

Simulation of fermion pair bremsstrahlung in decays

Extra pair
emissions with
PHOTOS

Extra pair
emissions with
TAUOLA

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Introduction

- ▶ With the increasing precision of measurements more detailed theoretical calculations are needed for interpretation of results in the language of physics parameters such as masses or couplings of Z and W bosons.
- ▶ My goal is to improve PHOTOS¹ algorithm for simulation of the additional pair emissions in decays of Z boson.
- ▶ Single lepton emission kernel can be used for other than Z to l^+l^- . As in the past for PHOTOS bremsstrahlung it may generalize to decay of any particle and into any decay channel.
- ▶ $\tau \rightarrow \mu\nu\bar{\nu}l^+l^-$ decay is added into TAUOLA², but it should be useful for l^+l^- pair emission in any decay. This decay is tested comparing to $\tau \rightarrow \mu\nu\bar{\nu}$ decay.

¹N. Davidson et al., [<http://photospp.web.cern.ch/photospp/>];
in the current study a version of PHOTOS dated 17.11.2017 is used.

²M. Chrzaszcz, T. Przedzinski, Z. Was and J. Zaremba, Comput. Phys. Commun. 232, 220 (2018) doi:10.1016/j.cpc.2018.05.017 [arXiv:1609.04617 [hep-ph]].

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- ▶ PHOTOS algorithm is of the after-burner type.
- ▶ For the previously generated event, with a certain probability, a decay vertex can be replaced with the one featuring additional photons. A similar solution for additional lepton pairs is installed.
- ▶ PHOTOS uses the exact phase space parametrizations thus work is to calculate and implement matrix elements.
- ▶ Such matrix elements could be exact ones or approximate ones, main requirement is that they should be applicable universally.

Numerical tests

- ▶ KORALW¹ Monte Carlo has been used to generate $e^+e^- \rightarrow 4f$ processes and has provided source of benchmarks for tests². For that purpose it is necessary to run the program for the Center of Mass Energy equal to Z boson mass and Z width is set to a very small value, effectively switching off emission of pair from the initial state.
- ▶ PYTHIA has been used to generate equal number of $e^+e^- \rightarrow Z \rightarrow e^+e^-$ and $e^+e^- \rightarrow Z \rightarrow \mu^+\mu^-$ events as input for PHOTOS, then PHOTOS has been used to generate $e^+e^-\mu^+\mu^-$ final states. Normalization for the sample size is fixed to assure 1M of four-fermion events.

¹S. Jadach, W. Placzek, M. Skrzypek, B. Ward, and Z. Was, *Comput.Phys.Commun.* 119 (1999) 272–311, [hep-ph/9906277](#).

²S. Antropov, A. Arbuzov, R. Sadykov and Z. Was, *Acta Phys. Polon. B* 48, 1469 (2017) doi:10.5506/APhysPolB.48.1469 [arXiv:1706.05571 [hep-ph]].

Extra pair emission

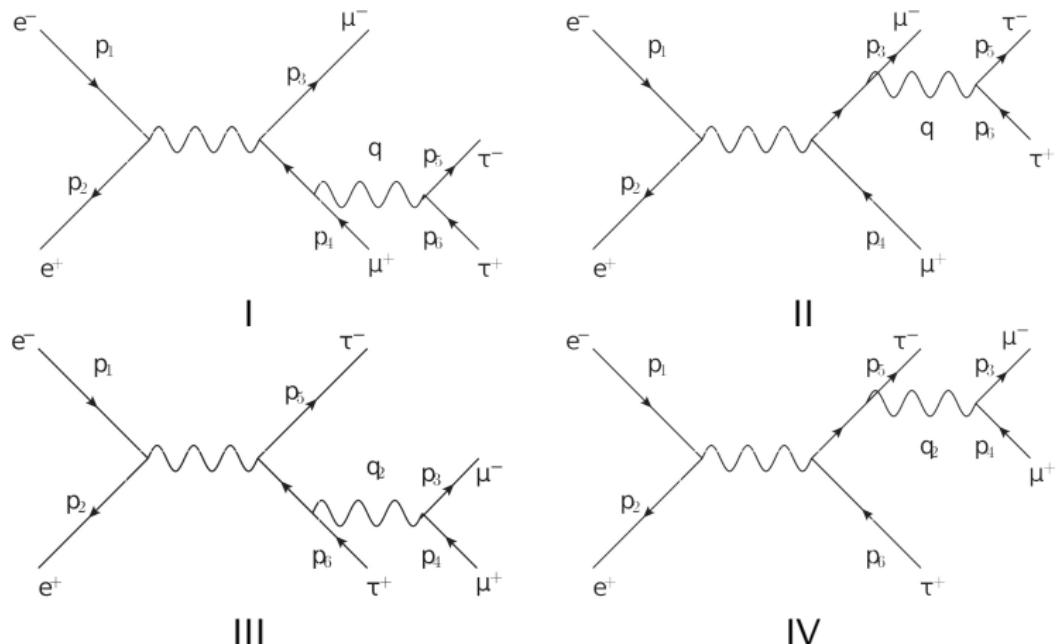
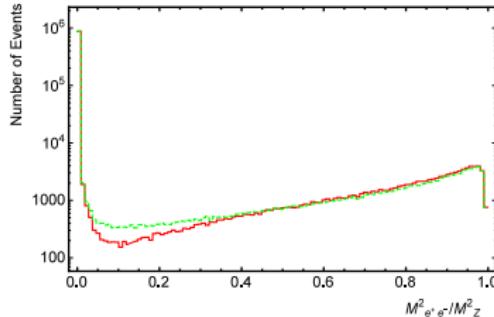
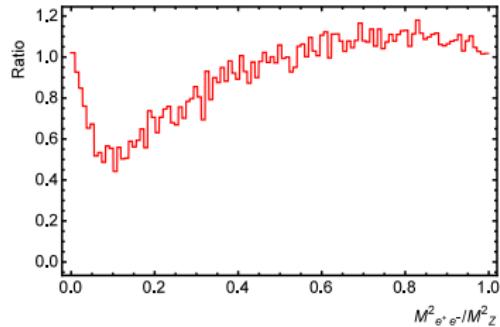


