Gravitational wave theory

Danièle Steer
APC, University of Paris

1) General introduction

2) From Einsteins equations to GWs: basic linearized theory
- – wave equation
- – polarizations, etc
- – generation of GWs
- – energy carried by GWs

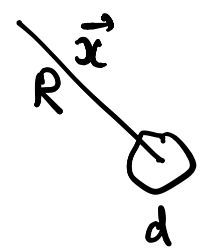
3) example: inspiral of compact binary system
- – wave form
- – characteristic scales, link with observations

4) The sources: on black holes (BH) and neutron stars (NS)

5) GWs in an expanding universe; cosmology
- – binaries at cosmological distance
- – basics on the SGWB

- Weak field expansion
  - $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad h_{\mu\nu} \ll 1$
  - $\partial_\mu T^{\mu\nu} = 0$
- Far from the source $R = |\vec{x}| \gg d$
- Then did a $(v/c) \ll 1$ expansion. $\leftarrow$
- $$h^{TT}_{ij}(t, \vec{x}) = \frac{4G}{c^4 R} \Lambda_{ij,kl}(\vec{n}) \, \partial_t^2 M_{ij}\left(t - R/c\right) + \dots$$

$$M_{ij} = \int d^3\vec{x}' \, T^{00}(t, \vec{x}') \, x'^i x'^j .$$



$$\vec{n} = \frac{\vec{x}}{|\vec{x}|}$$

- $$t^{GW}_{\alpha\beta} = \frac{c}{32\pi G} \left\langle \partial_\mu h_{\alpha\beta} \, \partial_\nu h^{\alpha\beta} \right\rangle$$ $\nwarrow$ average over several wavelengths of the GW

  $\rightarrow$ energy loss (shrinking of orbits)

  $\rightarrow$ angular momentum loss.

- λ_{GW} ? - Stress energy tensor $T_{\mu\nu}$ of source charac. by angular freq. Ω, typical velocity $v \sim \Omega d$

  - freq. of GWs $f \sim 2\Omega$

$$\lambda_{GW} \sim \frac{c}{f} \sim \frac{c}{\Omega}$$

$$\sim \left(\frac{c}{v}\right) d$$

$$\Rightarrow \lambda_{GW} \gg d$$

- For a consistent expansion, for self grav systems,

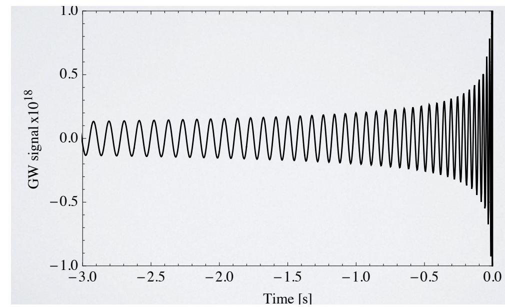
$$\left(\frac{v}{c}\right)^2 \sim \frac{R_s}{d}$$

$\Rightarrow$ $\boxed{\text{P. N expansion (post Newtonian)}}$ → see lectures by S. Babak.

Summary: taking into account the energy loss into gravitational radiation, and in linearized theory

$$h_+(t, i, \phi_c) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f(\tau))^{2/3}\frac{1+\cos^2(i)}{2}\cos(\phi(\tau))$$

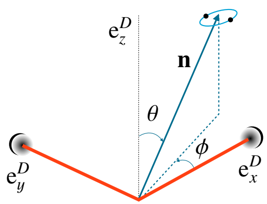
$$h_\times(t, i, \phi_c) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f(\tau))^{2/3}\cos(i)\sin(\phi(\tau))$$



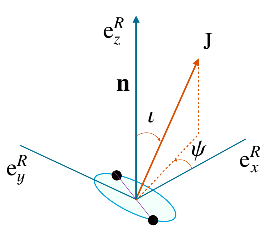
$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256\tau}\right)(G\mathcal{M})^{-5/8}$$

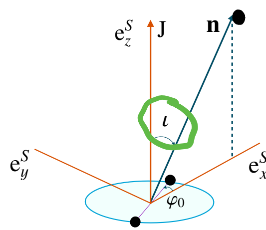
$$\phi = \phi_c + 2\pi\int_{t_c}^{t} dt' f(t') = -2\left(\frac{\tau}{5G\mathcal{M}}\right)^{5/8}$$



Detector frame



Radiation frame

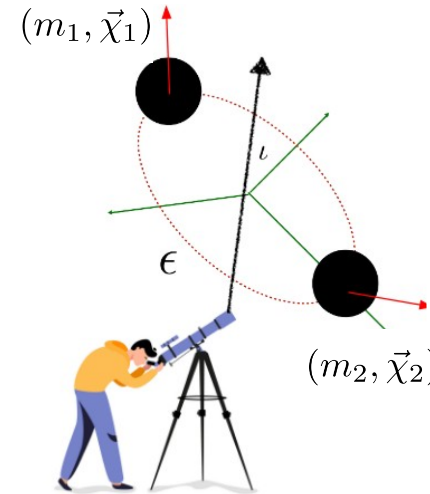


Source frame

- In an ideal world, one could measure two polarisations separately:

$$h_+(t,i,\phi_c) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f(\tau))^{2/3}\frac{1+\cos^2(i)}{2}\cos(\phi(\tau))$$

$$h_\times(t,i,\phi_c) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f(\tau))^{2/3}\cos(i)\sin(\phi(\tau))$$



– From their phase evolution determine redshifted chirp mass; $\mathcal{M}$
– from the ratio of the two the inclination angle; $\rightarrow i$
– and hence from their amplitude the distance R to the source

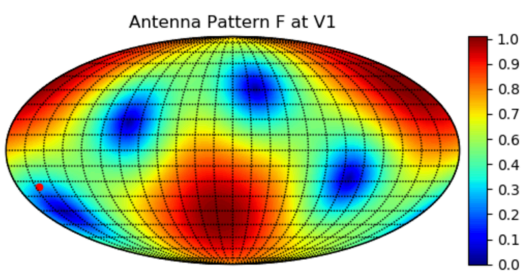
$\frac{h_+}{h_\times}$

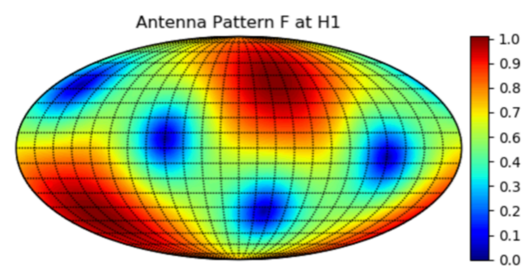
- But that's not the reality of GW interferometers
- Signal at detector *a*

$$h_a(t,\alpha,\delta,\ldots) = F_a^+(t,\alpha,\delta,\ldots)h_+ + F_a^\times(t,\alpha,\delta,\ldots)h_\times$$

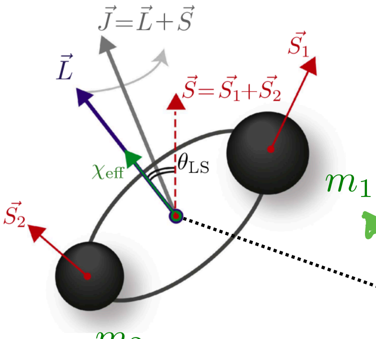
Large errors on R due to degeneracy with i

antenna pattern functions, depending on the geometry of the detector, position of the source in the sky defined by declination and right ascension, the polarisation angle etc (For Ligo-Virgo can ignore t dependence.)

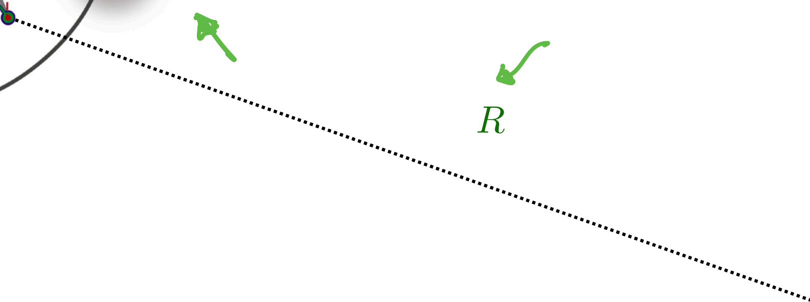




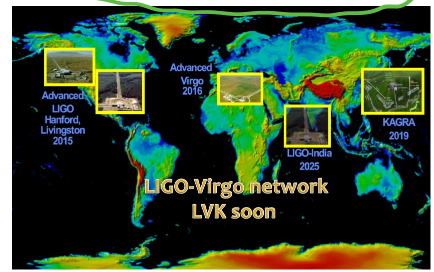
3) GWs from binaries: Characteristic scales

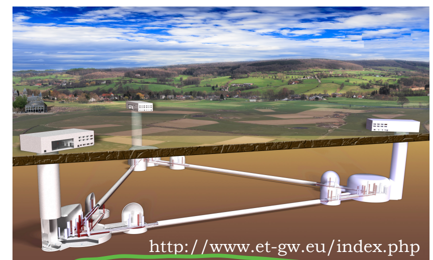


R

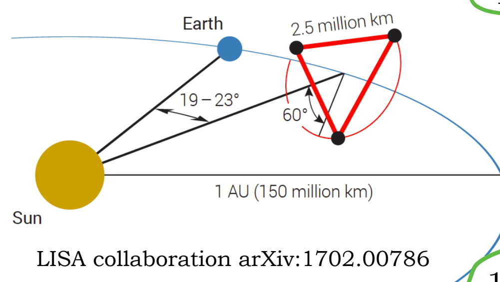


$10\,\mathrm{Hz} < f < 5\,\mathrm{kHz}$





$1\,\mathrm{Hz} < f < 10^4\,\mathrm{Hz}$



LISA collaboration arXiv:1702.00786

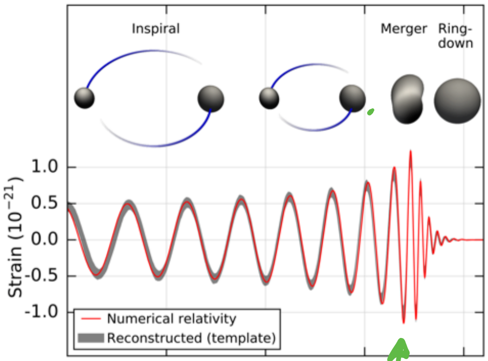
$10^{-4}\,\mathrm{Hz} < f < 1\,\mathrm{Hz}$

– Inspiral phase, neglecting expansion (z<<1)

$$f_{\rm GW} = \frac{1}{\pi}(GM_c)^{-5/8}\left(\frac{5}{256\,\tau}\right)^{3/8}$$

$$\tau = t - t_c$$

$$\mathcal{M} = M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$



(dominant quadrupolar mode calculated to lowest order in the Newtonian expansion, for *point particles of mass m1 and m2*; no tidal effects, no spins,…, assuming circular orbit; and using quadrupole formula)

• *Merger frequency:* Assuming merger at innermost stable circular orbit (ISCO)

$$\boxed{f_{\rm merger} = \frac{1}{6^{3/2}\pi}\left(\frac{c^3}{GM}\right)}$$

M=total mass $= m_1 + m_2$
Follows from Keplers laws

$\boxed{a \sim 6M}$

Ω

$\hookrightarrow f_{GW}$

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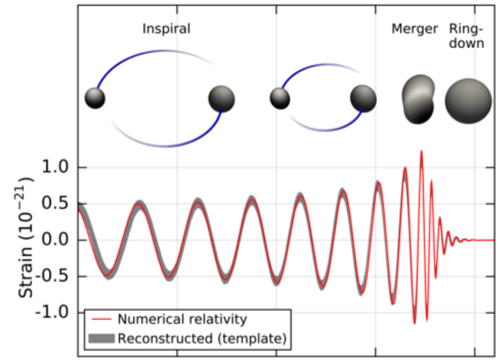

