up?view=theater

Translation: https://arxiv.org/abs/physics/9905030



PHYSICAL REVIEW VOLUME 110, NUMBER 4 MAY 15, 1958

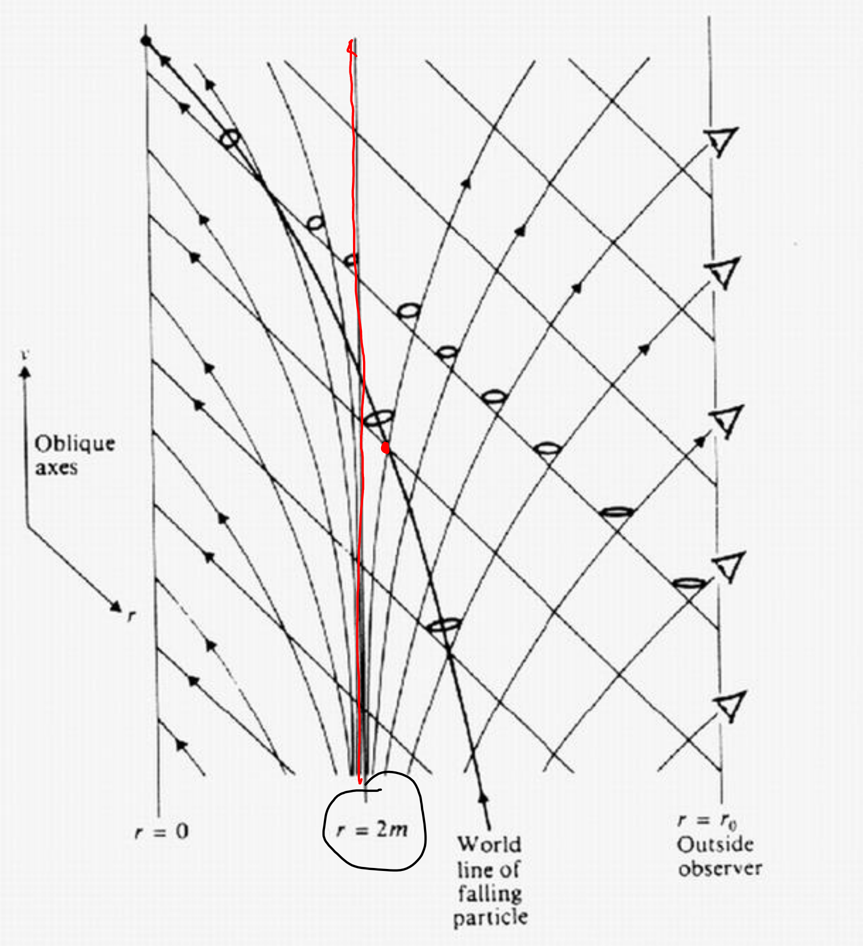
Past-Future Asymmetry of the Gravitational Field of a Point Particle

DAVID FINKELSTEIN
Stevens Institute of Technology, Hoboken, New Jersey, and New York University, New York, New York

(Received January 9, 1958)

The analytic extension of the Schwarzschild exterior solution is given in a closed form valid throughout empty space-time and possessing no irregularities except that at the origin. The gravitational field of a spherical point particle is then seen not to be invariant under time reversal for any admissible choice of time coordinate. The Schwarzschild surface $r=2m$ is not a singularity but acts as a perfect unidirectional membrane: causal influences can cross it but only in one direction. The apparent violation of the principle of sufficient reason seems similar to that which is associated with instabilities in other nonlinear phenomena.

- Eddington (1924)
- Lemaitre (1933)
- Finkelstein (1958) "*a perfect unidirectional membrane: causal influences can cross it in only one direction*"



Clock's Redshift:

$$\frac{ds_\infty^2}{ds_r^2} = \frac{\sqrt{-g_{00}(\infty)}}{\sqrt{-g_{00}(r)}} = \frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

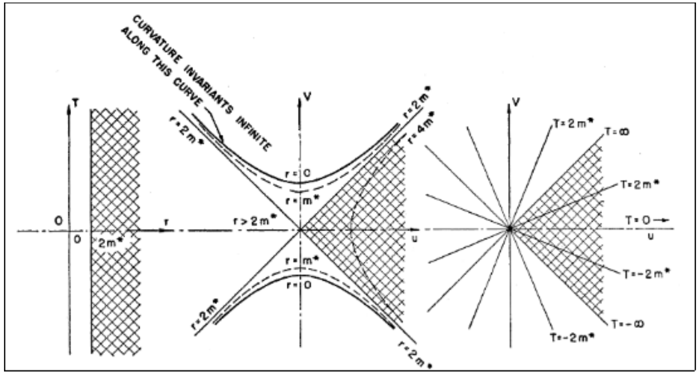
One way membrane:
Light cones tilt for $r < 2M$,
future directed paths are in the direction
of $r = 0$ (true singularity).
$r = 2M$ is a null surface called **event horizon**

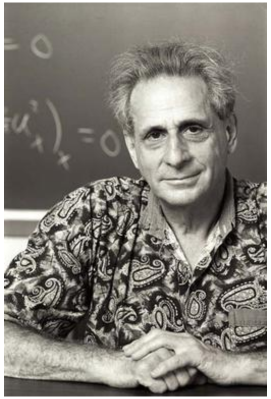
PHYSICAL REVIEW VOLUME 119, NUMBER 5 SEPTEMBER 1, 1960

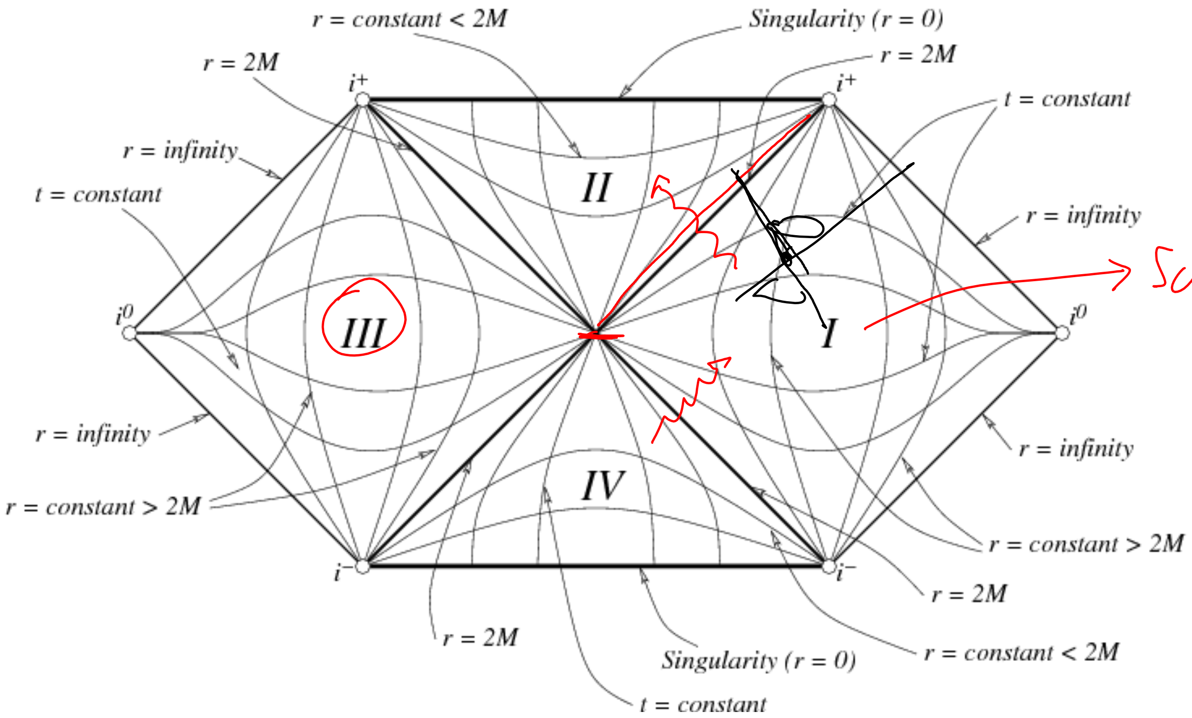
Maximal Extension of Schwarzschild Metric*

M. D. KRUSKAL†
Project Matterhorn, ~~Princeton University~~, Princeton, New Jersey
(Received December 21, 1959)

There is presented a particularly simple transformation of the Schwarzschild metric into new coordinates, whereby the "spherical singularity" is removed and the maximal singularity-free extension is clearly exhibited.



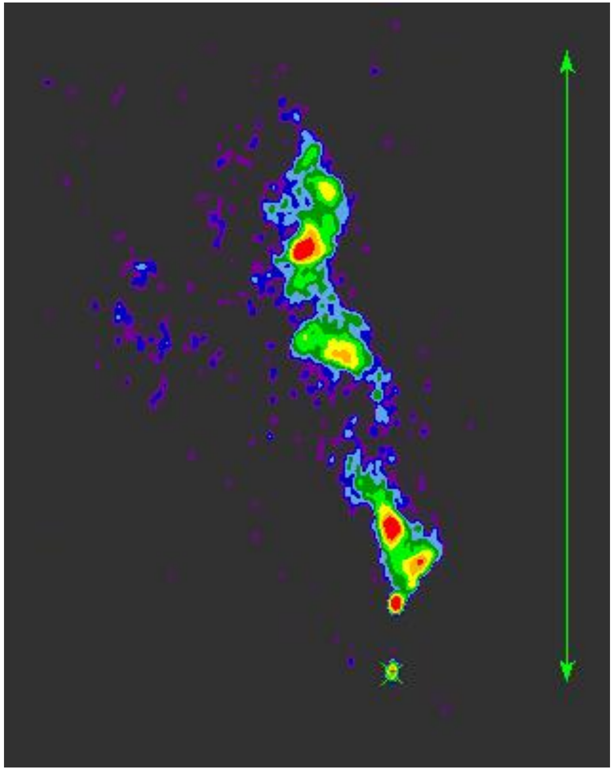




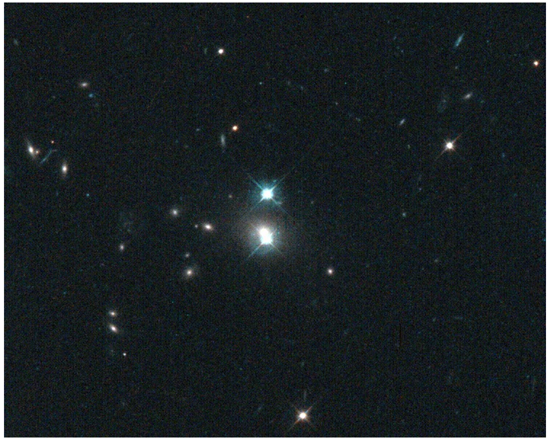
Schw. exterior

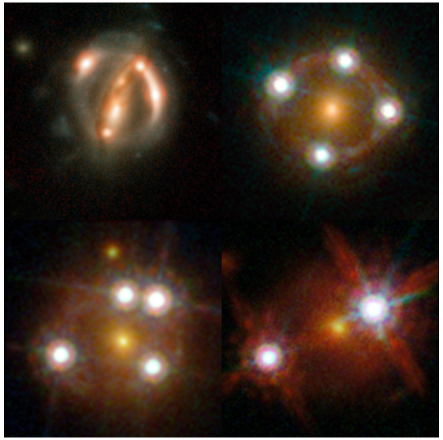
Quasar

C348 @ 1.4GHz



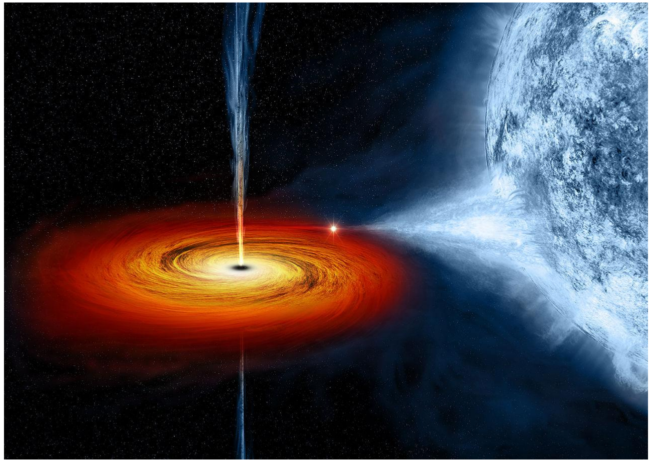
- 50s Radio sources of small size
- 60s Optical counterparts w\ High redshift (z ~ 7)
- Very luminous & extra-galactic? (> nuclear fusion, supernovae)
- 1964 Salpeter&Zeldovich: **Supermassive BH + accretion disk**
- Confirmed by
  - X ray observations of BH (*next slide*)
  - 1971 Peterson and Gunn: Galaxies containing quasars showed the same redshift as the quasars
  - 1979 Walsh,Carswell&Weyman: Grav. Lensing

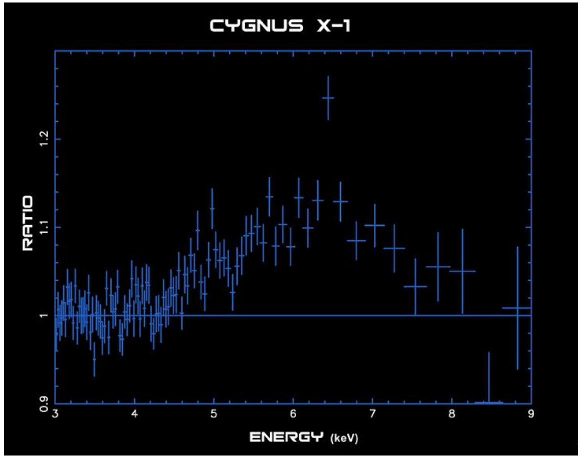




X-ray astronomy

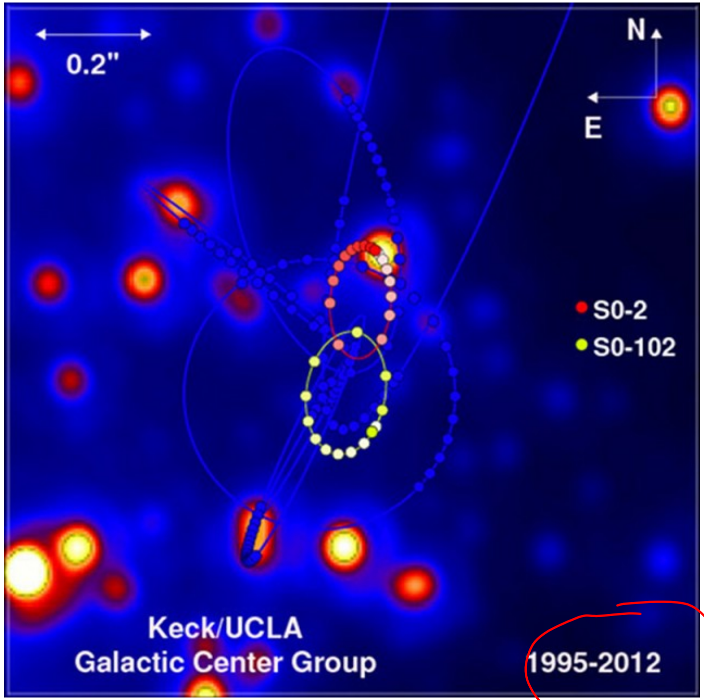
- Hot gases at T ~ 1,000,000K emit X-ray
- 1962 Scorpius X-1
  - Strongest X-ray source together the Sun.
  - Low-Mass-X-ray binary
  - $1.4M_{Sun}$ NS + 0.42 star
- 1964 Cygnus X-1
  - High-Mass-X-ray binary
  - $14.8M_{Sun}$ BH + 20-40M_{Sun} supergiant star

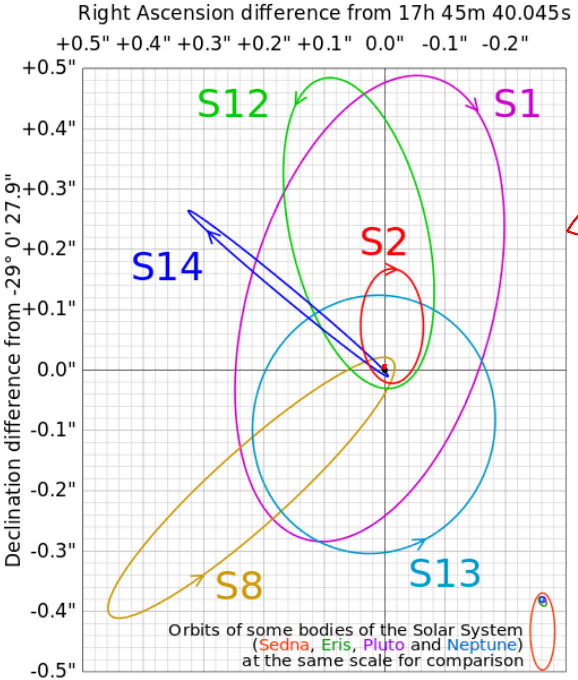




R.Giacconi (Nobel Prize 2002)

Sagittarius* A





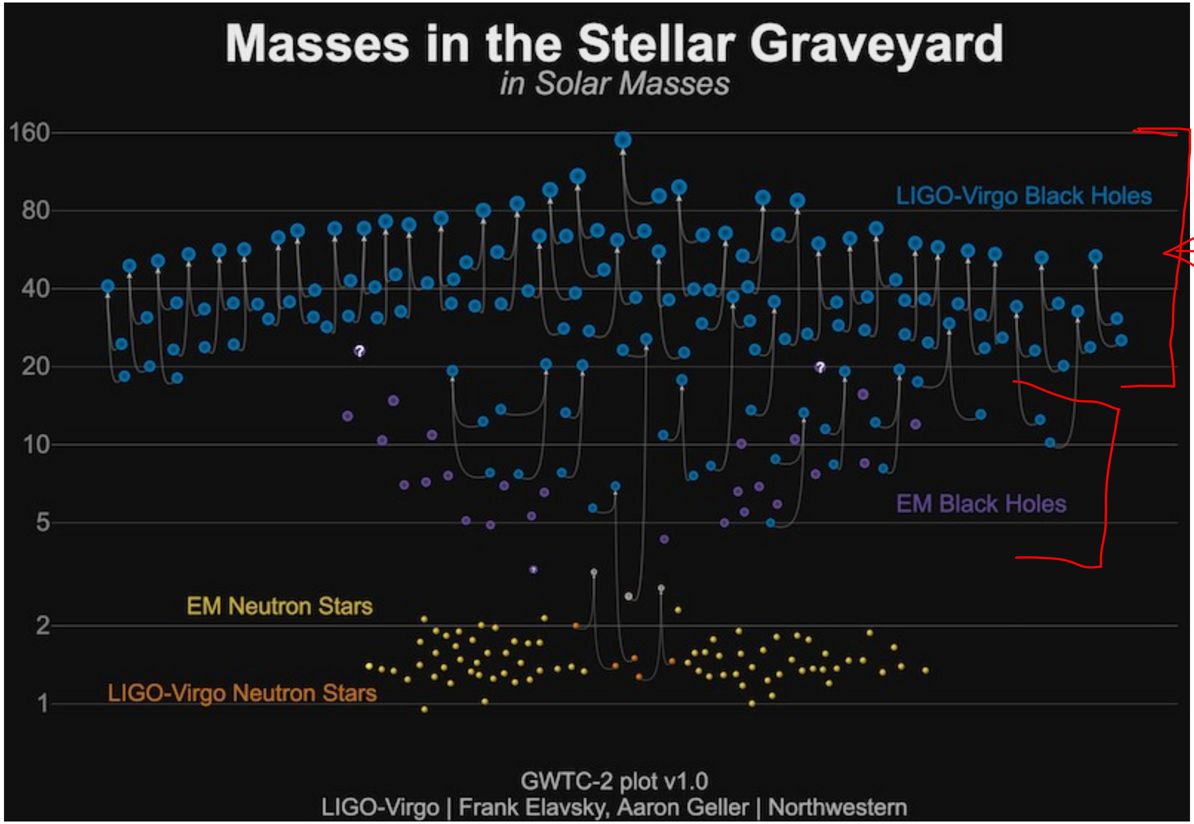
orbits

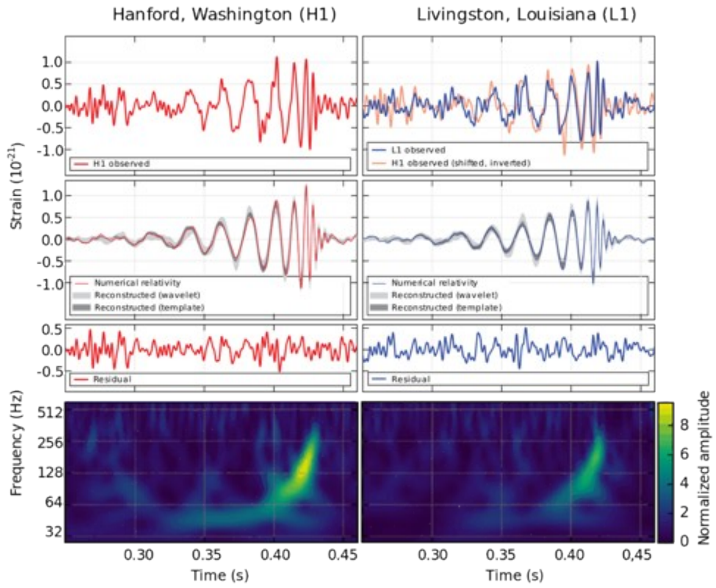
Galaxy center; Orbits' speed ~ 2% c

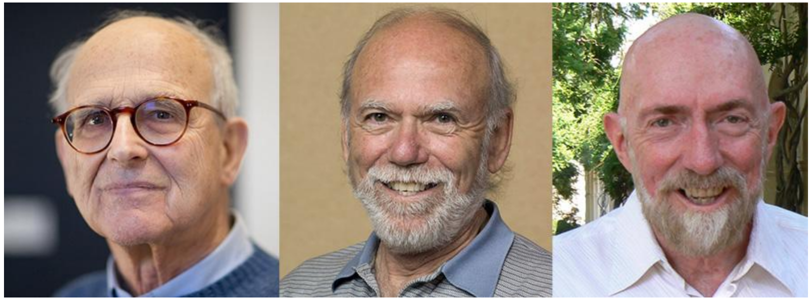
Mass ~ 4 million M_{Sun} ! => Supermassive BH

Gravitational-wave observations

Since 2015, LIGO-Virgo observations

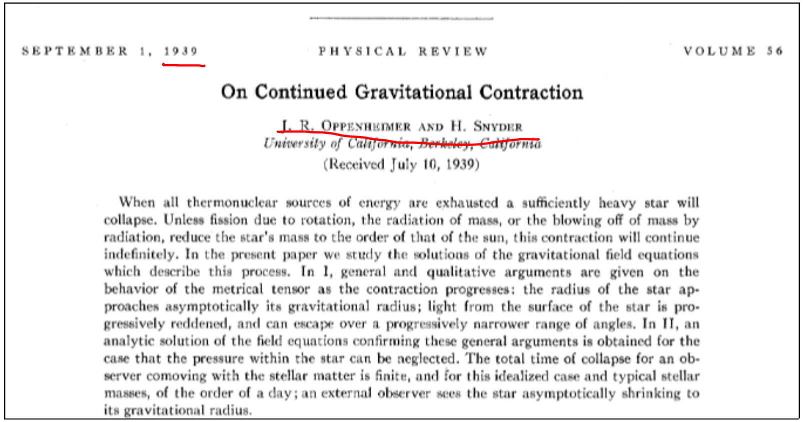






Weiss,Barish,Thorne Nobel Prize 2017

Gravitational collapse

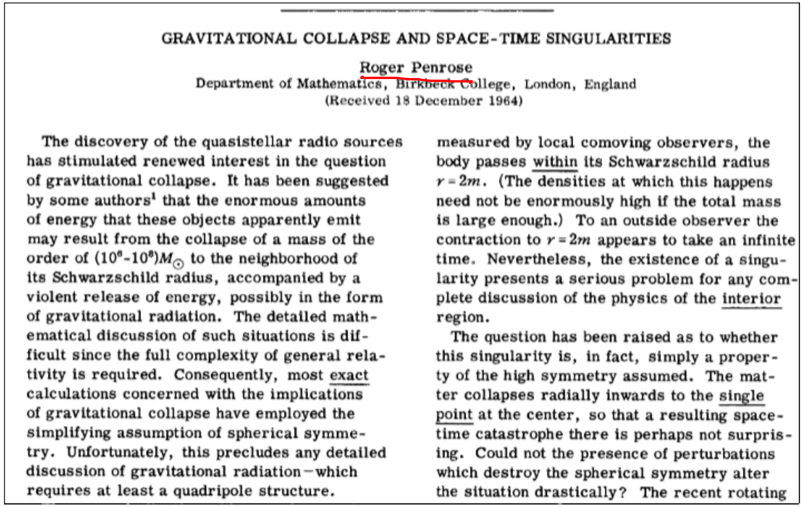


SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

On Continued Gravitational Contraction

J. R. OPPENHEIMER AND H. SNYDER
University of California, Berkeley, California
(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar masses, of the order of a day; an external observer sees the star asymptotically shrinking to its gravitational radius.



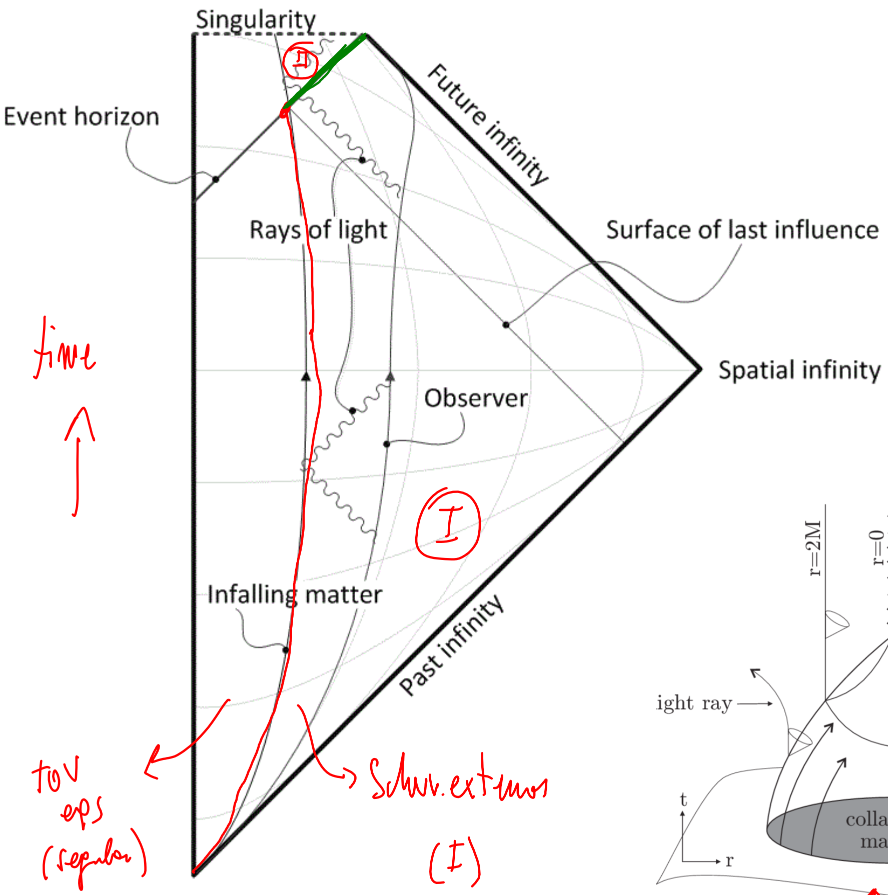
GRAVITATIONAL COLLAPSE AND SPACE-TIME SINGULARITIES

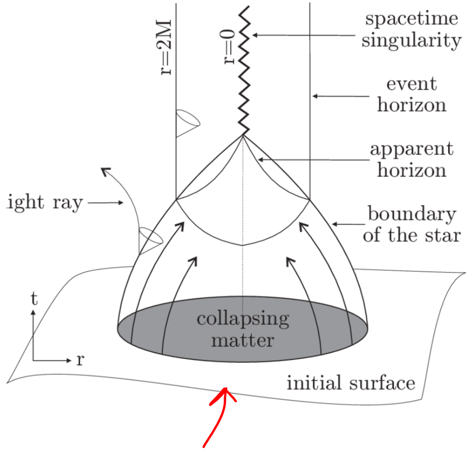
Roger Penrose
Department of Mathematics, Birkbeck College, London, England
(Received 18 December 1964)

The discovery of the quasistellar radio sources has stimulated renewed interest in the question of gravitational collapse. It has been suggested by some authors[1] that the enormous amounts of energy that these objects apparently emit may result from the collapse of a mass of the order of $(10^6\text{-}10^8)M_\odot$ to the neighborhood of its Schwarzschild radius, accompanied by a violent release of energy, possibly in the form of gravitational radiation. The detailed mathematical discussion of such situations is difficult since the full complexity of general relativity is required. Consequently, most exact calculations concerned with the implications of gravitational collapse have employed the simplifying assumption of spherical symmetry. Unfortunately, this precludes any detailed discussion of gravitational radiation—which requires at least a quadripole structure.