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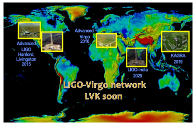
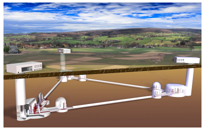
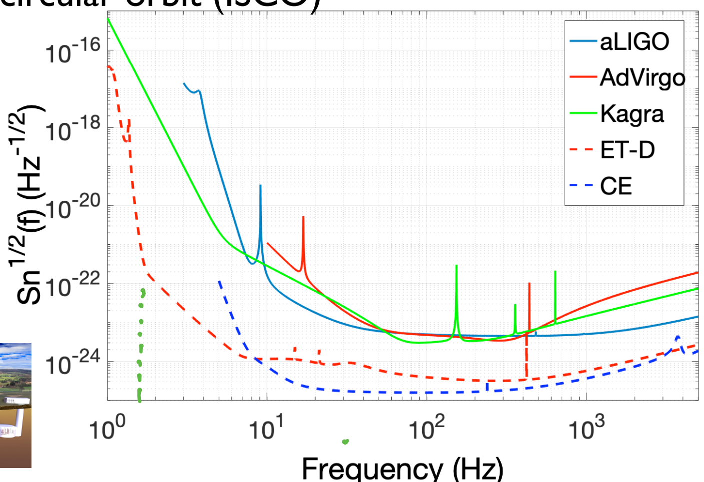
Follows from Keplers laws

- BNS, $m_{1,2} \sim 1.4 M_\odot$ $\quad f_{\rm merger} \sim 1.5\,{\rm kHz}$
- stellar mass BHs, $m_{1,2} \sim 35 M_\odot$ $\quad f_{\rm merger} \sim 60\,{\rm Hz}$







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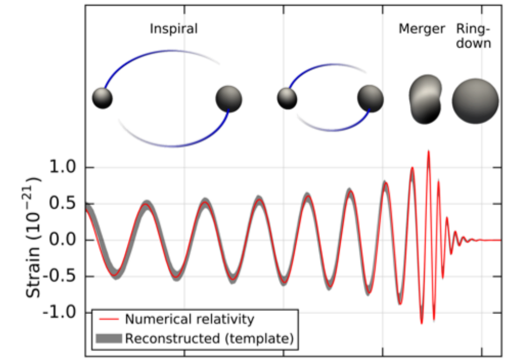

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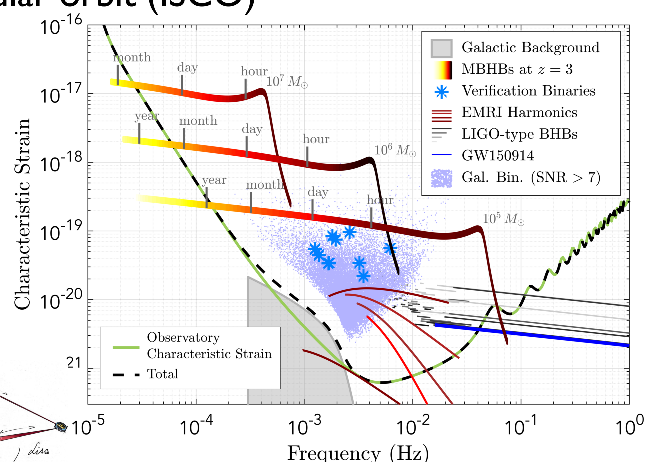
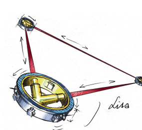


• BNS, $m_{1,2} \sim 1.4 M_\odot$ $\quad f_{\mathrm{merger}} \sim 1.5\,\mathrm{kHz}$

• stellar mass BHs, $m_{1,2} \sim 35 M_\odot$ $\quad f_{\mathrm{merger}} \sim 60\,\mathrm{Hz}$

• Supermassive BBHs, $m_{1,2} \sim 10^6 M_\odot$ $\quad f_{\mathrm{merger}} \sim 10^{-3}\,\mathrm{Hz}$

LISA



- *Time to merger*

$$f_{\rm GW} = \frac{1}{\pi}(GM_c)^{-5/8}\left(\frac{5}{256\,\tau}\right)^{3/8}$$

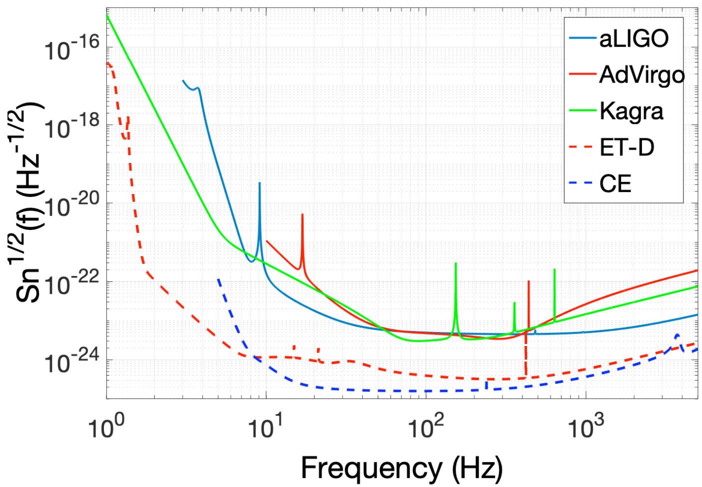
$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

If GWs enter frequency band of a detector at observed frequency $f_{\rm low}$

$$\boxed{T \sim 10^{-3} f_{\rm low}^{-8/3}\left(\frac{c^3}{G\mathcal{M}}\right)^{5/3}}$$

$f_{\rm low} < f < f_{\rm merger}$

- BNS, entering LIGO-Virgo detector window at observed frequency $f_{\rm low} \sim 20\,{\rm Hz}$ $T \sim 4\,{\rm min}$
  $m_{1,2} \sim 1.4 M_\odot$ $f_{\rm merger} \sim 1.5\,{\rm kHz}$
- BNS, entering ET detector window at observed frequency $f_{\rm low} \sim 1\,{\rm Hz}$ $T \sim 5\,{\rm days}$



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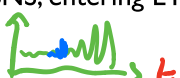
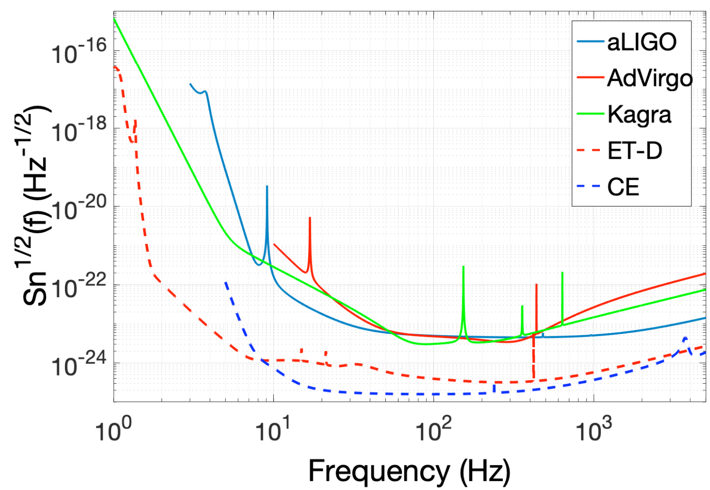


=> cannot neglect the rotation of the earth

=>Given the merger rates for BNS, BBH and BH-NS, expect a typical BNS signal will be overlapped by a number of BBH signals, which may merge at similar times

=> need very accurate waveforms.

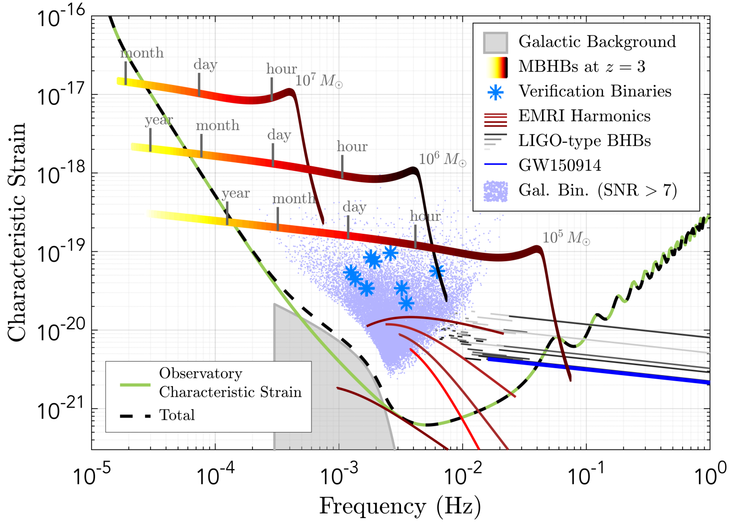


- stellar mass BHs entering LIGO-Virgo detector window

  $m_{1,2} \sim 35 M_\odot$ $\quad T \sim 0.1\,{\rm s}$

  $f_{\rm merger} \sim 60\,{\rm Hz}$

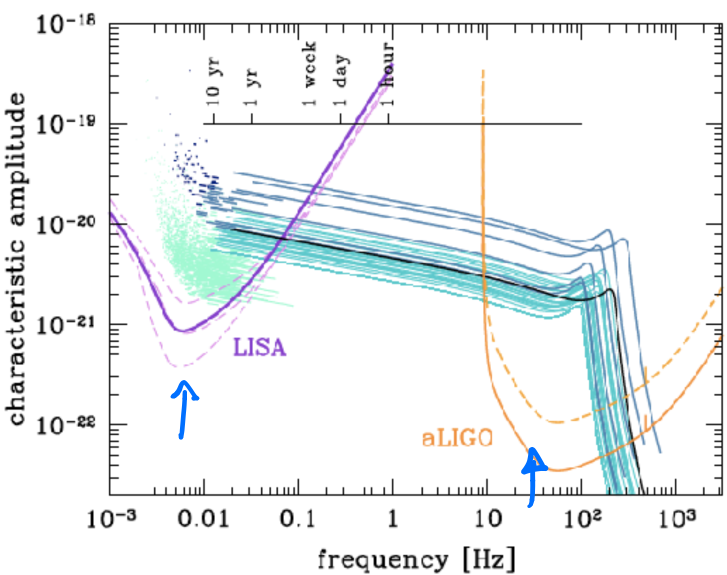
LISA



- Supermassive BBHs,

$$m_{1,2} \sim 10^6 M_\odot$$

$$T \sim 1\,\text{month}$$



- stellar mass BHs entering LISA detector window

$m \sim 30 M_\odot$

$$f_{\text{low}} \sim 10^{-2}\,\text{Hz}$$

$$T \sim 20\,\text{yrs}$$

Freqs in L-V freq. band.

- *Amplitude/distance*

$$\boxed{h \sim \frac{4c}{R}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(\pi f)^{2/3}}$$

- stellar mass BHs in LIGO-Virgo

- $m_{1,2} \sim 35 M_\odot$
- $f_{\rm merger} \sim 60\,\mathrm{Hz}$

$h \sim 10^{-21}$, $R \sim 400\,\mathrm{Mpc}$

converted to a redshift *assuming* the Planck values of cosmological parameters (z ~ 0.1)

$\left[\begin{array}{l} M_1 \sim 1.5 M_\odot \\ M_2 \sim 30 M_\odot \end{array}\right.$ $\left[\begin{array}{l} M \\ M. \end{array}\right.$

- at merger separated by small distances $a \sim \left(\frac{GM}{f^2}\right)^{1/3} \sim \mathcal{O}(10^2)\,\mathrm{km}$

tiny compared to normal length scale of stars (10^3km for WDwarfs -> millions of km for main sequence)

Dealing with small heavy objects, otherwise they'd have collided and merged before reaching such close proximity. Candidates: are BHs and NSs.

- ***Amplitude/distance***

$$\boxed{h \sim \frac{4c}{R}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(\pi f)^{2/3}}$$

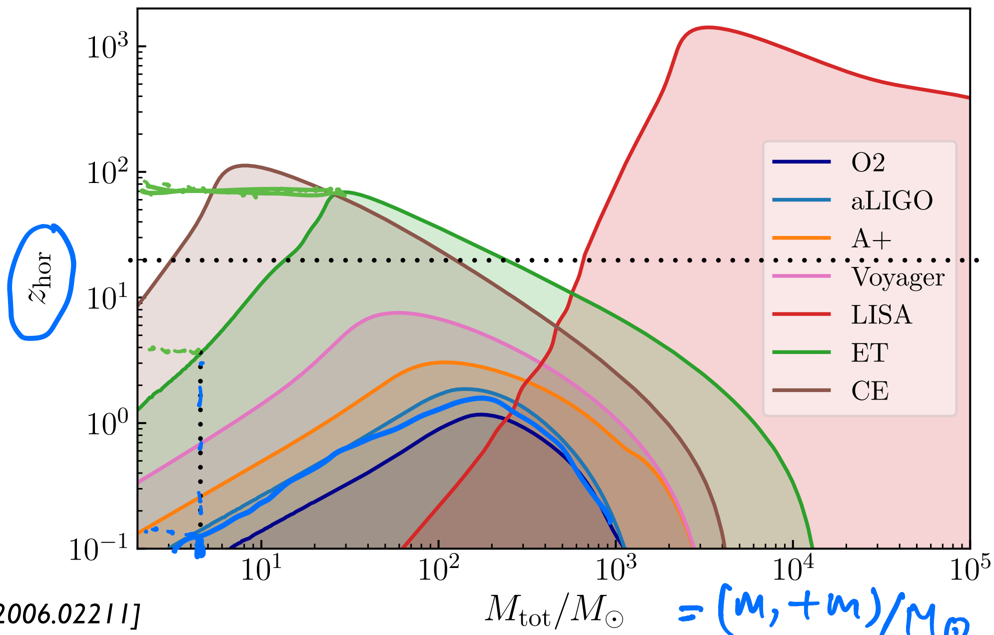
- stellar mass BHs in LIGO-Virgo

$$m_{1,2} \sim 35 M_\odot$$
$$f_{\mathrm{merger}} \sim 60\,\mathrm{Hz}$$

$$h \sim 10^{-21}, R \sim 400\,\mathrm{Mpc}$$

converted to a redshift *assuming* the Planck values of cosmological parameters (z ~ 0.1)

Horizon redshift as a function of total source frame mass for an SNR detection threshold of rho=8. For LISA assumes 4 yrs obsv.



[2006.02211]

z>20; dark era preceding birth of first stars: any detected BHs must be primordial

FLRW cosmo. metric

Conclusion: should not neglect expansion of the universe

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

4) A very brief word on the compact sources: Black holes (BH) and neutron stars (NS)

- To understand the astrophysical significance of LIGO-Virgo detections, important to discuss how stars *end their lives*.

- A normal star (e.g sun) supported against contracting under its own gravity by radiative pressure produced by thermonuclear reactions in its core.

- However, eventually all nuclear fuel will be used up.

- Star starts to shrink under gravitational self-attraction. For this to be balanced, different source of pressure required.

- If this balance is to last forever then this new source of pressure must be *non-thermal* because the star will eventually cool.

Quantum Fermi pressure (degeneracy pressure from Pauli exclusion principle)

– at sufficiently high densities, matter becomes degenerate.
– For a particle of mass m with number density n, can estimate temperature T when this occurs:

$$\Delta x \sim n^{-1/3} \qquad \Delta p \sim \sqrt{mkT}$$

Violate Heisenberg uncertainty principle $\Delta x \Delta p \geq \hbar$ when

$$T \leq T_F \qquad \text{Fermi temperature} \qquad T_F = \frac{\hbar^2}{2km}(3\pi^2 n)^{3/2}$$

WHITE DWARF

– a star in which gravity is balanced by *electron* degeneracy pressure. Electrons degenerate, not heavier ions
– Sun will end its life as a white dwarf
– More dense than normal stars
– Newtonian gravity with a fluid description of matter, with relevant equation of state (Chandrasekhar limit):

$$M \lesssim 1.5 M_\odot$$

– A star more massive than this cannot end its life as a white dwarf (unless it somehow sheds some of its mass, SN explosions).

$$\Phi \equiv \frac{Gm}{rc^2} \sim 10^{-4}$$

If $\Phi \ll 1$ can use Newtonian limit of GR

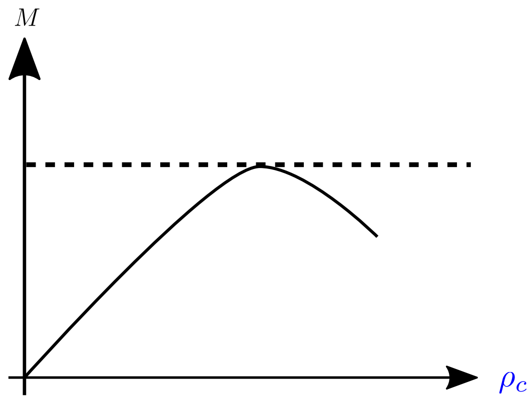
NEUTRON STAR

– when density approaches nuclear density, neutron degeneracy pressure. Neutrons, e- and protons degenerate
– tiny objects (most compact objects known without event horizons)
– Neutron star of mass $M_\odot$ would have radius of 10km ($R_\odot = 7 \times 10^5 \mathrm{km}$)
– Very dense $\bar{\rho} \gtrsim \rho_0 = 2{,}7 \times 10^{14}$ g cm^{-3} with strong magnetic fields $B \sim 10^4 - 10^{12}$ T
– and very relativistic. Newtonian potential

$$\Phi \equiv \frac{Gm}{rc^2} \sim 0.1 - 0.2$$

– Independently of the equation of state $P(\rho)$ of the NS

$$\boxed{M \lesssim 3M_\odot}$$



density at center
of the star

4) A word on on Black holes (BH) and neutron stars (NS)

- if stars of mass $M \gtrsim 3M_\odot$ do not shed matter sufficiently, they probably will end up as black holes.

Simplest: Schwarschild (static, non-rotating) solution:

In *Schwarzschild coordinates* (t, r, θ, ϕ), the Schwarzschild solution is

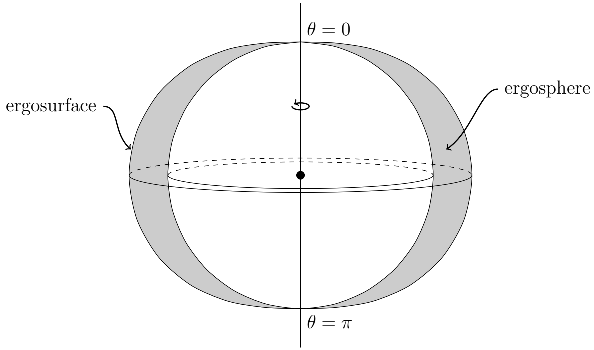
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$a = 0$ and event horizon at $R_{\text{sch}} = \dfrac{2GM}{c^2}$

Birkhoff's theorem: any spherically symmetric solution of the vacuum Einstein equations is isometric to the Schwarzschild solution

implies that the spacetime outside any spherical body is described by the time-independent (exterior) Schwarzschild solution. This is true even if the body itself is time-dependent. For example, consider a spherical star that "uses up its nuclear fuel" and collapses to form a white dwarf or neutron star. The spacetime outside the star will be described by the static Schwarzschild solution even during the collapse.

Next simplest: Kerr (unique stationary, rotating) solution:



$a_{\text{max}} = 1$ more complicated causal structure, with ergosphere.

$$R^{\text{max}}_{\text{Kerr}} = \frac{GM}{c^2}$$

• if stars of mass $M \gtrsim 3M_\odot$ do not shed matter sufficiently, expected to end up as black holes.

• BHs are very small: For

$$M = M_\odot$$

$$R_\odot \simeq 7 \times 10^5 \mathrm{km}$$
$$R_{WD} \simeq 7 \times 10^3 \mathrm{km}$$
$$R_{NS} \simeq 10 \mathrm{km}$$
$$R_{BH} \simeq 3 \mathrm{km}$$

• Newtonian potentials

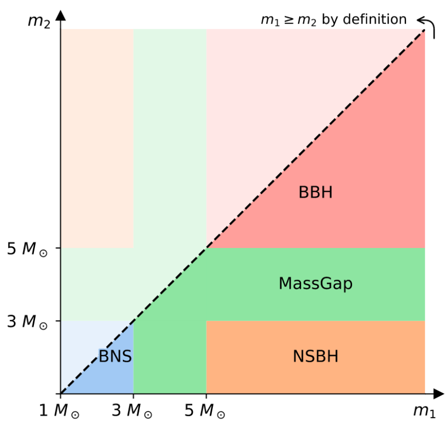
$$\Phi \equiv \frac{Gm}{rc^2}$$

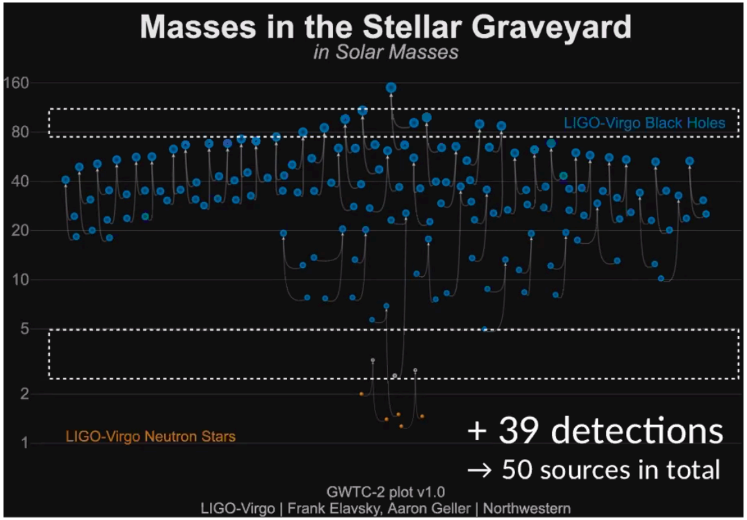
$$\Phi_\odot \simeq 10^{-6}$$
$$\Phi_{WD} \simeq 10^{-4}$$
$$\Phi_{NS} \simeq 0.2$$
$$\Phi_{BH} \simeq 1/2$$

to 10^{-3}

• Spectrum of BH masses expected to have some breaks…

• Mass gap?





•Physics of mass gap very interesting, and the upper one may be very useful for doing late-time cosmology with GWs.

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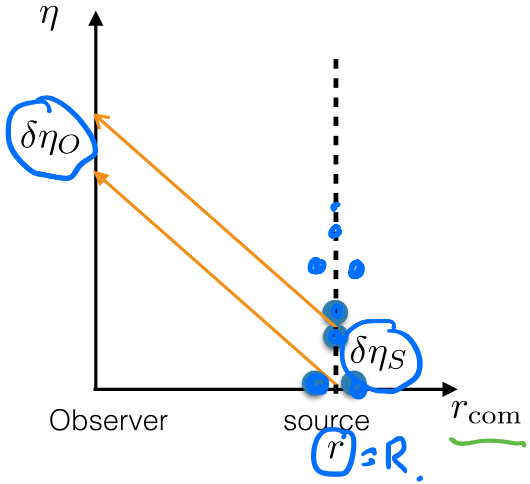
5) Inspiral of compact binaries at cosmological distances

Turn on expansion in FRW universe.