The general situation with regard to a spherically symmetrical body may be summarized as follows.[2] Once the body passes within its Schwarzschild radius ... $r = 2m$. (The densities at which this happens need not be enormously high if the total mass is large enough.) To an outside observer the contraction to $r = 2m$ appears to take an infinite time. Nevertheless, the existence of a singularity presents a serious problem for any complete discussion of the physics of the interior region.

The question has been raised as to whether this singularity is, in fact, simply a property of the high symmetry assumed. The matter collapses radially inwards to the single point at the center, so that a resulting space-time catastrophe there is perhaps not surprising. Could not the presence of perturbations which destroy the spherical symmetry alter the situation drastically? The recent rotating





Birkhoff's Theorem

Theorem 1. *Birkhoff (1923). The Schwarzschild metric is the unique vacuum solution in spherical symmetry.*

Sketch of the proof.

i. Any spherically symmetric spacetime (three spacelike rotational Killing vectors) can be foliated in 2-spheres

ii. The most general form of the metric is

$\mathcal{M} = M^2 \times S^2$

$$g = -e^{2\phi(t,r)}dt^2 + e^{2\lambda(t,r)}d^2r + r^2 d^2\Omega \tag{1}$$

iii. Use EFE to "eliminate" the time dependence

Corollary 2. *Any spherically symmetric vacuum spacetime is static.*

Physically, the staticity result can be understood as the absence of gravitational monopole radiation (analogous to the fact that the Coulomb solution is the only spherically symmetric solution of Maxwell equations in vacuum).

For example the exterior spacetime of a gravitationally collapsing spherical body is static always given by the static Schwarzschild metric.

Orbits

Geodesics of photons and particles in the Schwazrschild metric can be analyzed introducing the constants of motions associated to each Killing vector K^α of the spacetime. Exactly as in the Newtonian problem the motion is on a plane and the relevant equations are

$$-T_\alpha \frac{dx^\alpha}{d\tau} = \underbrace{\left(1 - \frac{2M}{r}\right)}_{=:A(r)} \dot{t} = \text{const} =: E \quad \text{energy} \tag{2}$$

$$\Phi_\alpha \frac{dx^{\,\alpha}}{d\tau} = r^2 \dot{\phi} = p_\phi = \text{const} =: L \quad \text{angular momentum} \tag{3}$$

$$-s = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{4}$$

where $s = 0, 1$ for photons and unit-mass test particles respectively. The key equation resulting from the ones above is remarkably simple:

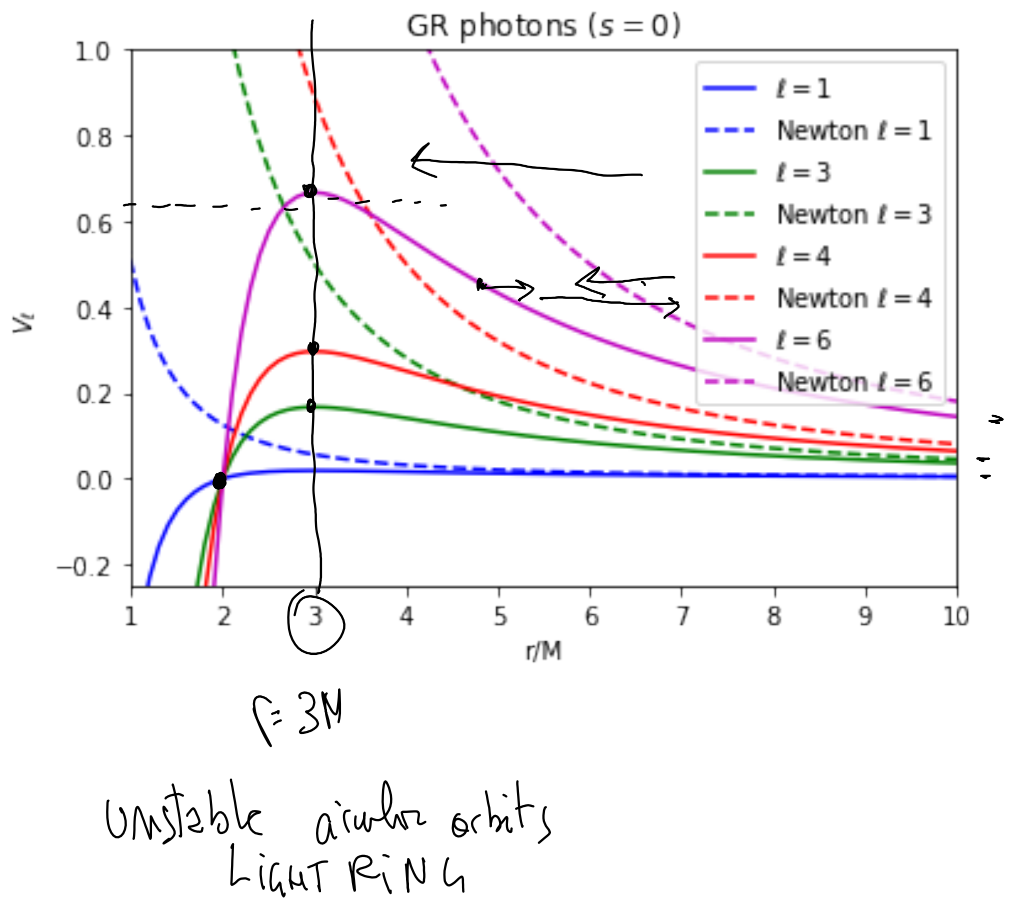
$$\boxed{\dot{r} + V = E^2} \tag{5}$$

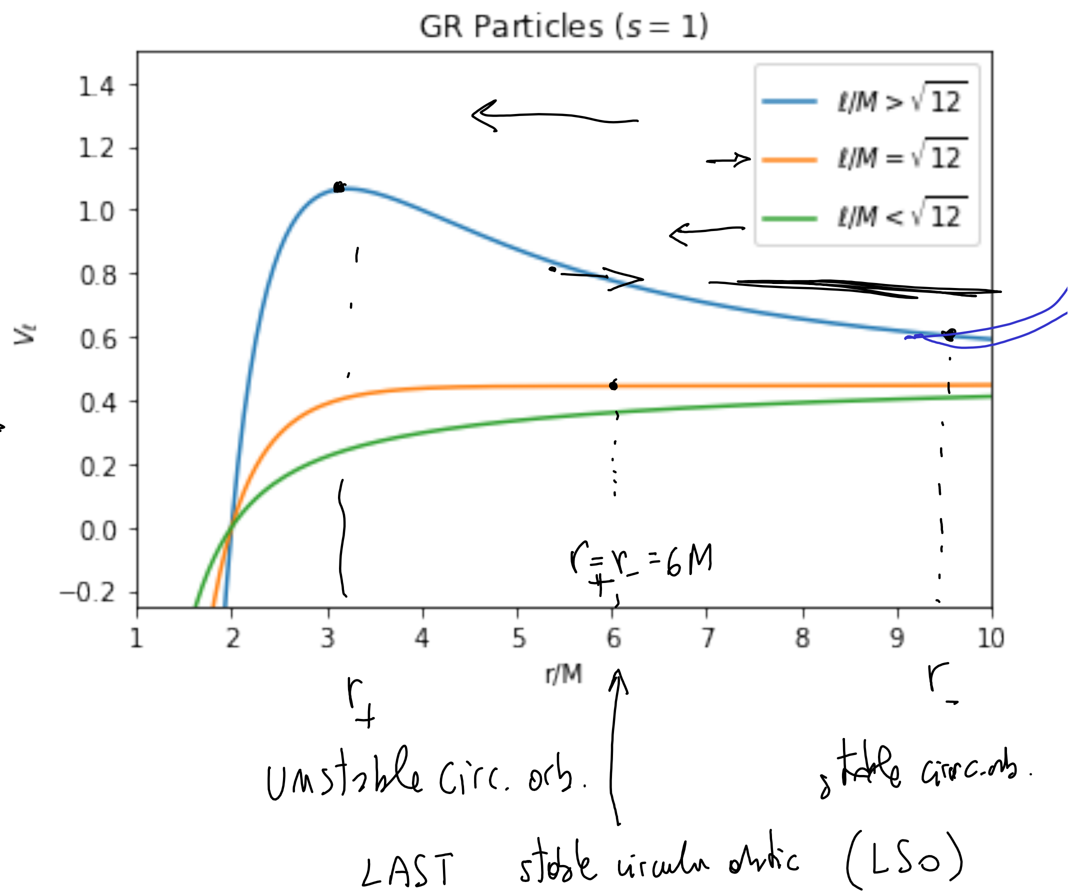
with the potential

$$V_L := A(r)\left(s + \frac{L^2}{r^2}\right) = s - s\frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3} \tag{6}$$

This result is analogous to the Newtonian motion in a central potential plus a GR term $\sim r^{-3}$.

Orbits


GR photons ($s = 0$)
$\ell = 1$
Newton $\ell = 1$
$\ell = 3$
Newton $\ell = 3$
$\ell = 4$
Newton $\ell = 4$
$\ell = 6$
Newton $\ell = 6$
V_ℓ
r/M
r = 3M
Unstable circular orbits
LIGHT RING


GR Particles ($s = 1$)
$\ell/M > \sqrt{12}$
$\ell/M = \sqrt{12}$
$\ell/M < \sqrt{12}$
V_ℓ
r/M
$r_+ = r_- = 6M$
r_+
r_-
Unstable circ. orb.
stable circ. orb.
LAST stable circular orbit (LSO)

The analysis of the orbits is thus performed by analyzing the stationary points of the potential (Note $\dot{r}^2 = E^2 - V_L \geq 0$):

$$0 = \frac{dV_L}{dr} = sMr^2 - L^2 r + 3ML^2 \qquad \frac{d^2V_L}{dr^2} = 2sMr - L^2 \tag{7}$$

Short summary of relevant feats:

Photons (s=0)

- The potential has maximum at $r = 3M$ (for $L > 0$), called *light ring* with energy $E_{\mathrm{LR}} = \sqrt{V(3M)} = \sqrt{L^2/(27M^2)}$
- The light ring corresponds to an *unstable circular orbit*
- Incoming photons with $E > E_{\mathrm{LR}}$ $(E < E_{\mathrm{LR}})$ continue to $r = 2M$ and below (hit a turning point at a minimum radius and reverse the trajectory)

Particles (s=1)

- The potential has extrema at $r_\pm = L^2 \pm \sqrt{L^2(L^2 - 12M^2)}$ with energies $E_\pm$
- The values $r_\pm$ correspond to an unstable and a stable circular orbit respectively
- $r = r_+ = r_- = 6M$ is the last stable orbit (LSO) or innermost stable circular orbits (ISCO)
- Incoming particles with $E > E_-$ $(E < E_+)$ continue to $R = 2M$ (hit a turning point and reverse)
- Particles with $E_- < E < E_+$ move on bound orbits (not necessarily closed; precession)

The simplest relativistic two-body problem

Imagine a small but finite mass on a circular orbit around a nonrotating black hole. The emission of gravitational radiation determines a deviation from geodesic motion. If initially $r \gg 2M$, the emission timescale is much longer than the orbital period and one can approximate the dynamics as a sequence of circular orbits with progressively smaller radius and higher frequency (*adiabatic approximation*). While at some point the adiabatic approximation will break, we can still analyze the motion and make some predictons/estimates.

The orbital radius will continue decreasing to the LSO. Below that point, no stable circular orbit is possible and the particle will fall to $r = 2M$ and then down to $r = 0$.

$\Rightarrow$ The "two bodies" collide and merge!

The orbital frequency of the LSO is easily found from the angular momentum value at $r_+ = 6M$:

$$\Omega^2 = \frac{L^2}{r^4} = \frac{M}{r^2(r-3M)} = 6^{-3/2} M^{-2} \tag{8}$$

or $\boxed{(M\Omega)^2 = 6^{-3/2}}$, the corresponding gravitational frequency is twice this value and provides an estimate of the merger frequency of the binary. Similarly, the energy of the LSO is

$$E = \frac{r-2M}{[r(r-3M)]^{1/2}} = \sqrt{\frac{8}{9}} \tag{9}$$

Thus, the energy emitted in gravitational waves (per unit mass) is $1 - E \approx 0.06$.