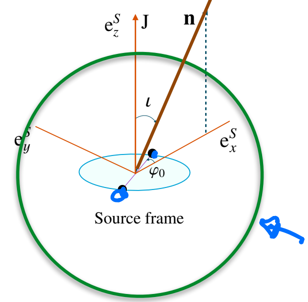
$$ds^2 = \underbrace{-dt^2 + a^2(t)d\vec{x}^2}_{\uparrow} = \underbrace{a^2(\eta)[-d\eta^2 + d\vec{x}^2]}$$

$$dt^2 = a^2(t)\, d\eta^2$$



Idea: in local wave-zone of the source (scales large relative to source, small relative to Hubble), know solution

$$\frac{1}{H_0} \gg R \gg d$$



$$r_{\rm phys} = a(t_{\rm emis})r$$

$$a(t) \sim \text{constant}$$

Then propagate it in FRW space-time to observer

– amplitude redshifts, plus time dilatation

$$dt_O = \frac{a(t_O)}{a(t_S)}dt_S = (1+z)dt_S$$

$$\boxed{f_O = \frac{f_S}{(1+z)}}$$

S = source
O = observer

Close to source, as measured by time t_S of clock of the source,

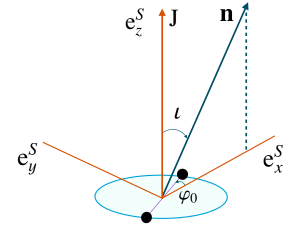
$$h_+(t_S, i) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f_S(t_S^{\rm ret}))^{2/3}\left(\frac{1+\cos^2 i}{2}\right)\cos(2\Phi_S(t_S^{\rm ret}))$$

$$h_\times(t_S, i) = \frac{4}{R}(G\mathcal{M})^{5/3}(\pi f_S(t_S^{\rm ret}))^{2/3}\cos i \sin(2\Phi_S(t_S^{\rm ret}))$$

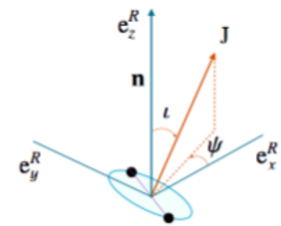
$R = r_{\rm phys} = a(t_{\rm emis})r$

frequency of GW

time to coalescence $t_S^{\rm ret} = t_S - t_c$



Source frame



Radiation frame

Quadruple formula

$$\frac{df_S}{dt_S} = \frac{96}{5}\pi^{8/3}(G\mathcal{M})^{5/3}f_S^{11/3}$$

source frame chirp mass

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$$f_S(t_S^{\rm ret}) = \frac{1}{\pi}\left(\frac{5}{256\, t_S^{\rm ret}}\right)^{3/8}(G\mathcal{M})^{-5/8}$$

$$\Phi_S(t_S) \equiv \Phi_c + 2\pi\int_{t_c}^{t_S} dt'_S f_S(t'_S) = -2\left(\frac{t_S^{\rm ret}}{5G\mathcal{M}}\right)^{5/8} + \Phi_c$$

From source to observer:

- Perturbed FRWL metric (ignoring scalars and vectors):

$$ds^2 = -dt^2 + a^2(t)[(\delta_{ij} + h_{ij})dx^i dx^j]$$

$= \bar{g}_{\mu\nu} + h_{\mu\nu}$

$|h_{ij}| \ll 1$

$h^i_i = \partial_j h^j{}_i = 0$

- from Einstein equations

$\dot{} = \frac{d}{dt}$

$dt^2 = a^2 d\eta^2$

$$\ddot{h}_{ij}(\vec{x},t) + \underbrace{3H\dot{h}_{ij}(\vec{x},t)}_{H = \dot{a}/a} - \underbrace{\frac{\nabla^2}{a^2}h_{ij}(\vec{x},t)}_{} = \cancel{16\pi G\Pi^{TT}_{ij}(\vec{x},t)} = 0$$

source: tensor anisotropic stress

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

- away from the source, in conformal time and Fourier space

$' = \frac{d}{d\eta}$

$$\boxed{h''(\vec{k},\eta) + 2\mathcal{H}h'(\vec{k},\eta) + k^2 h(\vec{k},\eta) = 0}$$

$\mathcal{H} = a'/a$

with $\mathcal{H} = \frac{a'}{a} = \frac{2}{\eta}$

$\boxed{\mathcal{H} = 2/\eta}$ in matter era.

- Change of variable $Q(\vec{k},\eta) = a(\eta)h(\vec{k},\eta)$

$$\boxed{Q'' + \left(k^2 - \frac{a''}{a}\right)Q = 0}$$

- Thus for sub-Hubble modes

$$\boxed{h(\vec{k},\eta) = \frac{A(\vec{k})}{a(\eta)}e^{ik\eta} + \frac{B(\vec{k})}{a(\eta)}e^{-ik\eta}}$$

<u>Plane waves with redshifting amplitude</u>

- **from source to observer**

$$h_+(t_S,i) = \frac{4}{a(t_O)R}(G\mathcal{M})^{5/3}(\pi f_S(t_S^{\rm ret}))^{2/3}\left(\frac{1+\cos^2 i}{2}\right)\cos(2\Phi_S(t_S^{\rm ret}))$$

$$h_\times(t_S,i) = \frac{4}{a(t_O)R}(G\mathcal{M})^{5/3}(\pi f_S(t_S^{\rm ret}))^{2/3}\cos i\,\sin(2\Phi_S(t_S^{\rm ret}))$$

still measured in source clock; want it in the observers clock.

$$f_S = (1+z)f_O$$

$$dt_O = (1+z)dt_S$$

$$\Phi_S(t_S) = 2\pi\int_{t_{c,S}}^{t_S} dt'_S f_S(t'_S) = 2\pi\int_{t_{c,O}}^{t_O} dt'_O f_O(t'_O) = \Phi_O(t_O)$$

phase is constant along null geodesics

$$\boxed{h_\times(t_S,i) = \frac{4}{a(t_O)R}(G\mathcal{M})^{5/3}(\pi f_S(t_S^{\rm ret}))^{2/3}\cos i\,\sin(2\Phi_S(t_S^{\rm ret}))}$$

$$f_S = (1+z) f_O$$

$$h_\times(t_O,i) = \frac{4}{a(t_O)R}(G\mathcal{M})^{5/3}(\pi f_O(t_O^{\rm ret})(1+z))^{2/3}\cos i\,\sin(2\Phi_O(t_O^{\rm ret}))$$

$$h_\times(t_O,i) = \frac{4}{a(t_O)R(1+z)}(G\mathcal{M}(1+z))^{5/3}(\pi f_O(t_O^{\rm ret}))^{2/3}\cos i\,\sin(2\Phi_O(t_O^{\rm ret}))$$

$$\boxed{h_\times(t_O,i) = \frac{4}{d_L}(G\mathcal{M}_z)^{5/3}(\pi f_O(t_O^{\rm ret}))^{2/3}\cos i\,\sin(2\Phi_O(t_O^{\rm ret}))}$$

Redshift absorbed in a shift in the chirp mass

$$m_{1,2}^{\rm det}(z) = (1+z)m_{1,2}$$

$$\mathcal{M}_z = (1+z)\mathcal{M}$$

redshifted / detector frame masses

BNS system with

$m_{1,2} \sim 1.4 M_\odot \quad \mathcal{M} \sim 1.21 M_\odot$

at z=1 has $\mathcal{M}_z \sim 2.42 M_\odot$

Luminosity distance

$$d_L(z) = a(t_O)R(1+z) = \sqrt{\frac{L}{4\pi F}}$$

$$d_L(z) = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\left[\Omega_m(1+z')^3 + \Omega_\Lambda(1+z')^{3(1+w(z'))}\right]^{1/2}}$$

at observer

$$h_+(t,i) = \frac{4}{d_L(z)} (G\mathcal{M}_z)^{5/3} (\pi f(t^{\rm ret}))^{2/3} \left(\frac{1+\cos^2 i}{2}\right) \cos(2\Phi(t^{\rm ret}))$$

$$h_\times(t,i) = \frac{4}{d_L(z)} (G\mathcal{M}_z)^{5/3} (\pi f(t^{\rm ret}))^{2/3} \cos i \sin(2\Phi(t^{\rm ret}))$$

where from

$$\frac{df_S}{dt_S} = \frac{96}{5}\pi^{8/3}(G\mathcal{M})^{5/3} f_S^{11/3}$$ with $$f_S = (1+z)f$$

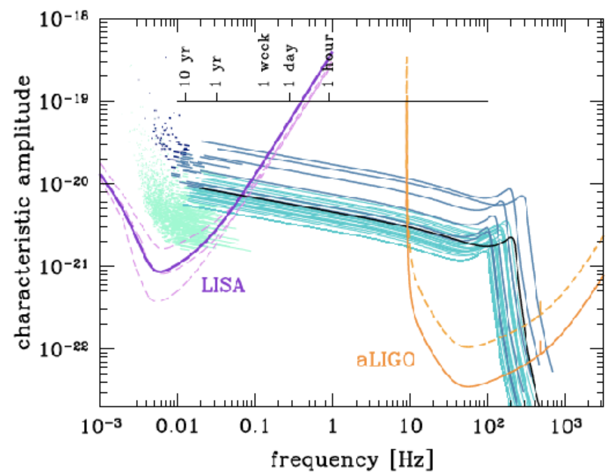
$$(1+z)\frac{d[f(1+z)]}{dt} = \frac{96}{5}\pi^{8/3}(G\mathcal{M})^{5/3} f^{11/3}(1+z)^{11/3}$$

assuming z is constant during the observation time
(could lead to a bias for stellar-mass binaries entering
LISA band and then coalesce in LIGO band),

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}(G\mathcal{M}_z)^{5/3} f^{11/3}$$

$$\Phi(t^{\rm ret}) = -2\left(\frac{t^{\rm ret}}{5G\mathcal{M}_z}\right)^{5/8} + \phi_c$$

Phase
depends
on redshifted
chirp mass



at observer

$$h_+(t,i) = \frac{4}{d_L(z)}(G\mathcal{M}_z)^{5/3}\,(\pi f(t^{\rm ret}))^{2/3}\left(\frac{1+\cos^2 i}{2}\right)\cos(2\Phi(t^{\rm ret}))$$

$$h_\times(t,i) = \frac{4}{d_L(z)}(G\mathcal{M}_z)^{5/3}\,(\pi f(t^{\rm ret}))^{2/3}\cos i\,\sin(2\Phi(t^{\rm ret}))$$

$$\mathcal{M}_z = (1+z)\mathcal{M}$$

Phase depends on redshifted chirp mass

- phase information cannot extract z (unless source frame masses known for some reason)
- amplitude information cannot extract z

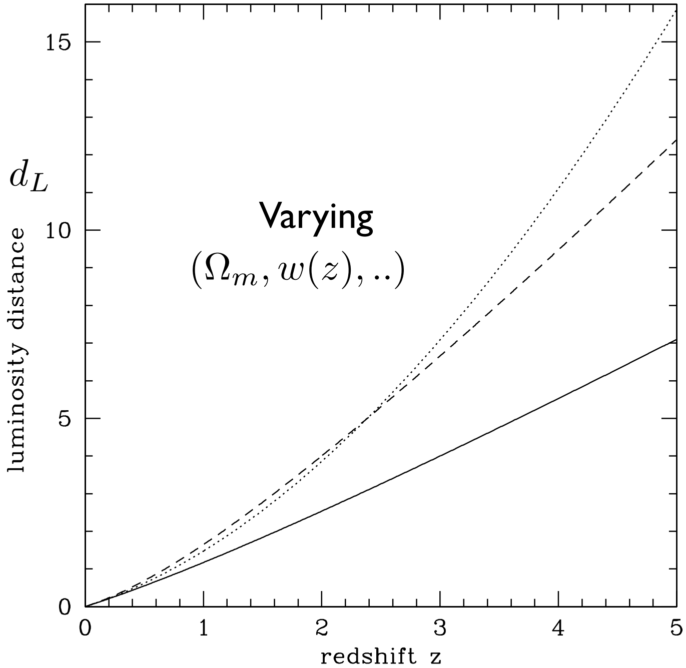
perfect degeneracy between source masses, redshift, spins.. (gravity is scale-free)

The redshift does change the waveform, but in a way that can be exactly compensated by a shift of the masses from their source to detector values and by replacing the comoving distance with the luminosity distance.

<u>Some extra non-gravitational information is necessary to determine z.</u>

$$d_L(z) = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{\left[\Omega_m(1+z')^3 + \Omega_\Lambda(1+z')^{3(1+w(z'))}\right]^{1/2}}$$

$z \ll 1$



For $z \ll 1$ (LIGO-Virgo): $d_L = \frac{cz}{H_0} + \mathcal{O}(z^2)$

For larger z (ET, LISA), dL depends on other cosmological parameters. Can potentially access

$$H(z) = H_0\left[\Omega_m(1+z')^3 + \Omega_\Lambda(1+z')^{3(1+w(z'))}\right]^{1/2}$$

Cosmological parameter determination:

1) their accuracy will depend on accuracy of dL measurement

2) and require z, with some accuracy.

$$cz = H_0\, D_L$$

$$\frac{\Delta H_0}{H_0} \sim \frac{\Delta z}{z} + \frac{\Delta D_L}{D_L}$$

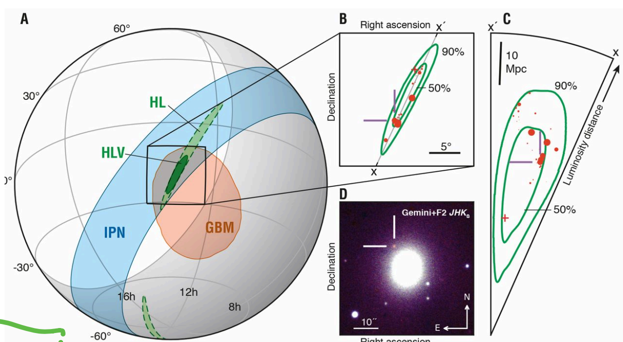
GW170817

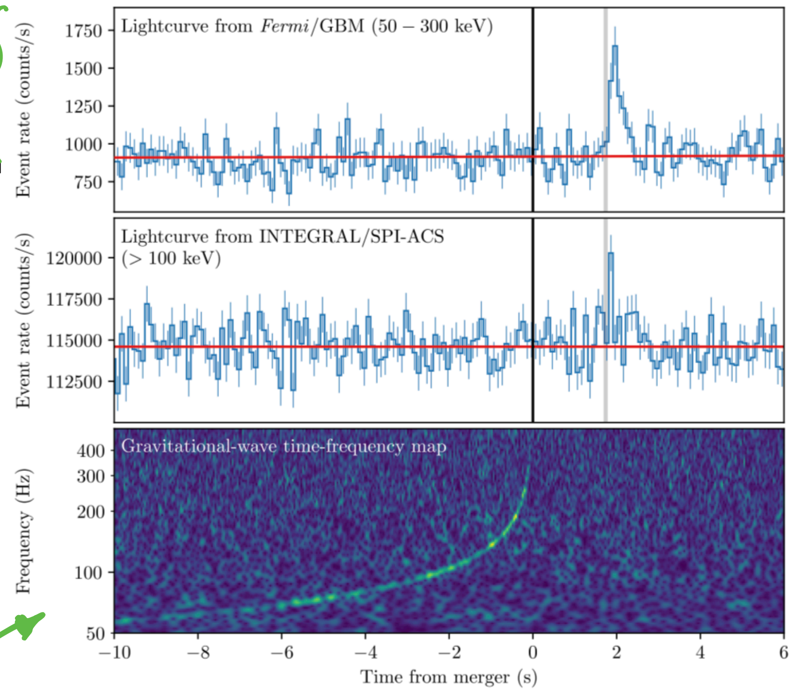
- Observed on august 17th 2017, by Adv LIGO + Virgo, binary NS event.
- Loud, network signal-to-noise ratio of 32.4
- Coincident GRB observed by Fermi
  1.7 seconds after the GW merger time

- followed up by multiple EM observatories, and an optical counterpart found
- host galaxy (NGC4993) in the Hydra constellation

$z \sim 0.01$

1.74 s





B. P. Abbott +, APJL, 848:L13 (2017)

$\boxed{H_0} = \frac{cz}{d_L}$ $\qquad d_L = \frac{cz}{H_0}$

redshift of NGC4993, $cz = 3327 \pm 72\ \mathrm{km\,s^{-1}}$

Luminosity distance from GW170817 $d_L = 43.8^{+2.9}_{-6.9}$ Mpc

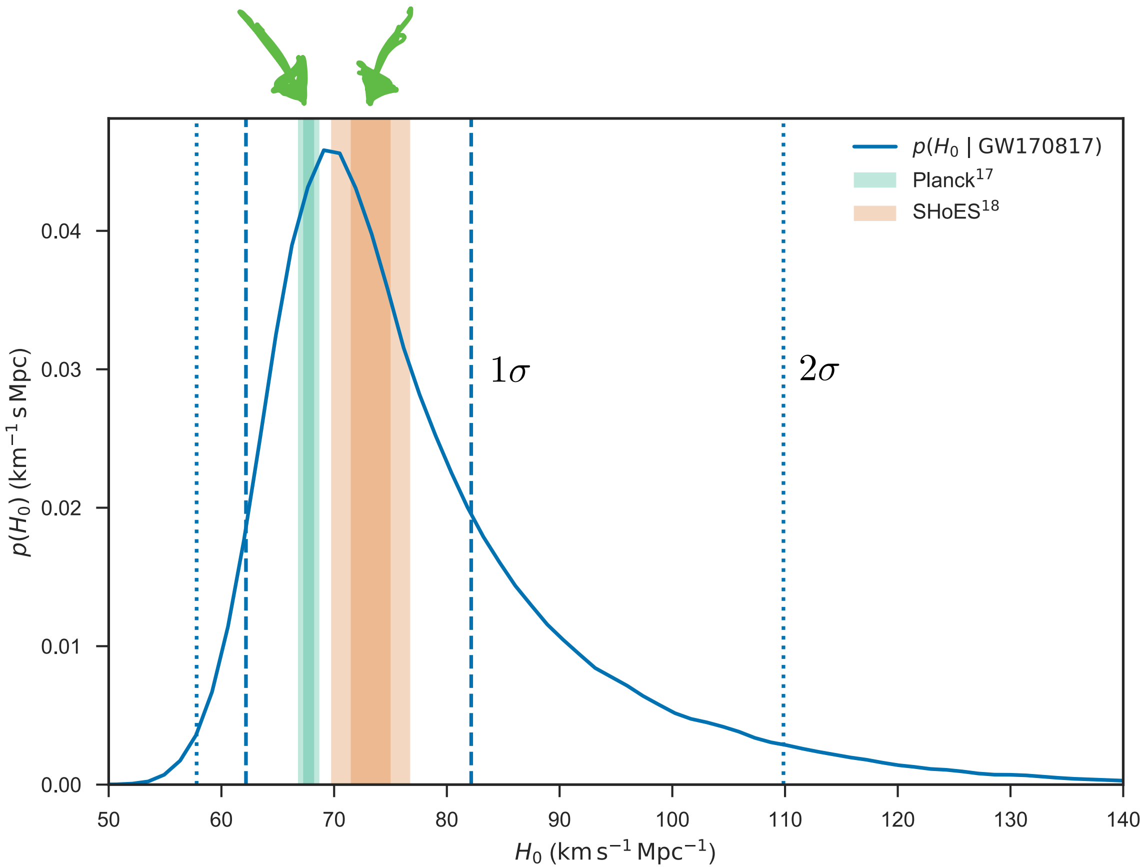
$$-3 \times 10^{-15} \leq \frac{c_{GW} - c}{c} \leq +7 \times 10^{-16}$$

First measurement of H0 using GW170817

$v_H = H_0 \times d$

H_0

[LIGO-Virgo: 1710.05835]



Errors:

1) Hubble flow velocity: need to account for peculiar velocities, caused by local grav field, attraction to nearby galaxies

2) Distance error large,

$$d = 43.8^{+2.9}_{-6.9}\,\mathrm{Mpc}$$

3) statistical measurement error from noise in detectors + instrumentation calibration uncertainties:

$$\boxed{H_0 = 70.0^{+12.0}_{-8.0}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}$$

Does not resolve H0 tension….yet

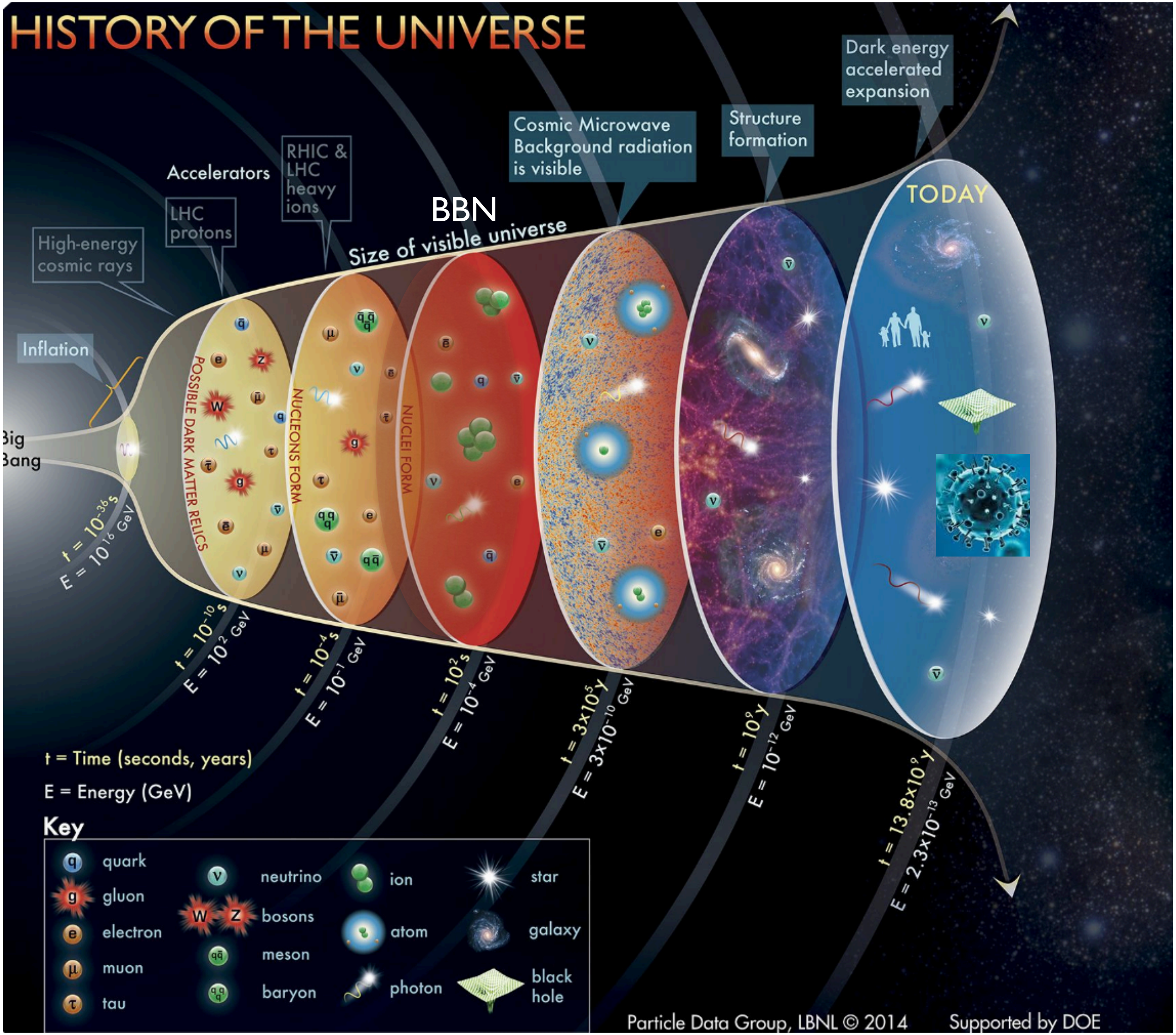
1) General introduction

2) From Einsteins equations to GWs: basic linearized theory
- wave equation
- polarizations, etc
- generation of GWs
- energy carried by GWs

3) example: inspiral of compact binary system
- wave form
- characteristic scales, link with observations

4) The sources: on black holes (BH) and neutron stars (NS)

5) GWs in an expanding universe; cosmology
- binaries at cosmological distance
- basics on the SGWB



Tested cosmology,

$$t \gtrsim 10^{-3}\text{sec}$$

$$T \lesssim 100\,\text{MeV}$$

Generalities

- Different sources for the SGWB can have very different characteristics properties.