Exercises

1. Derive the formulas used above to discuss the orbits.
2. Derive the Hamiltonian of particles [Hint: Start from circular orbits]
3. Estimate the gravitational-wave merger frequency of a binary neutron star made of two euqal-masses neutron stars of $1.4 M_{\odot}$. Comment about the result.
4. Estimate the gravitational-wave merger frequency of an equal-mass binary black hole system of stellar-mass black holes of $30 M_{\odot}$ and supermassive black holes of $10^6 M_{\odot}$.
5. The correct result of the previous exercise is $2M\Omega \simeq 0.36$. Can you say why it holds for both cases (i.e. why there is a trivial mass scale)?

Perturbations & Stability

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

Stability of a Schwarzschild Singularity

Tullio Regge, *Istituto di Fisica della Università di Torino, Torino, Italy*

and

John A. Wheeler, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

$$\boxed{-\Psi^{\ell m}_{tt} + \Psi^{\ell m}_{r_* r_*} - V_\ell(r_*(r))\Psi^{\ell m} = S_{\ell m}}$$

$$r_* = r + R_s \ln\left(\frac{r}{R_s} - 1\right)$$

$(2M, +\infty) \to (-\infty, +\infty)$

GW

$$h_+ - i h_\times = \frac{G}{c^4 r}\sum_{\ell=2}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\,{}^{(-2)}Y_{\ell m}(\theta,\varphi)\left(\Psi^{(e)}_{\ell m} + i\Psi^{(o)}_{\ell m}\right) + O(1/r^2)$$

Quasi Normal Modes (QNMs)

Scattering of Gravitational Radiation by a Schwarzschild Black-hole

THE discovery of pulsars and the general conviction that they are neutron stars resulting from gravitational collapse have strengthened the belief in the possible presence of Schwarzschild black-holes—or Schwarzschild horizons—in nature, the latter being the ultimate stage in the progressive spherical collapse of a massive star. The stability of these objects, which has been discussed in a recent report[1], ensures their continued existence after formation. Inasmuch as the infinite redshift associated with it and its behaviour as a one-way membrane make the

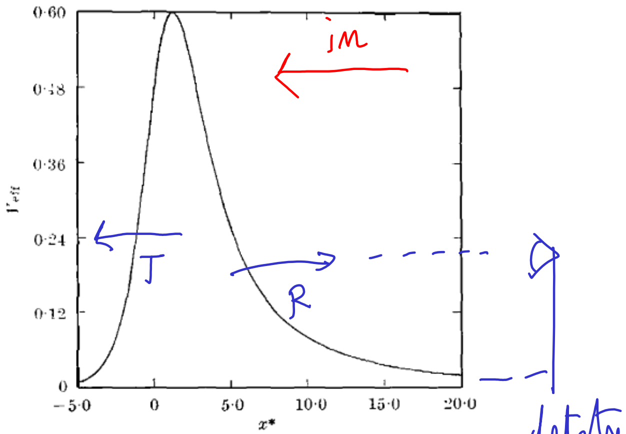
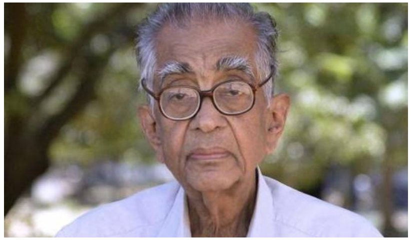


Fig. 1. The effective potential V_{eff} for the odd-parity gravitational waves of the lowest mode $l=2$ plotted against x^*.



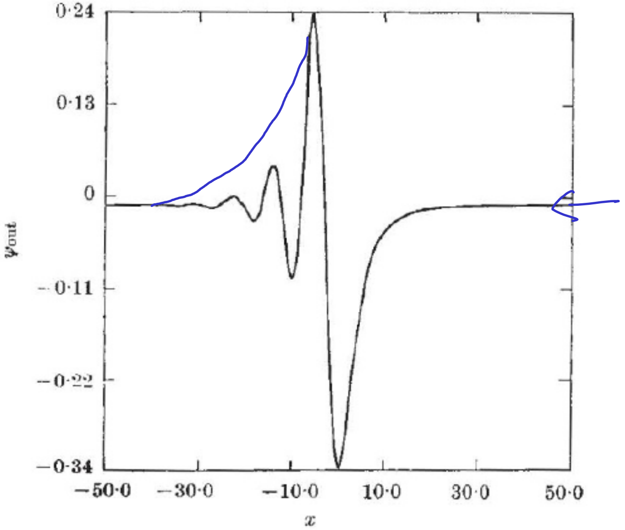


Fig. 3. The outgoing wave packet $\psi_{out}(x)$ at spatial infinity corresponding to the incident Gaussian wave packet $\psi_{in}(x)=e^{-ax^2}$ with $a=1$.

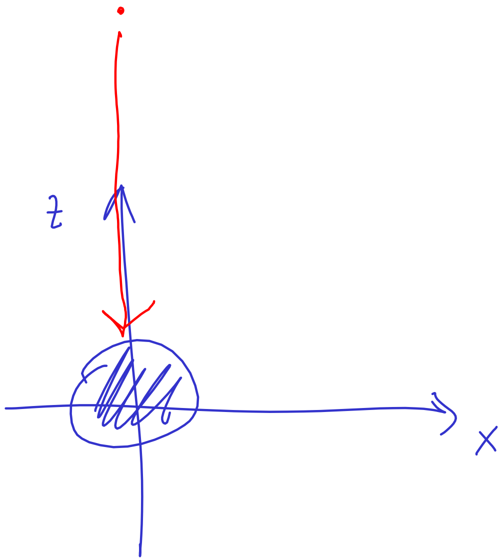
detector

PHYSICAL REVIEW D VOLUME 5, NUMBER 12 15 JUNE 1972

Pulses of Gravitational Radiation of a Particle Falling Radially into a Schwarzschild Black Hole*

Marc Davis, Remo Ruffini, and Jayme Tiomno†
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 20 December 1971)

Using the Regge-Wheeler-Zerilli formalism of fully relativistic linear perturbations in the Schwarzschild metric, we analyze the radiation of a particle of mass m falling into a Schwarzschild black hole of mass $M \gg m$. The detailed shape of the energy pulse and of the tide-producing components of the Riemann tensor at large distances from the source are given, as well as the angular distribution of the radiation. Finally, analysis of the energy going down the hole indicates the existence of a divergence; implications of this divergence as a testing ground of the approximation used are examined.



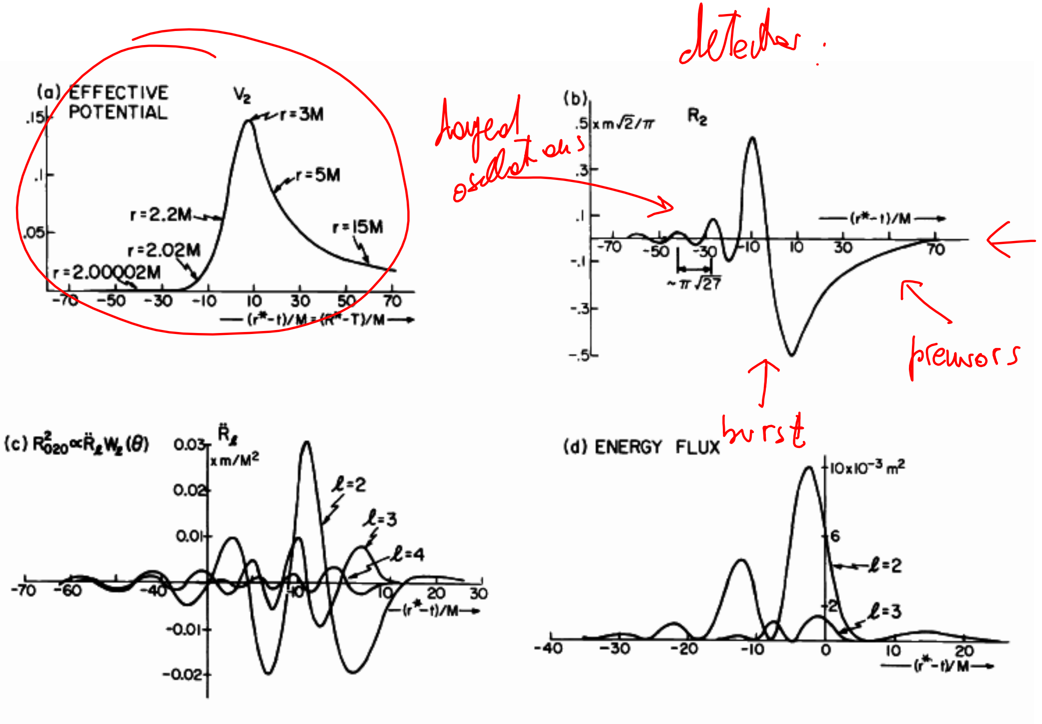


FIG. 1. Asymptotic behavior of the outgoing burst of gravitational radiation compared with the effective potential, as a function of the retarded time $(t-r^*)/M$. (a) Effective potential for $l=2$ in units of M^2 as a function of the retarded time $(t-r^*)/M = (T-R^*)/M$. For selected points the value of the Schwarzschild coordinate r is also given. (b) Radial dependence of the outgoing field $R_l(r,t)$ as a function of the retarded time for $l=2$. (c) $\ddot{R}_l(r^*,t)$ factors of the Riemann tensor components (see text) given as a function of the retarded time for $l=2,3,4$. (d) Energy flux integrated over angles for $l=2,3$; the contributions of higher l are negligible.

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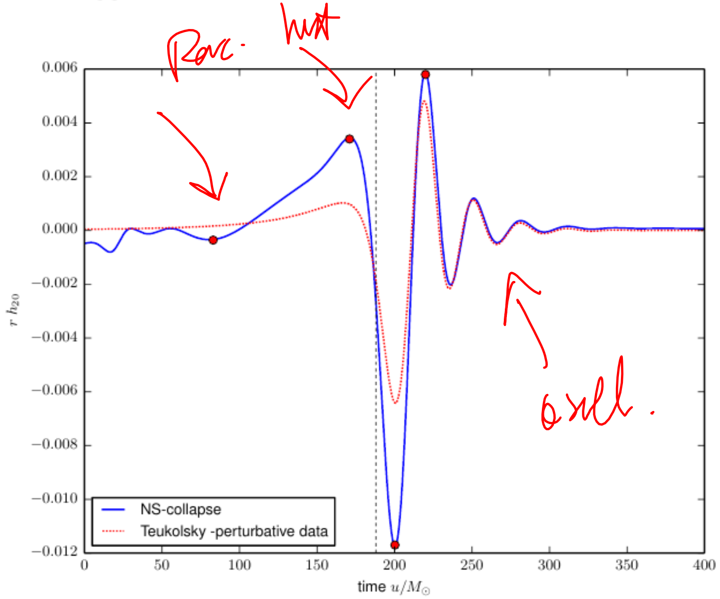