- To discuss them (and at the same time recall some basic notation), useful to go back to basics

- Unperturbed FLRW metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = a^2(\eta)[-d\eta^2 + d\vec{x}^2]$$

with $a(t) \sim t^p$ $\quad p = \frac{1}{2}$ (radiation era) $\quad p = \frac{2}{3}$ (matter era) $\quad p > 1$ (inflation)

Hubble radius $\frac{1}{H} = \frac{a}{\dot{a}}$ and comoving Hubble radius $\frac{1}{aH} = \frac{1}{\dot{a}}$

$$q = \frac{p}{1-p}$$

- In terms of conformal time $\eta = \int \frac{dt}{a(t)}$

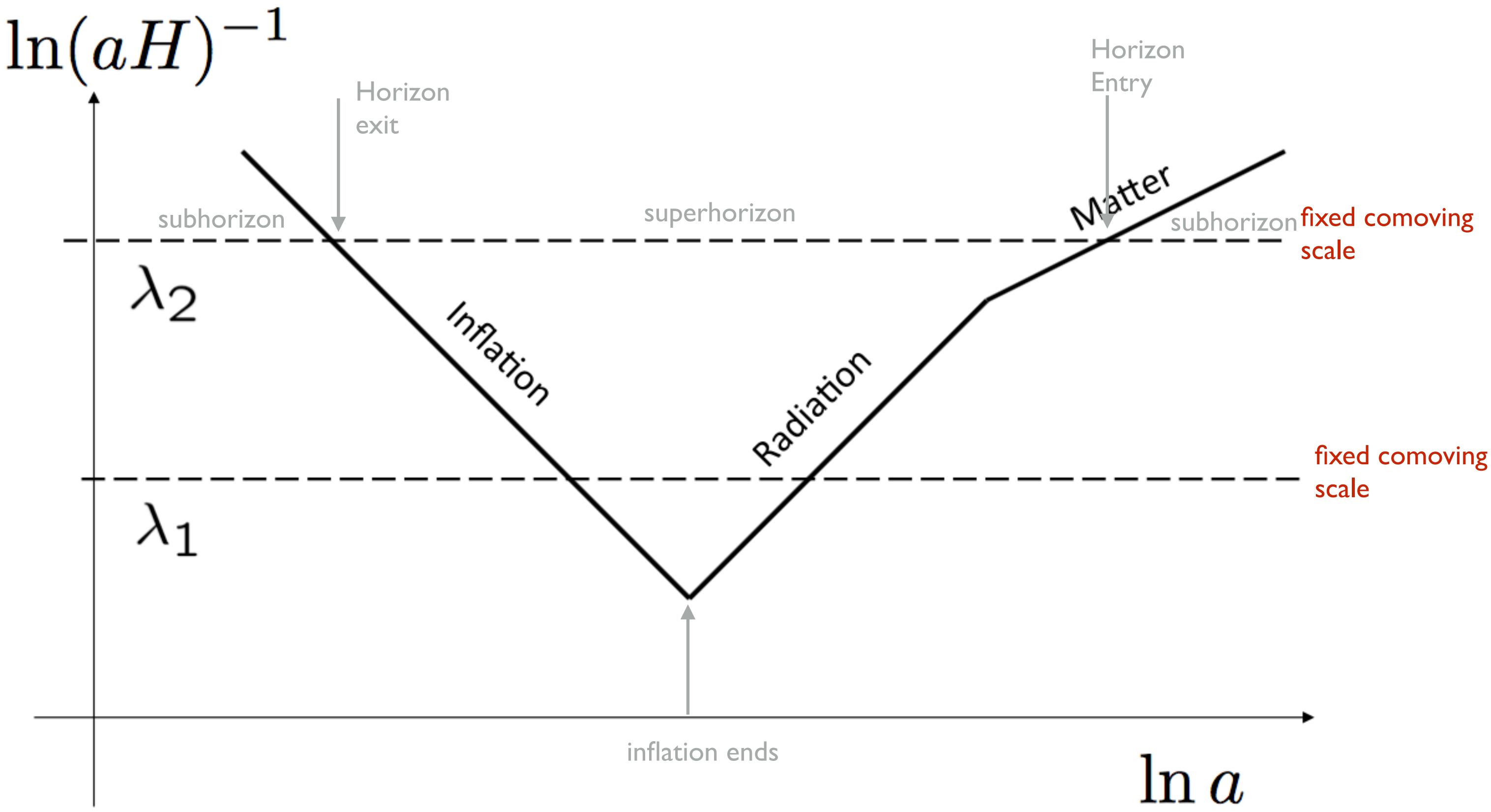
$a(\eta) \sim \eta^q$ $\quad q = 1$ (radiation era) $\quad q = 2$ (matter era) $\quad q < 0$ (inflation)

comoving Hubble radius $\frac{1}{aH} = \frac{1}{\dot{a}} = \frac{a}{a'} = \frac{1}{\mathcal{H}}$

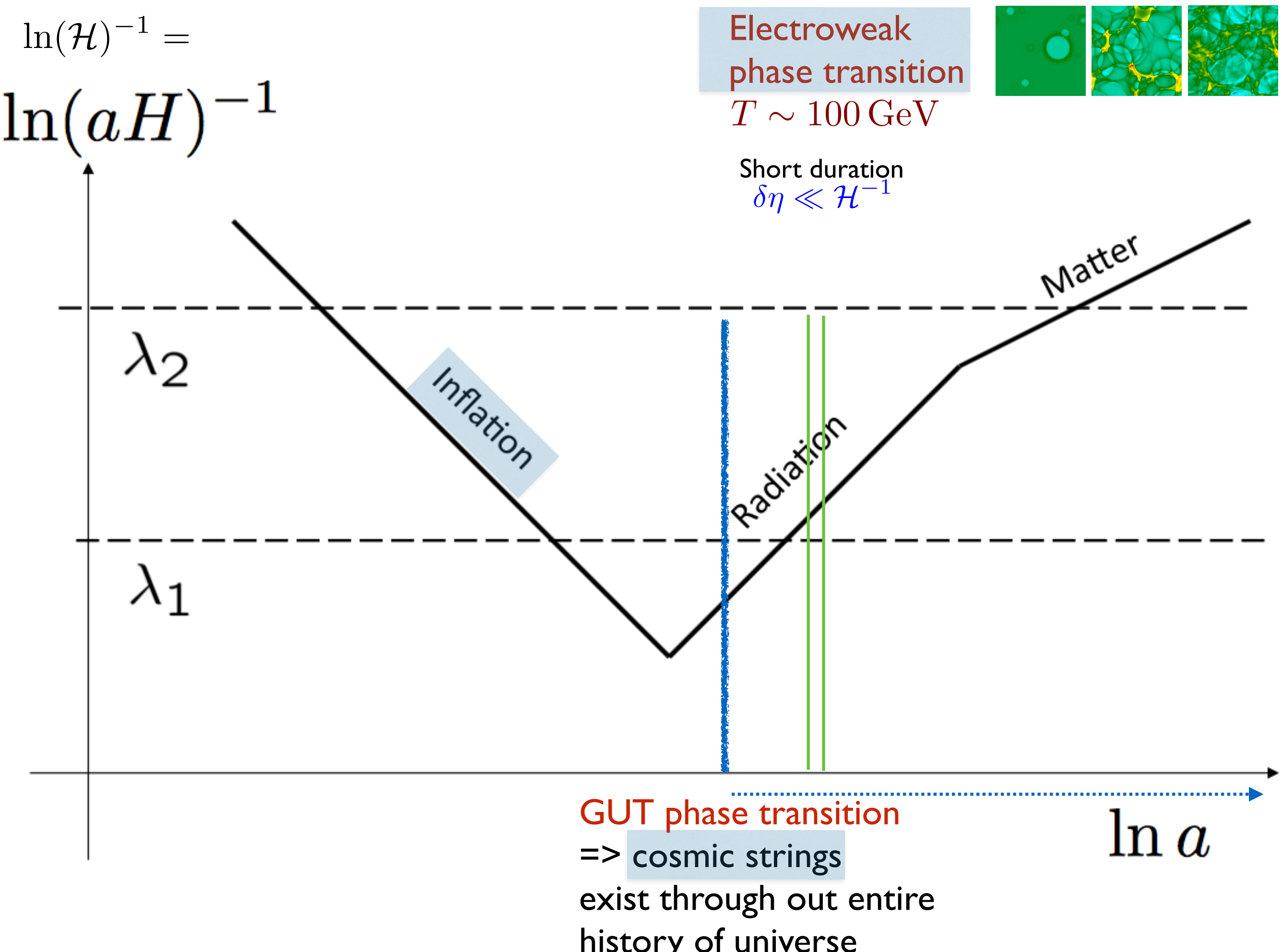
- Hence $\mathcal{H}^{-1} \sim a^{1/q}$ so that $\boxed{\ln(\mathcal{H}^{-1}) \sim \frac{1}{q} \ln a}$

Generalities

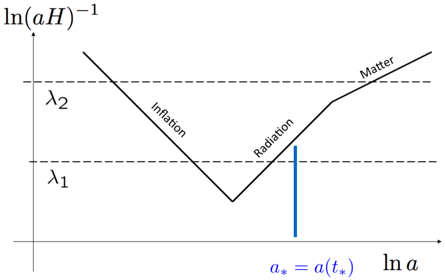
$$\ln(\mathcal{H})^{-1} =$$



Generalities



Generalities: why a SGWB?
[standard phase of cosmic expansion]



- assume the source operates at some time $t = t_\star$
  Why does it generate a SGWB?

- Causality: produced signal correlated on length/time scales at most as large as the causal horizon at that time

$$\ell_* \leq H^{-1}(t_*)$$
$$\Delta t_* \leq H^{-1}(t_*)$$

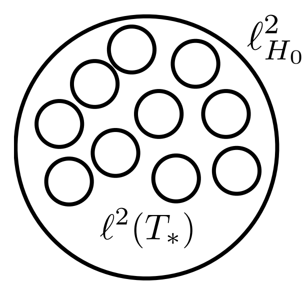
ℓ_* characteristic length-scale of the source
(typical size of variation of the tensor anisotropic stresses)

- Length scale redshifted to today corresponds to scale $\ell_*^0 = \ell_* \left(\frac{a(t_0)}{a(t_*)}\right)$

- Compared to the Hubble radius today. Using the Friedmann equations
(in the radiation era, neglecting changes in effective number of relativistic d of f, and saturating the inequality)

$$\frac{\ell_*^0}{H^{-1}(t_0)} \sim 10^{-11} \left(\frac{\text{GeV}}{T_*^4}\right)$$

correlation length tiny compared to present size of universe today.

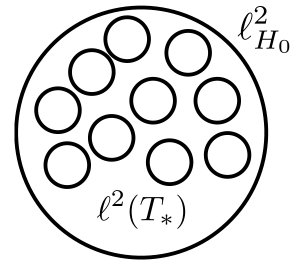


- Angular size of that patch on the sky today?

Generalities: b) why a SGWB?
[standard phase of cosmic expansion]

• Angular size of that patch on the sky today $\Theta_* = \dfrac{\ell_*}{d_A(z_*)}$ $\qquad \ell_* \leq H^{-1}(t_*)$

$$d_A(z_*) = \frac{1}{H_0(1+z_*)} \int_0^{z_*} \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_r(1+z')^4 + \Omega_\Lambda}}$$



• number of uncorrelated regions today from which we are receiving independent GW signals $\sim \Theta_*^{-2}$

EW scales $\qquad \Theta(T_* = 100\,\mathrm{GeV}) \simeq 10^{-12}\mathrm{deg}$

equality $\qquad \Theta(z_* = 1090) \simeq 0.9\,\mathrm{deg}$

GW signal (as received today on Earth) due to a causal process operating at the EW epoch, is composed by the superposition of independent signals emitted by at least 10^{24} uncorrelated regions

• a GW signal from the early universe cannot possibly be resolved beyond its stochastic nature

$\Theta(z_* \lesssim 17) \lesssim 10\,\mathrm{deg}$
(GW detector angular resolution)

– Can only the statistical properties of the signal
– Must treat $h_{ij}(\vec{x}, t)$ as a random variable

• By observing large enough regions of the Universe today (or a given region for long enough time), have access to many realisations of the system: replace *ensemble averages* with *volume/time averages*
(over a length scale much larger than the typical GW wavelength, and much smaller than the Horizon)

[exception in the case of inflation]

2) Characterisation of the SGWB

- Perturbed FRWL metric (ignoring scalars and vectors):

$$ds^2 = -dt^2 + a^2(t)[(\delta_{ij} + h_{ij})dx^i dx^j]$$

$$|h_{ij}| \ll 1$$
$$h^i_i = \partial_j h^j{}_i = 0$$

- from Einstein equations

$$\ddot{h}_{ij}(\vec{x},t) + 3H\dot{h}_{ij}(\vec{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{x},t) = 16\pi G \Pi^{TT}_{ij}(\vec{x},t) \longleftarrow$$

source: tensor anisotropic stress

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

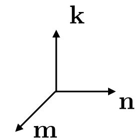
- Fourier transform, and polarisation components +, x

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

$$\Pi_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Pi_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

k= comoving wave number

$$e^+_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{m}_j - \hat{n}_i\,\hat{n}_j$$
$$e^\times_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{n}_j + \hat{n}_i\,\hat{m}_j$$
$$e^r_{ij}(\hat{\mathbf{k}})\, e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\,\delta_{rr'}$$

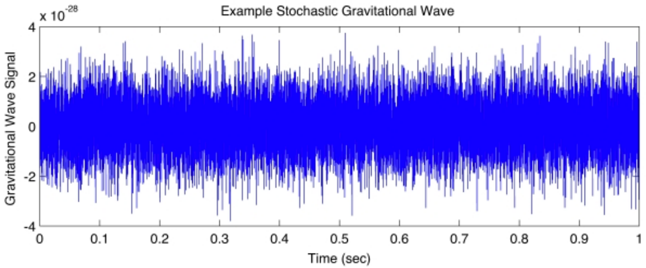


- The equation decouples for each polarisation mode. In terms of conformal time

$$h''_r(\mathbf{k},\eta) + 2\mathcal{H}\, h'_r(\mathbf{k},\eta) + k^2 h_r(\mathbf{k},\eta) = 16\pi G a^2 \Pi_r(\mathbf{k},\eta)$$

Note from Friedmann equation: $\mathcal{H}^2 = H^2 a^2 = \frac{8\pi G}{3} a^2 \bar{\rho}$

Statistical properties



• In general the SGWB is

– *homogenous and isotropic*
(inherited from FLRW universe)

$$\langle h_{ij}(\mathbf{x},\eta_1)\, h_{lm}(\mathbf{y},\eta_2)\rangle = \xi_{ijlm}(|\mathbf{x}-\mathbf{y}|,\eta_1,\eta_2)$$

– *unpolarised*
(absence of significant source of parity violation
in the universe)

$$\langle h_{+}(\mathbf{k},\eta) h_{\times}(\mathbf{k},\eta)\rangle = 0.$$

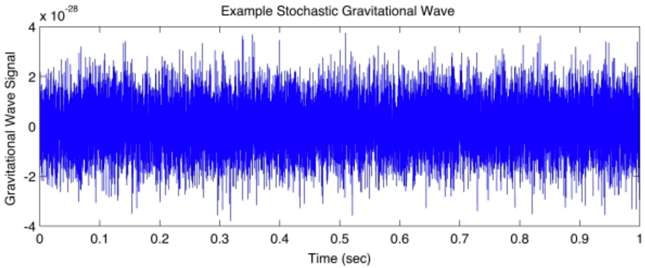
– *gaussian*
(formed by emission from many uncorrelated regions)

therefore characterized by the 2-point function

• In terms of the Fourier amplitudes $h_r(\mathbf{k},\eta)$:

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^{r}_{ij}(\hat{\mathbf{k}})$$

Statistical properties



- the 2-point function

homogeneity and isotropy, unpolarised

$$\boxed{\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\, h_c^2(k,\eta)}$$

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

- Hence

$$\boxed{\langle h_{ij}(\mathbf{x},\eta)\, h_{ij}(\mathbf{x},\eta)\rangle = 2\int_0^{+\infty}\frac{dk}{k}\, h_c^2(k,\eta)}$$

characteristic GW amplitude per logarithmic wave-number interval and per polarization state, at time η

- In terms of which can express GW energy density, given by

$$\boxed{\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \frac{\langle h'_{ij}(\mathbf{x},\eta)\, h'_{ij}(\mathbf{x},\eta)\rangle}{32\pi G\, a^2(\eta)} \equiv \int_0^{+\infty}\frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}}$$

spectrum of GW energy density per logarithmic wave-number interval

- GW energy density parameter. In terms of present day physical frequency $f = k/(2\pi\, a_0)$

$$\boxed{\Omega_{\rm gw}(t_0,f) = \frac{f}{\rho_c}\frac{d\rho_{\rm gw}}{df}(t_0,f)}$$

$$\rho_c = 3H_0^2/8\pi G$$

critical energy density of universe.

GW background from cosmic strings

Cosmic strings: some basics

[Kibble '76]

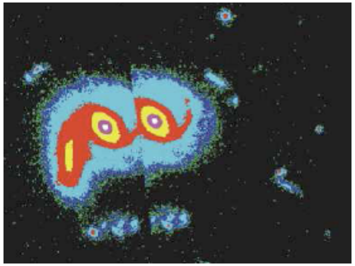
– line-like *topological defects*, formed in a symmetry breaking phase transition $G \rightarrow H$
provided the vacuum manifold contains non-contractible loops $\Pi_1(G/H) = \Pi_1(\mathcal{M}) \neq 1$.

– A lot of input/interplay with other branches of physics:
- difficult to see cosmic strings in the sky
- "easier" to see strings in the lab (vortex loops in He4, He3, superconductors, strings in NLC...)

– Generically formed at the end of hybrid-like inflation [Jeanneret et al 03]
or in brane inflation [Jones et al, Sarangi and Tye]
(cosmic super-strings)

– if formed, should still exist today, they cannot disappear!

– Numerous potentially observable signatures:



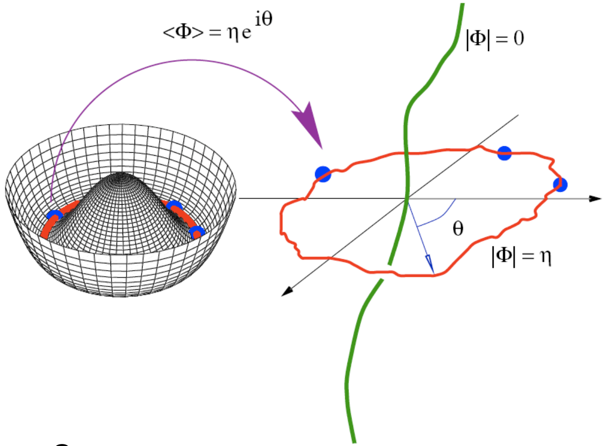
Gravitational wave emission;
CMB anisotropies & B-modes;
lensing,....
particle emission
electromagnetic radiation

$$G\mu \leq \text{few} \times 10^{-7}$$

[Planck paper XXV]

– Typical example: strings in the Abelian Higgs model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - \frac{\lambda}{4}(|\phi|^2 - \eta^2)$$



- Degenerate vacuum/ground state with $\langle\phi\rangle = \eta e^{i\alpha}$
- U(1) invariance $\phi \to \phi e^{i\alpha}$ broken by choice of phase
- String/vortex is a linear defect around which α changes by $2n\pi$

infinite, or *closed loops*

(n = winding number)

$G = U(1) \quad \mathcal{M} = S^1 \quad \Pi_1(\mathcal{M}) = Z$

- Energy/unit length of string: μ_n

- Scales: $r \sim M^{-1}$ $\quad G\mu \sim G\eta^2 \sim GM^2$

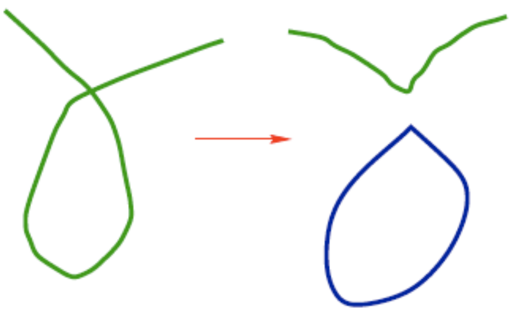
  GUT: $\sim 10^{-32}\mathrm{cm}$ $\quad \sim 10^{-7}$

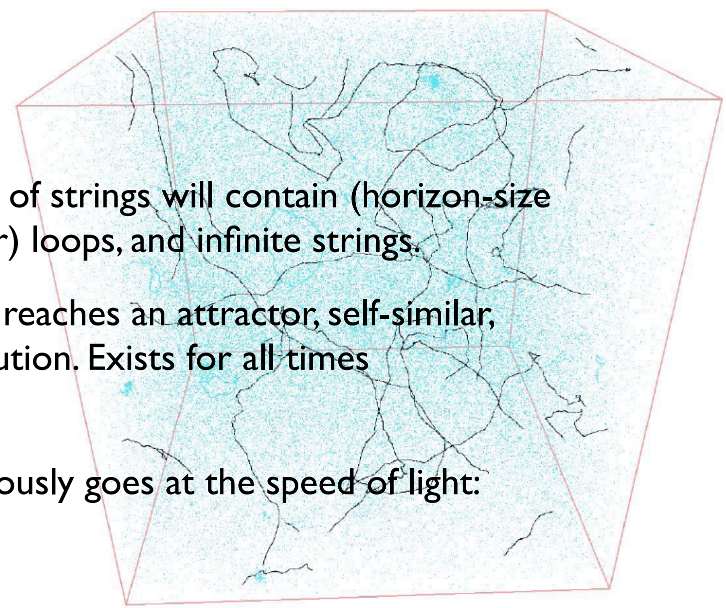
- Prototypical model of infinitely thin strings: Nambu-Goto strings
  - Approx. dynamics of relativistic string: action = area of world-sheet

$$S = -\mu \int d^2\sigma \sqrt{-\det(\gamma_{ab})}$$

- <u>only one free parameter</u> $G\mu$

- intercommutation:





- Network of strings will contain (horizon-size and smaller) loops, and infinite strings.

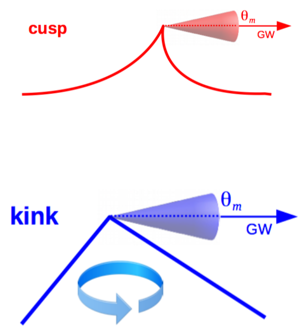
- network reaches an attractor, self-similar, scaling solution. Exists for all times

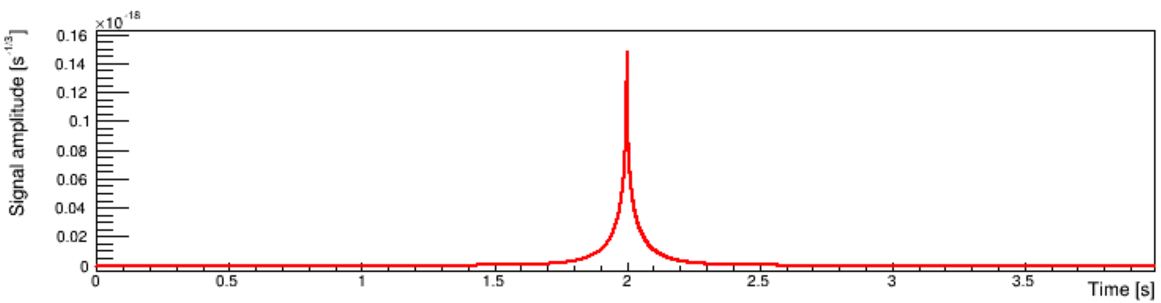
- <u>cusps</u>: points at which the string itself instantaneously goes at the speed of light:
- **kinks** (discontinuity in tangent vector of string)

- Cosmic string loops produce 2 types of GW signals

1) sharp, non-gaussian **bursts** of gravitational waves from kinks and cusps. Their characteristic form is directly searched for by LIGO and Virgo.
   [use match-filtering techniques]

[Damour+Vilenkin 2001]





Predicted gravitational waveform produced by a cosmic string cusp.