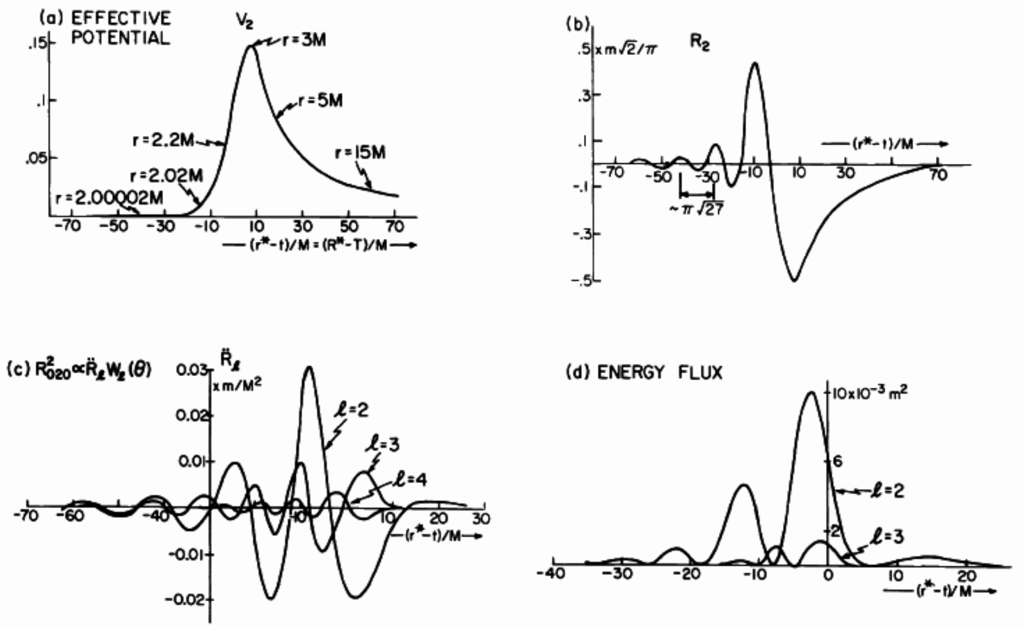

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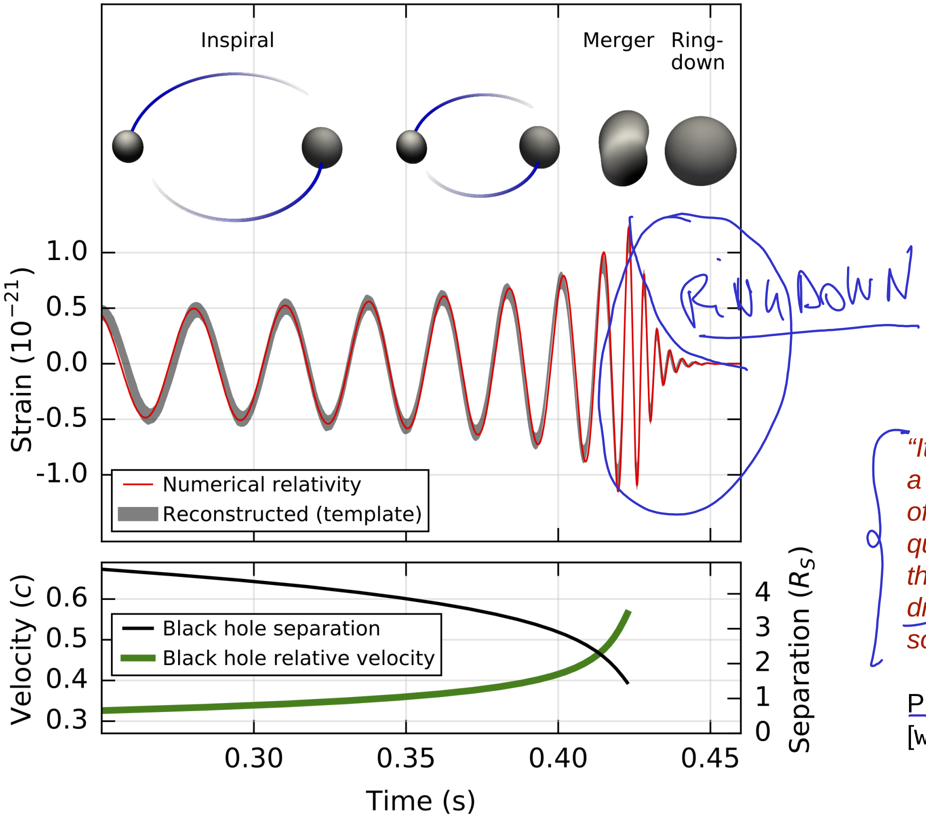
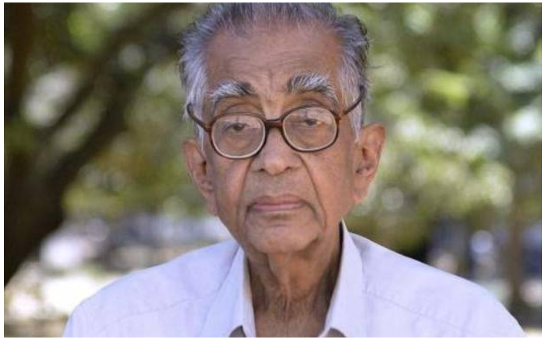
Signal from NS rotating collapse

QNMs in binary black holes remnants





"It was a natural question then to ask: how does one see a black hole? So, using a computer, I scattered packets of gravitational waves from a black hole and the quasinormal modes emerged carrying the signatures of the black hole… this was theoretical. I had never dreamed that this would take on an aspect of reality some day,"

Prof. Vishveshwara (6 March 1938 – 16 January 2017)
[www.thehindu.com]

Origin of QNMs

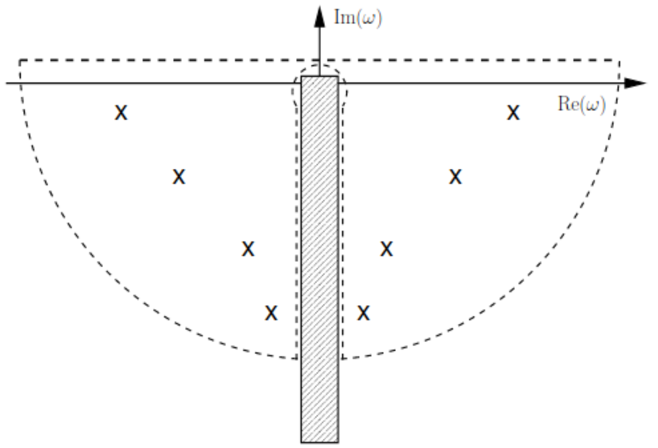
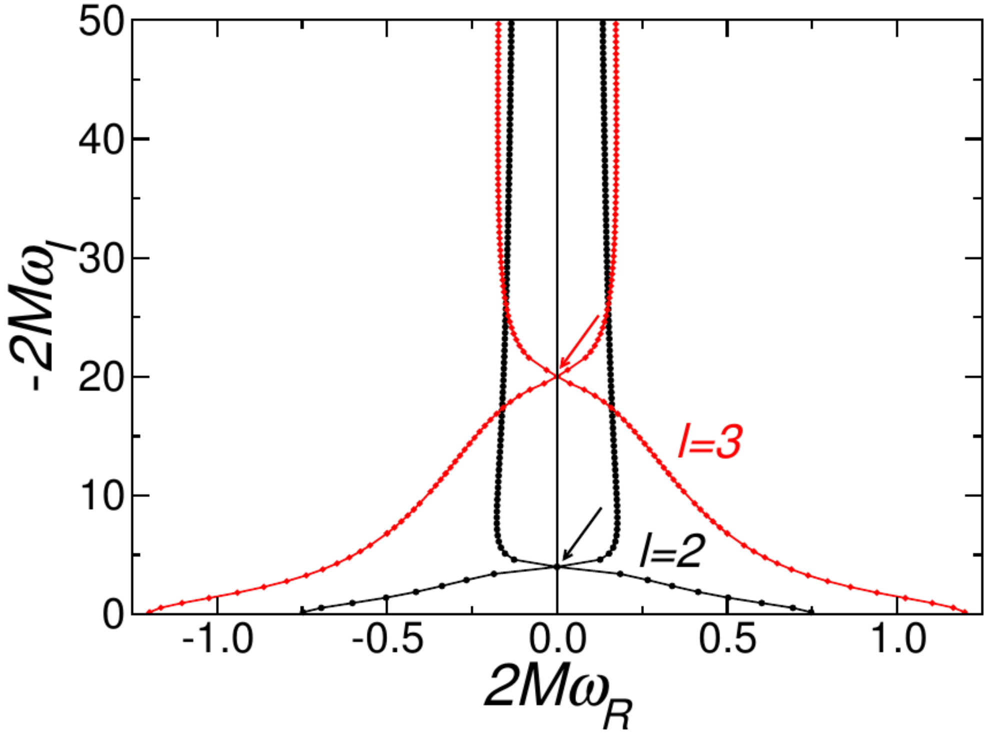


Figure 2. Integration contour for Eq. (36). The hatched area is the branch cut and crosses mark zeros of the Wronskian W (the QNM frequencies).



Review: https://arxiv.org/abs/0905.2975

Perturbations of spherical spacetimes

Consider the perturbation $h_{\alpha\beta}$ of a spherically symmetric background metric $g^{(0)}_{\alpha\beta}$ ($\mathcal{M} = M^2 \times S^2$) in some suitable coordinates (e.g. Schwarzschild):

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta} \tag{1}$$

Because of the background, the perturbation can be decomposed in scalar $Y_{\ell m}$, vectors $Z^{\ell m}_{a}$ and tensor $Z^{\ell m}_{ab}$ spherical harmonics with indexes (ℓ, m) and further separated between *even* (*electric-type*) and *odd* (*magnetic-type*) parity according to the behaviour under reflection trhough the origin: $(-1)^{\ell}$ and $(-1)^{\ell+1}$:

$$h_{\mu\nu} = h^{(e)}_{\mu\nu} + h^{(o)}_{\mu\nu} \tag{2}$$

For example, the decomposition of the even parity part reads ($A = 0, 1; a = 2, 3$)

M^2

$$h^{(e)}_{\mu\nu} = \begin{pmatrix} H_0 Y_{\ell m} & H_1 Y_{\ell m} & h^{(e)}_{A} Z^{\ell m}_{a} \\ '' & H_2 Y_{\ell m} & \\ '' & '' & r^2 K Y_{\ell m} \Omega_{ab} + r^2 G Z^{\ell m}_{ab} \end{pmatrix} \tag{3}$$

where Ω_{ab} is the metric on S^2 and the metric coefficients do not carry multipolar indexes for simplicity (A sum on (ℓ, m) is also understood).

Gauge invariant quantities (under infinitesimal coordinate transformations) can be identified from the above metric. Of particular importance are the two *scalar functions* for each multipole (suffix (ℓ, m) understood):

$$\Psi^{(e)}(t,r) \text{ and } \Psi^{(o)}(t,r) \tag{4}$$

The perturbed EFE lead to the *Regge-Wheleer-Zerilli (RWZ)* wave equation for the above scalar functions (one for each multipole (ℓ, m) that are all decoupled from each other):

$$\Psi_{tt} - \Psi_{xx} + V_\ell = S_{\ell m} \tag{5}$$

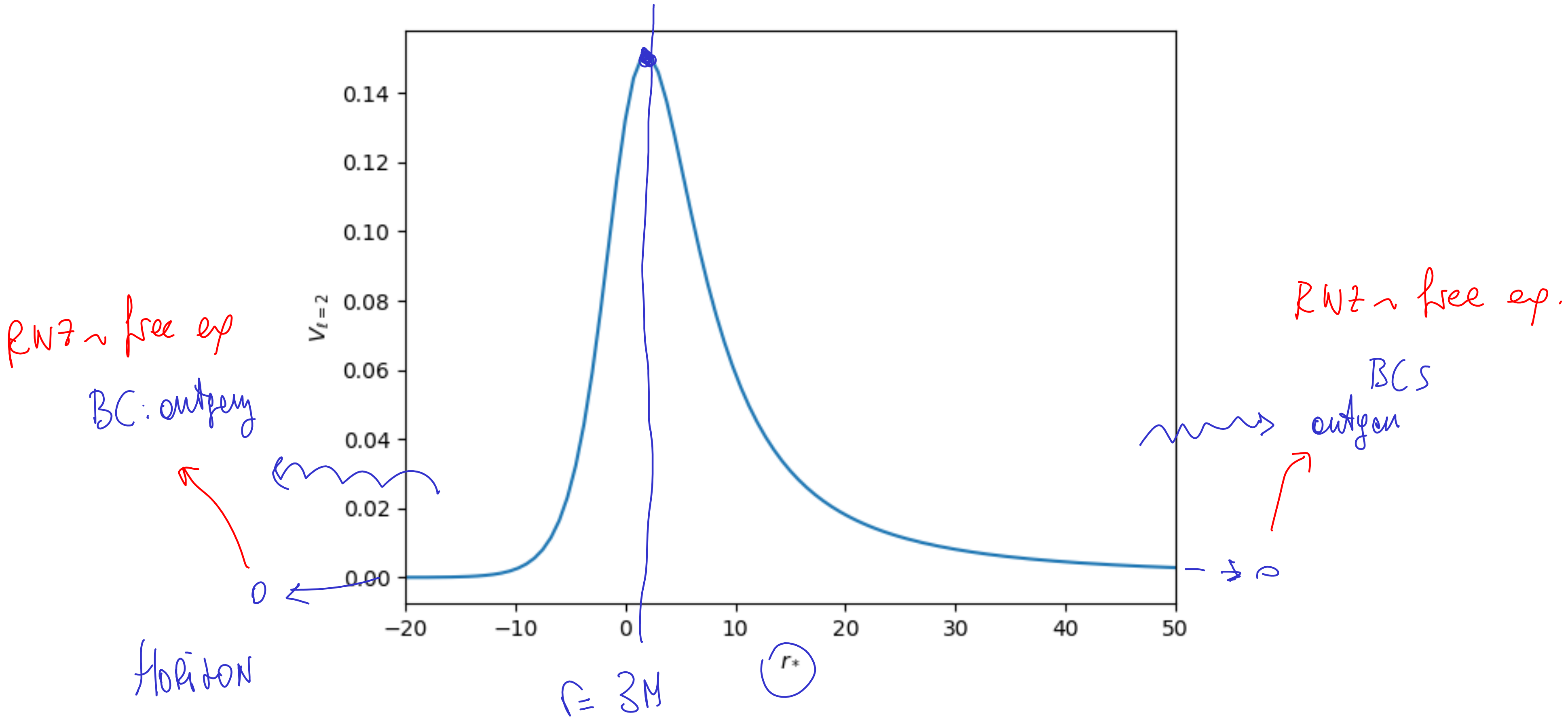
where x is the tortoise coordinate that maps $[2M, \infty)$ to $(-\infty, \infty)$

$$x = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{6}$$

$S_{\ell m}$ is a source term from the stress-energy tensor, and V_ℓ is a potential determined by the background metric that for even and odd parity reads, respectively ($\Lambda := \ell(\ell+1)$)

$$V_\ell^{(e)} = A(r) \frac{\Lambda(\Lambda-2)^2 r^3 + 6(\Lambda-2)^2 M r^2 + 36(\Lambda-2) M^2 r + 72 M^3}{r^3 ((\Lambda-2) r + 6M)^2} \tag{7}$$

$$V_\ell^{(o)} = A(r)\left(\frac{\Lambda}{r^2} - \frac{6M}{r^3}\right) \tag{8}$$



RWZ ~ free eq
BC: outgoing
0
Horizon
r = 3M
RWZ ~ free eq.
BCs
outgoing
→ 0
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$V_{\ell=2}$
−20
−10
0
10
20
30
40
50
r_*

There is no dependence on m due to the spherical symmetry of the background. Among the linearized EFE, Eq.(5) plays a special role because its asymptotic solutions for large r represent the gravitational-wave degrees of freedom in the spin weighted spherical harmonics decomposition

$$h_+ - i h_\times = \frac{G}{c^4}\frac{1}{r}\sum_{\ell=2}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\Psi^{(e)}_{\ell m}(t) + i\,\Psi^{(o)}_{\ell m}(t)\right){}^{-2}Y_{\ell m}(\theta,\phi) \tag{9}$$

The RWZ problem in vacuum

The initial-boundary value problem with the RWZ requires to chose appropriate initial and boundary conditions. Because the RWZ potential tends to zero for both $x \to \pm\infty$ (horizon and spatial infinity), the asymptotic solutions at large (tortoise) radii are the solution of the "free" wave equation on the light cones.

By considering solutions with time dependence $\Psi \sim e^{-i\omega t}$ (or, equivalently, the Fourier modes), the RWZ equation can be cast in a form similar to the Schroedinger equation for stationary states,

$$\frac{d^2\tilde{\Psi}}{dx^2} + [\omega^2 - V_\ell]\tilde{\Psi} = 0 \tag{10}$$

However, since the RWZ potential is positive, no "bound states" can exists, and the spectrum is continuous. The physical requirement that no signals can come out from the the horizon, implies that the boundary condition at $x \to -\infty$ is an ingoing wave,

$$\tilde{\Psi} \sim e^{-i\omega x} \quad (x \to -\infty) \tag{11}$$

This boundary condition also follows from requiring smoothness. Requiring instead hat no signal can come in from spatial infinity, implies an outgoing wave for $x \to +\infty$

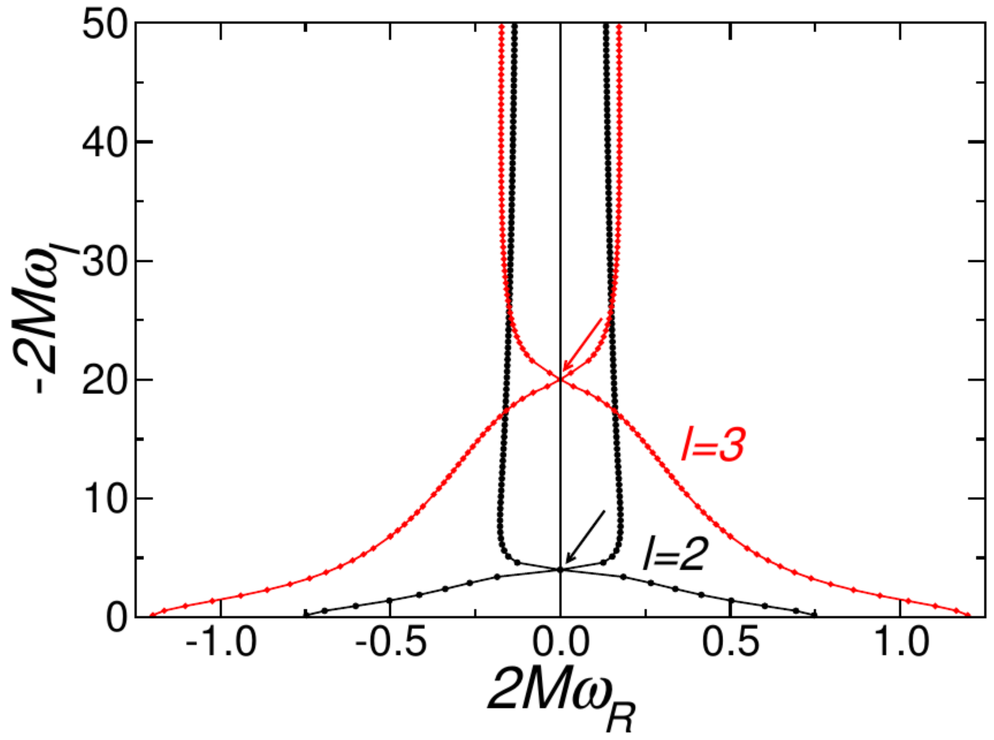
$$\tilde{\Psi} \sim e^{i\omega x} \quad (x \to +\infty) \tag{12}$$

With these boundary conditions, Eq.(10) admits solutions for a discrete infinity of *complex* frequencies ω_n with negative imaginary frequencies $\mathrm{Im}(\omega_n) < 0$. These damped modes are called *quasi-normal modes* (QNMs) appear also in other wave problems with *open* boundaries, and generically characterize *dissipative systems*. Differently from the normal modes of a vibrating string with "fixed" boundary conditions, QNMs do not form a complete set of eigenfunctions for the solution.

The presence of damped QNMs in the solutions suggests the stability of Schwatrzschild black holes under small perturbations (mode stability). The conclusion is correct, although the story is richer.

Some steps:

- Regge-Wheeler (1957) use a WKB analysis to argue that odd perturbations of the Schwarzschild spacetime are stable under the boundary conditions Eq.(11-12)
- Zerilli (1970) obtains the master equation for even parity (same WKB as above applies)
- Vishveshwara (1970) rules out perturbations growing in time because they diverge at the event horizon, if they fall at infinity.
- Chandrasekhar (1975) finds a map between the odd and even parity perturbations, and proves the QNMs are "isospectral"



50
40
30
20
10
0
-1.0
-0.5
0.0
0.5
1.0
$-2M\omega_I$
$2M\omega_R$
$l=3$
$l=2$

- Leaver (1986) formally identifies QNM as pole of a Green function
- Kay & Wald (1987) show that solutions with data of compact support are bounded
- Bachelot and Motet-Bachelot (1993) prove the existance of infinite number of QNMs

Solution by Laplace transform

The Cauchy problem specified by Eq.(5), boundary conditions (like Eq.(11-12)) and initial data $\Psi(0,x)=\psi(x)$ and $\Psi_t(0,x)=\psi_t(x)$ with compact support (or sufficiently localized) can be solved introducing the Laplace transform

$$\phi(s,x)=\int_0^\infty e^{-st}\Psi(t,x)\,dt \tag{13}$$

The Laplace transform is defined for positive, real $s>0$ and can be analytically continued into the positive complex plane. The equation for ϕ can be immediately found by integrating the RWZ,

initial data

BCs

$$\phi_{xx}-(s^2+V(x))\phi=F(s,x):=-s\psi(x)-\psi_t(x) \tag{14}$$

Two independent solutions $f_\pm(s,x)$ of the homogeneous equation ($F\equiv 0$) determine the unique Green function of the problem; the solution is

$$\phi(s,t)=\int_{-\infty}^{+\infty}G(s;x,x')F(s,x')dx'=\int_{-\infty}^{+\infty}\frac{f_-(s,x_-)f_+(s,x_+)}{W(s)}F(s,x')dx' \tag{15}$$

where $x_{\pm} = \frac{\max}{\min}(x, x')$ and $W(s)$ is the Wrongskian. The formal solution of the Cauchy problem is then obtained from the inverse Laplace transform

$$\Psi(t,x) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\epsilon-iR}^{\epsilon+iR} e^{st}\,\phi(s,x)\,ds \tag{16}$$

where ϵ (real) is greater than the real part of all the singularities of ϕ .

The Laplace solution contains both the initial data (in $F(s,x)$) and the boundary conditions. The latter are implemented in the choice of the homogeneous solutions $f_{\pm}$. The integral in Eq.(16) can be performed using the residue theorem by chosing an appropriate contour in the complex plane, as determined by the analytical properties of ϕ .

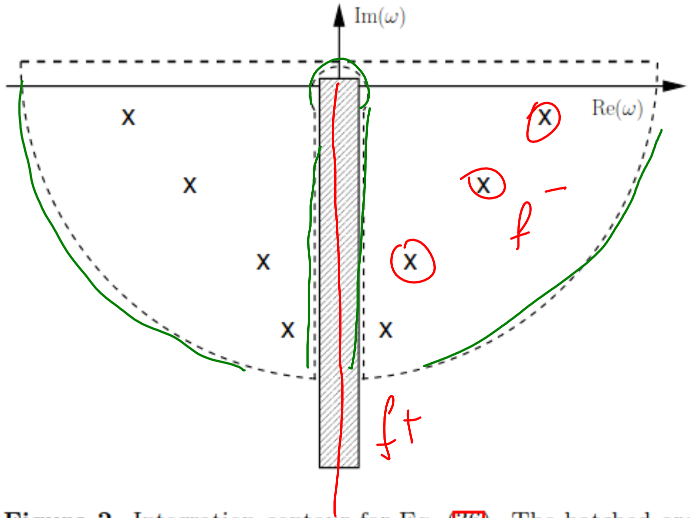
The RWZ potential decays exponentially for $x \to -\infty$, it reaches a maximum and then decays as $1/x^2$ for $x \to \infty$. For the RWZ potential can be proven that (Bachelot&Motet-Bachelot 1993):

- f_- has poles only at negative real integers
- f_+ has a branch cut in the negative real axis due to the r^{-2} decay at large radii

The solution is then determined by different contributions

$$\Psi \sim \int_{\epsilon-iR}^{\epsilon+iR}(.) = \underbrace{\int_{\mathrm{large\,half-circle}}(.)}_{\mathrm{source\,term}} + \underbrace{\sum_k \mathrm{res}(.,s_k)}_{\mathrm{QNMs}} + \underbrace{\int_{\mathrm{branch\,cut}}(.)}_{\mathrm{Late-time\,tails}} \tag{17}$$

$\psi \sim t^{-p_\ell}$



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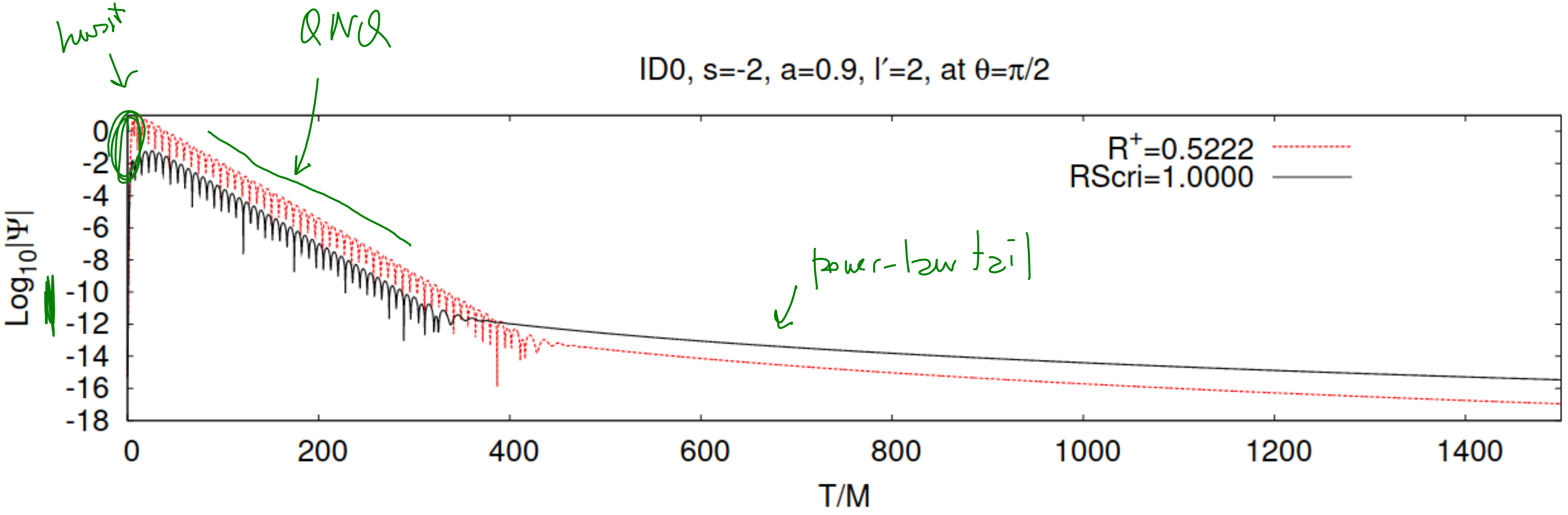


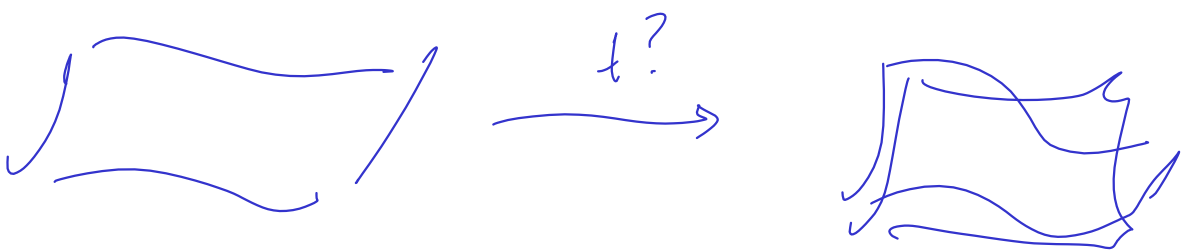
Figure 3. Evolution of the perturbation field at the horizon and $\mathscr{I}^+$ ($\theta = \pi/2$). The field is characterized by the quasi normal mode ringdown and a power law tail. The plot refers to a simulation of an axisymmetric gravitational perturbation ($s = -2$ and $m = 0$) with ID0, $l' = 2$ and $a = 0.9$.

Stability of black hole spacetimes

A fundamental question about exact stationary solutions of Einstein's field equations (EFE) like Minkowski, Schwarzschild or Kerr is their stability under small perturbations. Rigorous proofs are very nontrivial and usually built on several results.

A very rough scheme is the following:

- Linear mode stability: within linear perturbation theory one proves that the time evolution of each mode, say $\Psi_{\ell m}$, is bounded (in some norm) for a suitable class of initial data (say, with compact support).
- Linear stability: mode stability does not, in general, guarantee that a solution composed of an infinite sum of modes remains bound. Here one proves that all solutions to the linearised EFE remain bounded for all times by a suitable norm of their initial data. Mode stability is a necessary condition to linear stability.
- Nonlinear stability: here one considers the more general Cauchy problem in GR with initial data "near" Minkowski, Schwarzschild or Kerr, and shows that the solution remains bound.