- Cosmic string loops produce 2 types of GW signals

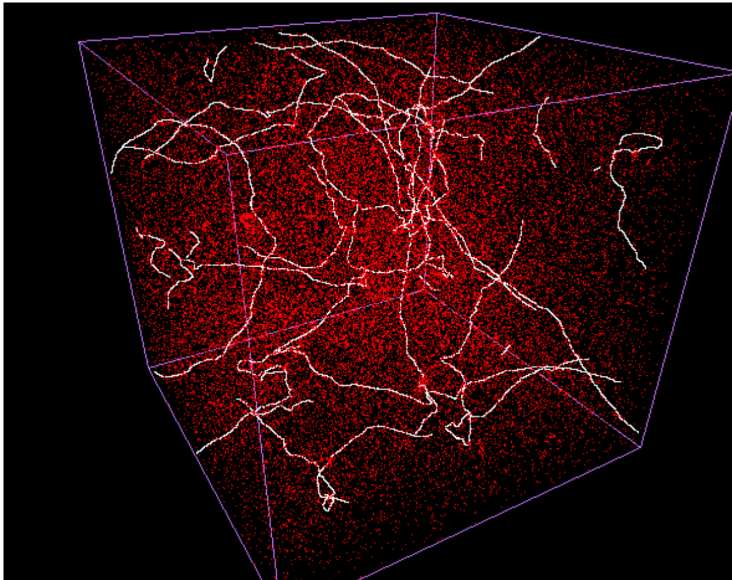
1) sharp, non-gaussian **bursts** of gravitational waves from kinks and cusps. Their characteristic form is directly searched for by LIGO and Virgo.
[use match-filtering techniques] *[Damour+Vilenkin 2001]*

2) A **stochastic GW background** ranging over many decades in frequency

$$\Omega_{GW}(f) = \frac{f}{\rho_c}\frac{d\rho_{GW}}{df},$$

which can be probed by e.g. pulsar timing at nHz frequencies, LIGO/Virgo…,
Sum of all the GWs emitted by oscillating loops

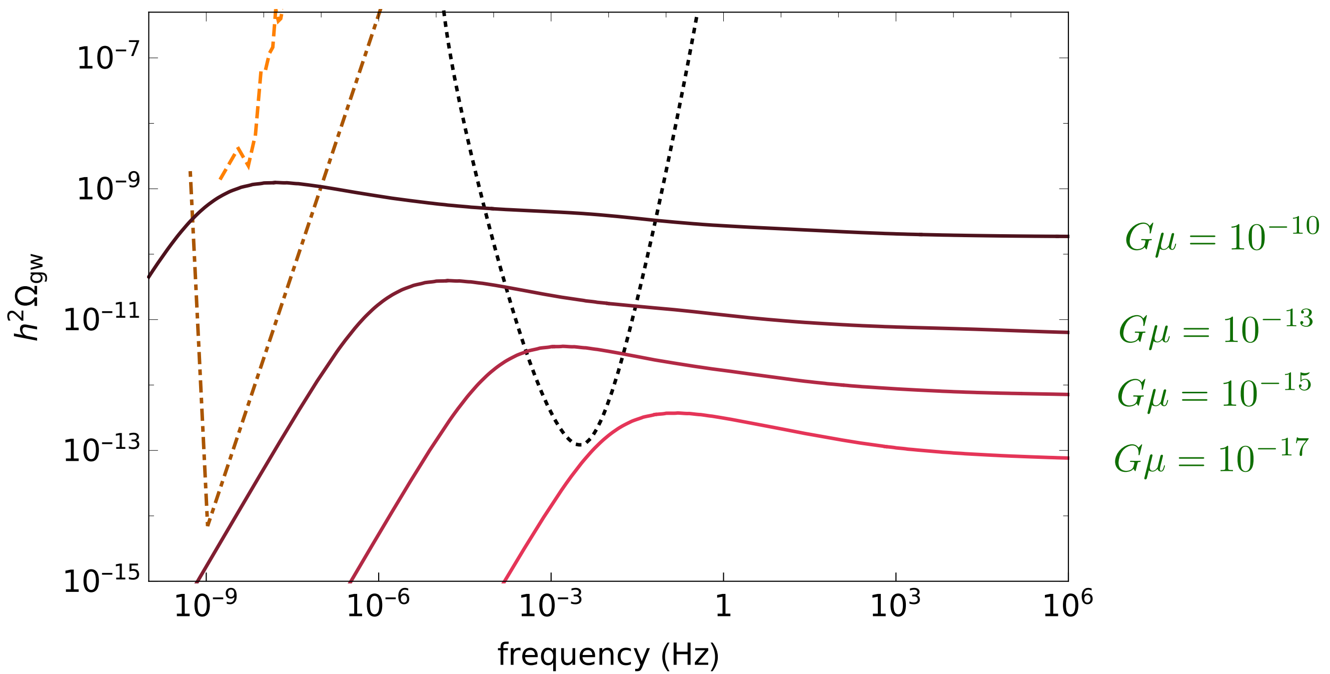
[C.Ringeval]



- Crucial quantity: $n(\ell,t)d\ell$

number density of loops with length between $\ell \to \ell + d\ell$ at time t

Model “A”

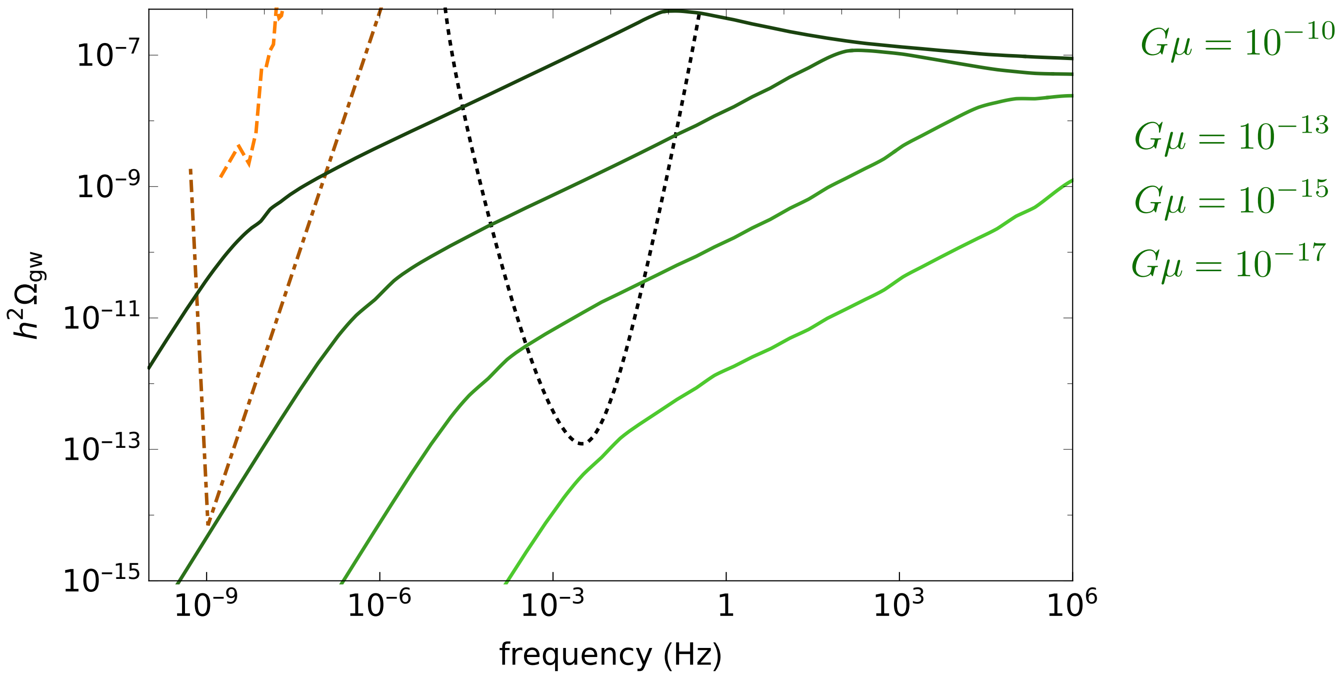


Constraints in the LISA band.

[1909.00819]

broad spectrum spanning many decades in frequency.

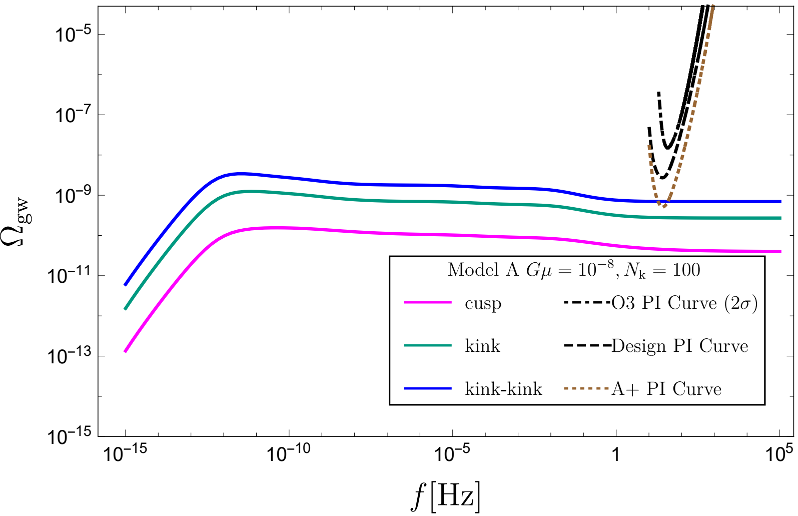
Model "B"



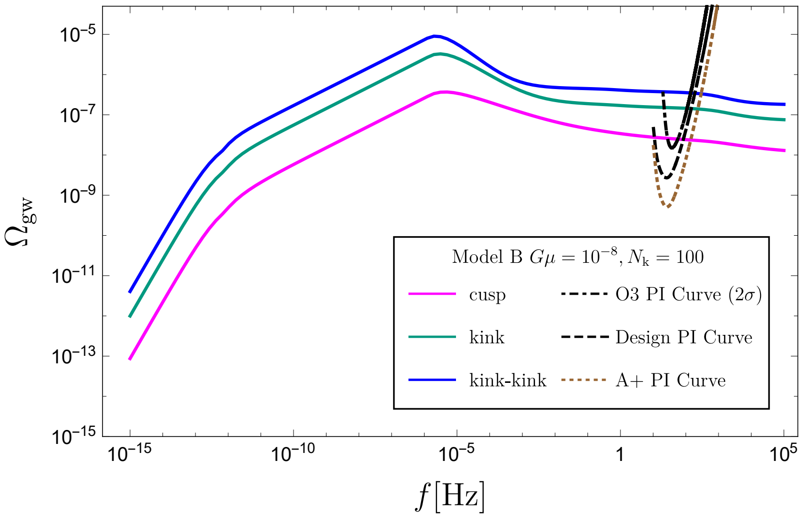
Constraints in the LISA band

[1909.00819]

Model "A"

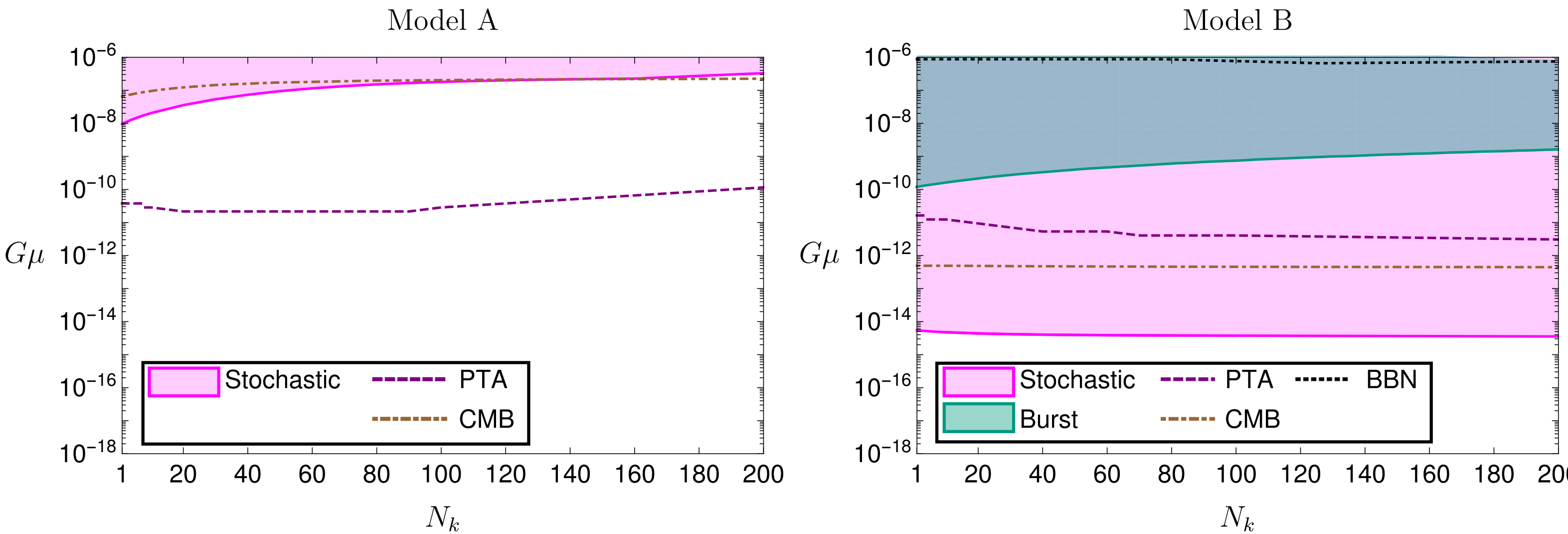


Model "B"



Constraints in the LIGO band

[2101.12248]



Constraints in the LIGO band

[2101.12248]