The question of nonlinear stability of Kerr black holes is still open, although many positive results are available. An incomplete list is:

- [7] First argument for mode stability of Schwarzschild
- [5] Linear stability of scalar perturbation of Schwarzschild
- [8] Mode stability of Kerr
- [2] Nonlinear stability of Mikowski for asymptotically flat vacuum initial data
- [3] Linear stability of Schwarzschild
- [4] Linear stability of scalar perturbation of nonextremal Kerr BH
- [6] Nonlinear stability of Schwarzschild proven for a class of nontrivial perturbations
- [1] All extremal Kerr BH are unstable to gravitational perturbationalong their event horizon
- [?] Nonlinear stability of Schwarzschild

Dafermos + 2021

Open problem: Nonlin stability Kerr

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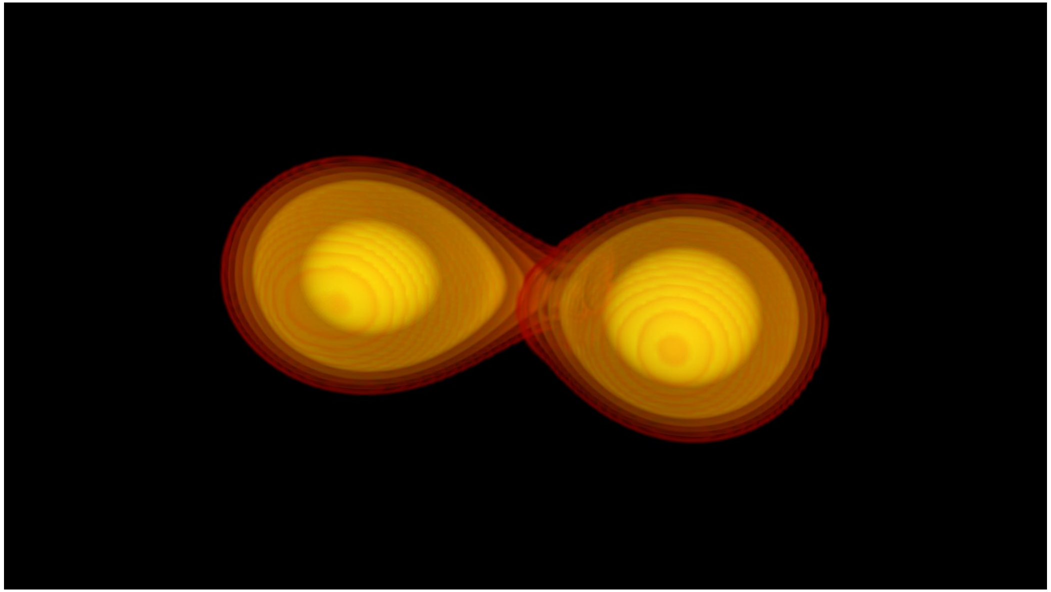
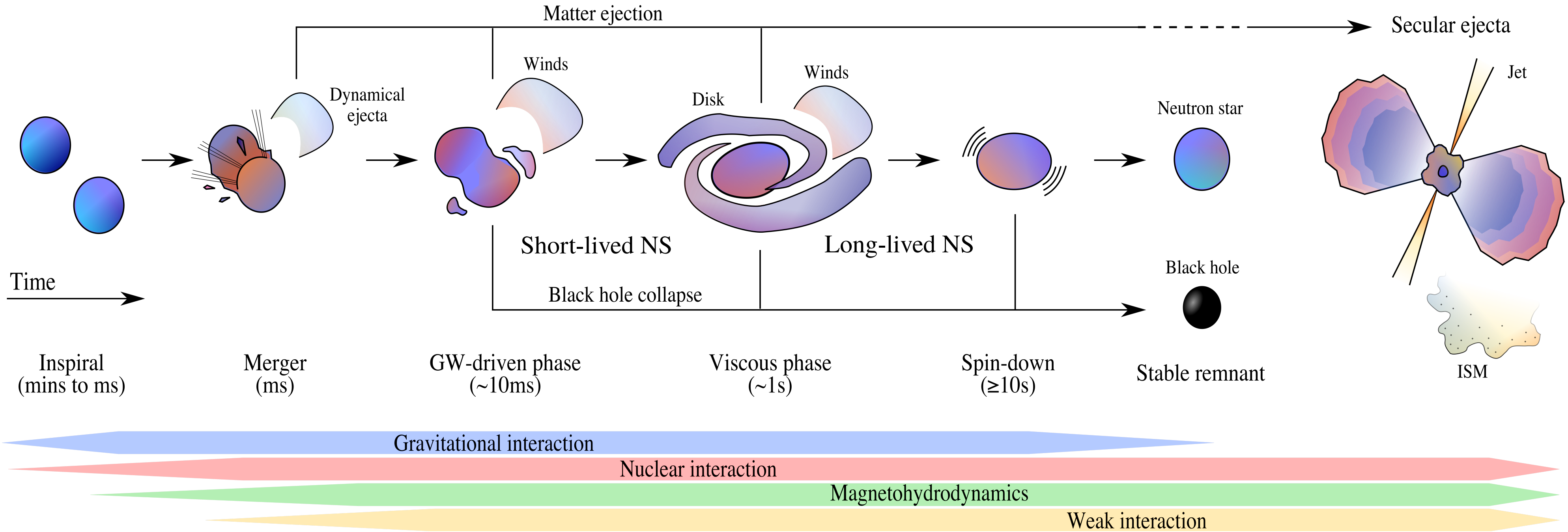
NS in binaries: tides



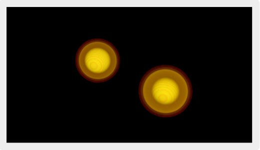
BNS mergers "(2-body dynamics)^4"

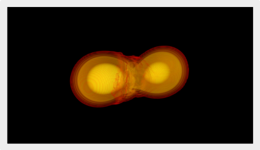


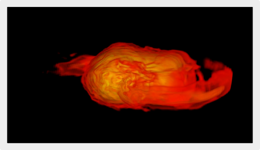
Annual Reviews of Nuclear and Particle Science
Radice, SB, Perego [https://arxiv.org/abs/2002.03863]

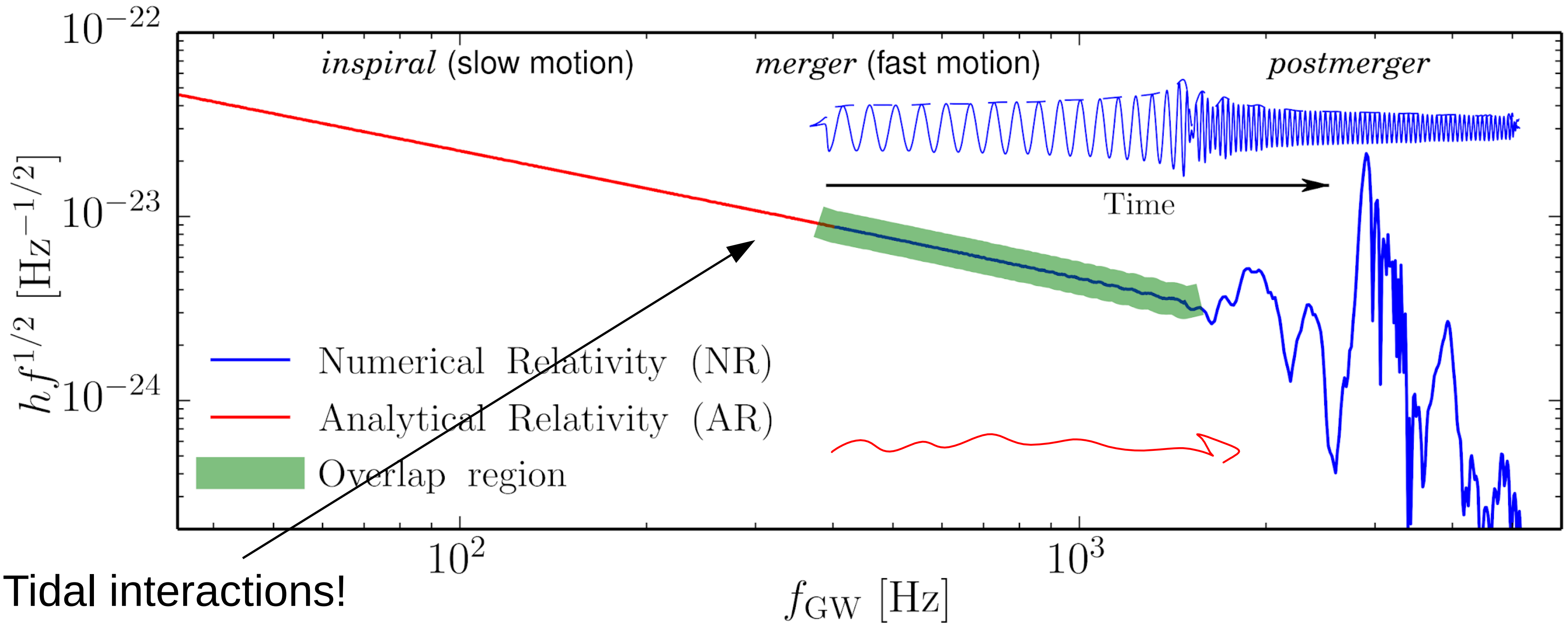
GERG Collection on Binary Neutron Star mergers
SB [https://arxiv.org/abs/2002.03863]

The gravitational-wave spectrum





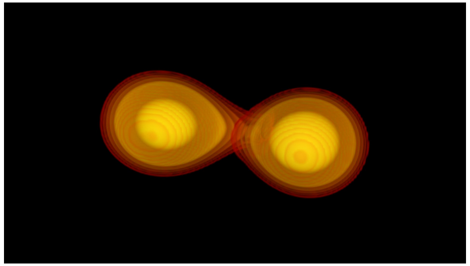




Tidal interactions!

$\tilde{\Lambda}$

Tidal interactions in BNS



$$\kappa_2^T = \frac{3}{2}\left[\Lambda_A\left(\frac{M_A}{M}\right)^4\frac{M_B}{M} + (A \leftrightarrow B)\right]$$

(Damour&Nagar 2009a 2009b)

Hamiltonian (Newtonian limit):

$$H_{\rm EOB} \approx \cancel{Mc^2} + \frac{\mu}{2}\left[p^2 + A(r) - 1\right]$$

$$A(r) = 1 - \frac{2GM}{c^2 r} - \frac{\kappa_2^T}{r^6}$$

✓ short range

✓ attractive

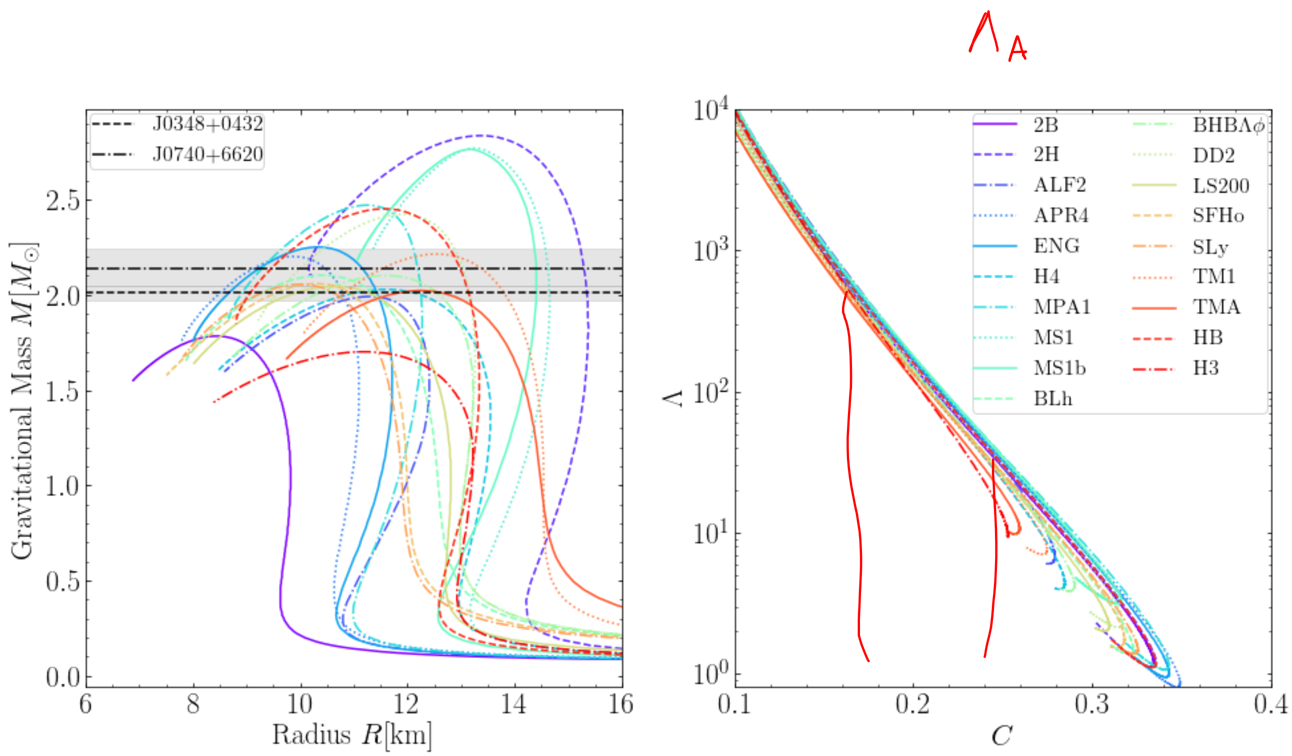
✓ coupling constant ~ Λ_A

Waveform:

$$h \approx A f^{-7/6} e^{-i\Psi(f)} \approx A f^{-7/6} e^{-i\Psi_{\rm p.m.}(f) + i\frac{39}{4}\kappa_2^T (x(f))^{5/2}}$$

Key point: No other binary parameter (mass, radii, etc) enter separately the formalism at LO

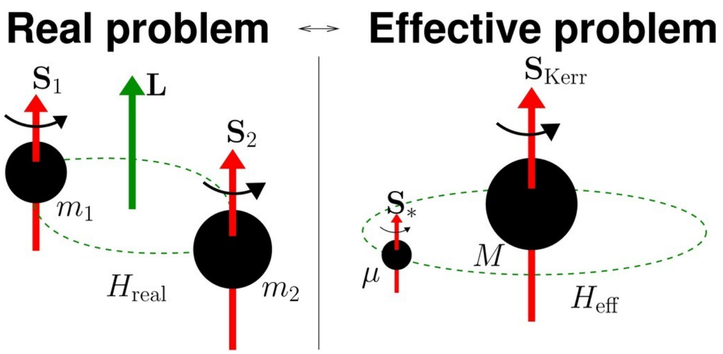
Tidal polarizability coefficients



[Hinderer 2007, Damour&Nagar 2009a, Binnington&Poisson 2009]

Effective-one-body framework in a nutshell

[Buonanno&Damour PRD 2000a, 2000b]



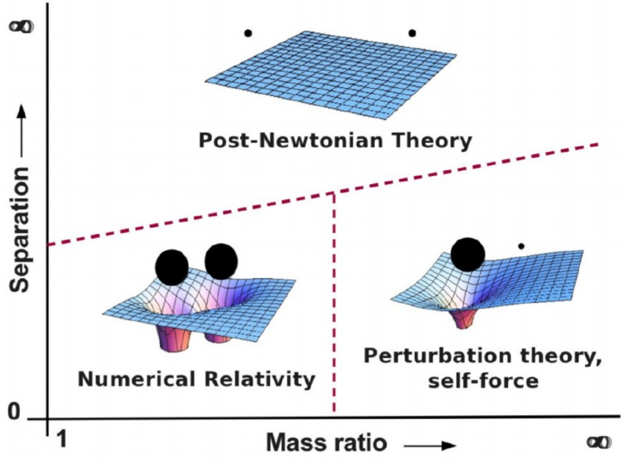
Credit: A.Taracchini/AEI

$$H_{\rm eff} \sim \mu\sqrt{A(u)(1+p_\phi^2 u^2)+p_{r*}^2}$$

$$A(u;\nu;\kappa_2^T) = A^0(u;\nu) + A^T(u;\nu;\kappa_2^T)$$

$$A^0(u;\nu) = 1 - 2u + \nu(\ldots$$

Factorized (resummed) PN waveform [Damour,Iyer,Nagar 2008]
Includes test-mass limit (i.e. particle on Schwarzschild)
Includes post-Newtonian and self-force results
Uses resummation techniques → predictive strong-field regime
Includes tidal interactions (→ BNS) [Damour&Nagar PRD 2010]
Flexible framework → NR informed



Credit: L.Barak

Compact binaries dynamics

The motion and radiation of a system of well separated, strongly self-gravitating ("compact"), bodies can be described by a "matching" approach which consists in splitting the problem into two (Damour 1983; Damour, Soffel, Xu 1991)

(i) the outer problem where one solves field equations in which the bodies are "skeletonized" by worldlines endowed with some global characteristics (such as mass, spin or higher-multipole moments)

(ii) the inner problem where one obtains the near-worldline behavior of the outer solution from a study of the influence of the other bodies on the structure of the fields in an inner world tube around each body

This approach can be used to obtain binary black hole dynamics in post-Newtonian (PN) formalism and to prove that the bodies' finite-size correction enters at 5PN.

Inner problem

How star respond to an external field ?

Definition of multipolar tidal coefficients

Consider a static, spherically symmetric star of mass M perturbed by a stationary, external gravitational quadrupolar field $E_{ij} \sim \partial_i \partial_j \phi^{\text{external}}$. The star is expected to respond to the external field by developing a quadrupole moment Q_{ij}. This phenomeon is analogous to the electric polarizability of a medium that, placed in an external electric field, develops a dipole moment. Assuming a linear response, an (eletric-type) quadrupolar tidal coefficient is defined as

$$\boxed{Q_{ij} = \mu_2 E_{ij}} \tag{1}$$

A more general definition of μ_2 (valid also for other other multipoles) and the framework for the actual calculation can be obtained by the following argument.

In the star local frame and for large radii, the metric coefficient g_{00} (gravitational potential in the weak field) can be written as

$$\frac{1-g_{00}}{2} = -\frac{M}{r} + \frac{3}{2}\frac{Q_{ij}}{r^3}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) + \frac{1}{2}E_{ij}\overbrace{x^i x^j}^{r^2} + \mathcal{O}(r^3) \tag{2}$$

The above expression shows that the tidal coefficient μ_2 in Eq.(1) can be obtained by matching the term growing as $\sim r^2$ to the term falling as $\sim 1/r^3$ of the asymptotics expression of the (perturbed) metric coefficient.

This procedure can be generalized. In the local frame of the body, define the *external gravito-electric and gravitomagnetic tidal moments*

$$G_L := G_{i_1\ldots i_\ell} \quad H_L := H_{i_1\ldots i_\ell} \tag{3}$$

as those *multipoles* of the perturbed metric that grow as r^ℓ. Similarly, the *internal mass and spin multipoles moments*

$$M_L \qquad S_L \tag{4}$$

are those that decay as $r^{-(\ell+1)}$. The multipolar tidal coefficients of the body are then postulated as those relating the linear response of the internal moments to the external ones

$$M_L = \mu_\ell G_L \qquad S_L = \sigma_\ell H_L \tag{5}$$

In a linearly perturbed, stationary star spacetime the asymptotics behaviour of the field uniquely defines these moments (Note: this is different from the vacuum case). In the following only the gravitoeletric sector is discussed since the magnetic sector is analogous.

Calculation of tidal Love numbers

Consider even parity, stationary perturbations of the TOV metric $g^{(0)}_{\alpha\beta}$

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h^{(e)}_{\alpha\beta} \tag{6}$$

The $h^{(e)}_{00}$ coefficient of the perturbed metric can be expressed in terms of a function H that is directly related to the logarithm of the enthalpy perturbation. The perturbative equation for H is

$$\boxed{H'' + C_1 H' + C_0^{(\ell)} H = 0} \quad \text{interior of star} \tag{7}$$

with

$$\begin{aligned} C_0^{(\ell)} &= e^{2\lambda}\left[\frac{\ell(\ell+1)}{r^2} + 4\pi\,(\rho + P)\frac{d\rho}{dP} + 4\pi\,(5\rho + 9P)\right] - 4\phi'^2 \\ C_1 &= \frac{2}{r} + e^{2\lambda}\left[\frac{2m}{r} + 4\pi r\,(p - \rho)\right] \end{aligned}$$

(A similar equation hold for odd parity perturbations). In the star interior, Eq.(7) needs to be solved numerically together with the background equations and by specifying a EOS. In the star exterior, $\rho = P = 0$ and $m = M$, and the equation reduces to the *associated Legendre equation* with variable $x = r/M - 1$. The general solution can be expressed in terms of the *associated Legendre functions*

$$H^{\text{outer}} = a_P \hat{P}_{\ell 2}(x) + a_Q \hat{Q}_{\ell 2}(x) \qquad \text{exterior} \tag{8}$$

The coefficients a_P and a_Q are to be determined by the boundary conditions, in particular by the matching with the interior solution. The ratio $a_\ell := a_Q / a_P$ can be determined by requiring the continuity of the logarithmic derivative at the surface

$$y_\ell(r) = \frac{r H'(r)}{H(r)} \qquad \text{Radius match (log derivatives)} \tag{9}$$

i.e.

$$y_\ell^{\text{inner}}(R) = y_\ell^{\text{outer}}(R) = (1 + x_R) \frac{\hat{P}'_{\ell 2}(x_R) + a_\ell \hat{P}_{\ell 2}(x_R)}{\hat{Q}'_{\ell 2}(x_R) + a_\ell \hat{Q}_{\ell 2}(x_R)} \tag{10}$$

with $x_R = R/M - 1 = 1/C - 1$. Note this is a nontrivial statement to check, since it depends on the EOS (and the regularity of the matter fields at the surface, e.g. the sound speed) and on the fact that the perturbed star surface does not coincide with the background star radius. Solving Eq.(10) for a_ℓ gives

$$a_\ell = -\frac{\hat{P}'_{\ell 2}(x_R) + C y_\ell(R) \hat{P}_{\ell 2}(x_R)}{\hat{Q}'_{\ell 2}(x_R) + C y_\ell(R) \hat{Q}_{\ell 2}(x_R)} \tag{11}$$

This coefficient can be now directly related to the tidal coefficient μ_ℓ. The asymptotic behaviour of the outer solution is determined by

$a_\ell \sim \frac{\hat{P}}{\hat{Q}}$

$$\hat{P}_{\ell 2}(x) \sim \left(\frac{r}{M}\right)^{\ell+1} \qquad \hat{Q}_{\ell 2} \sim \left(\frac{M}{r}\right)^{\ell+1} \tag{12}$$

such that the growing and falling part of the perturbation are

$$\left(h_{00}^{(e)}\right)^{\text{growing}} \sim a_P \left(\frac{r}{M}\right)^{\ell+1} Y_{\ell m} \qquad \left(h_{00}^{(e)}\right)^{\text{falling}} \sim a_P \left(\frac{r}{M}\right)^{-(\ell+1)} Y_{\ell m} \tag{13}$$

The matching gives (reintroducing the constants G and c)

$$\boxed{(2\ell-1)!!\,G\mu_\ell = a_\ell \left(\frac{GM}{c^2}\right)^{2\ell+1}} \tag{14}$$

$G\mu_\ell$ has dimension of $[\mathrm{length}]^{2\ell+1}$. The *tidal Love numbers* are defined as the dimensionless combination

$$\boxed{k_\ell := \frac{1}{2} a_\ell C^{2\ell+1} = -\frac{1}{2} C^{2\ell+1} \frac{\hat{P}'_{\ell 2}(R/M-1) - C y_\ell(R) \hat{P}_{\ell 2}(R/M-1)}{\hat{Q}'_{\ell 2}(R/M-1) + C y_\ell(R) \hat{Q}_{\ell 2}(R/M-1)}} \tag{15}$$

The *tidal polarizability parameters* of a star often employed in gravitational-wave astronomy are defined as

$$\Lambda_\ell := \frac{2\,k_\ell}{(2\ell-1)!!C^{2\ell+1}} \qquad (16)$$

Λ_2 for star A

Outer problem

Effective action

Up to 5PN order ($\mathcal{O}(v/c)^{10}$) the motion of two body compact bodies of mass M^A $A=1,2$ is described by the effective action

$$S = \int \frac{R}{16\pi G} - \sum_{A=1}^{2} \int M^A d\tau_A \qquad (17)$$

where an "apportune regularization" must be introduced to deal with the point-mass source term in the EFE. Note that the calculation of the 5PN dynamics is not yet completed: for nonspinning bodies, the conservative dynamics is fully known at 4PN, while the waveform at 3.5PN.

Finite-size effects enter at 5PN and the action needs to be augment with the term

$$S_{\mathrm{nonminimal}} = \sum_{A,\ell} \left[\frac{1}{2}\frac{\mu_\ell}{\ell!}\int (G_L^A)^2\, d\tau_A + \frac{1}{2}\frac{\ell}{(\ell+1)}\frac{\sigma_\ell^A}{\ell!}\frac{1}{c}\int (H_L^A)^2 d\tau_A \right] \qquad (18)$$

Tidal Lagrangian at leading order

At leading order the tidal Lagrangian for body A is given by

$$L_T^A \sim \mu_2 \, (G_L^A)^2 \tag{19}$$

where the external tidal moment is calculated on the worldline z_A^a of the body as

$$G_L^A \sim \partial_{_L} U^{\text{external}} = \partial_L \left(\frac{G M^B}{|r_{\mathrm{AB}}|} \right) \tag{20}$$

with $r_{\mathrm{AB}} = |z_A^a - z_B^a|$. The calculation uses the formalism of symmetric trace-free (STF) tensors for multipolar expansions, and in particular the expression

$$\partial_L^A \left(\frac{1}{r_{\mathrm{AB}}} \right) = (-1)^\ell (2\ell - 1)!! \, \frac{\hat{n}^L}{r_{\mathrm{AB}}^{\ell+1}} \tag{21}$$

where $\hat{n}^L$ is the STF projection of the of the unit vectors $n^a = (z_A^a - z_B^a) / r_{\mathrm{AB}}$. The result is

$$L_T^A \sim + \sum_\ell \frac{(2\ell - 1)!!}{2} \mu_A \frac{(G M^B)^2}{r_{\mathrm{AB}}^{2\ell+2}} = + \sum_\ell k_\ell^A G \, (M^B)^2 \frac{R_A^{2\ell+1}}{r_{\mathrm{AB}}^{2\ell+2}} \tag{22}$$

The interaction is proportional to the Love numbers (or to the tidal polarizability parameters Λ_ℓ^A), it is attractive, and it is *short-range*, e.g. the first term scales as $\sim 1 / r^6$.

Was this useful? Quick self-check:

- What are the two main characteristics of compact objects?
- What is compactness? Make a table with order of magnitude values for mass, radius, compactness, average density for WD, NS, BH (without books/googling)
- What is the origin of pressure support in WD? What is the order of magnitude of the critical density? What are the values of the adiabatic index above/below the critical density?
- What is the physical origin of the Chandrasekhar mass? Can you provide an order of magnitude argument for the existance of $M_{\rm Ch}$?
- What is the Buchdal limit?
- What is the maximum NS mass $M_{\rm max}$? Can you give an upper bound?
- What is the difference between the WD and the NS EOS?
- What is a black hole? Why and How black holes form from stars?
- What is the LSO? And how can you estimate the merger frequency of black hole binaries?
- What is the RWZ equation? What boundary conditions are usually imposed?
- Are black holes stable? How can one formulate the stability problem?
- What are the Love numbers and the tidalpolarizability parameters? How can they be computed?

Now you can do the proposed exercises and check the references mentioned in the lectures!