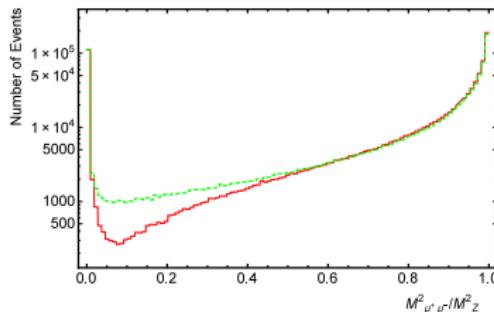
Figure 1: Feynman graphs corresponding to real pair emissions for an example process.



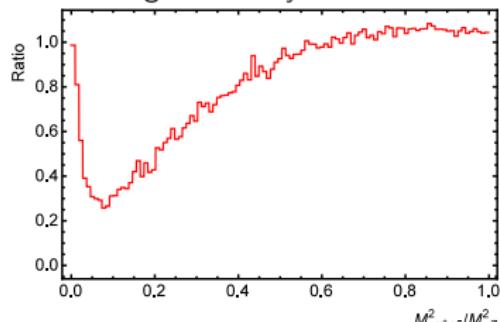
a) Spectrum of electron pair mass squared.



b) Ratio of PHOTOS generated spectrum to the one generated by KORALW.



a) Spectrum of muon pair mass squared.



b) Ratio of PHOTOS generated spectrum to the one generated by KORALW.

Figure 2: Lepton pair invariant mass spectra in the channel $Z \rightarrow \mu^+ \mu^- e^+ e^-$.

Results generated by PHOTOS (solid red line) are obtained from samples of equal number of $Z \rightarrow e^+ e^-$ and $Z \rightarrow \mu^+ \mu^-$ decays. They are compared with results from KORALW (dashed green line) where four fermion final state matrix elements are used.

Matrix Element

$$\begin{aligned}
 M_1 + M_2 + M_3 + M_4 &= \frac{-e^4}{(p_1 + p_2)^2 q^2} [\bar{v}(p_2) \gamma_\mu u(p_1)] \times \quad (1) \\
 &\times \left[\bar{u}(p_3) \left(\gamma^\alpha \frac{\not{p}_3 + \not{q} + m_\mu}{(p_3 + q)^2 - m_\mu^2} \gamma^\mu - \gamma^\mu \frac{\not{p}_4 + \not{q} - m_\mu}{(p_4 + q)^2 - m_\mu^2} \gamma^\alpha \right) v(p_4) \right] [\bar{u}(p_5) \gamma_\alpha v(p_6)] - \\
 &- \frac{e^4}{(p_1 + p_2)^2 q_2^2} [\bar{v}(p_2) \gamma_\mu u(p_1)] \times \\
 &\times \left[\bar{u}(p_5) \left(\gamma^\alpha \frac{\not{p}_5 + \not{q}_2 + m_\tau}{(p_5 + q_2)^2 - m_\tau^2} \gamma^\mu - \gamma^\mu \frac{\not{p}_6 + \not{q}_2 - m_\tau}{(p_6 + q_2)^2 - m_\tau^2} \gamma^\alpha \right) v(p_6) \right] [\bar{u}(p_3) \gamma_\alpha v(p_4)]
 \end{aligned}$$

Complete amplitude

$$\begin{aligned}
 |M_1 + M_2 + M_3 + M_4|^2 &= \frac{e^8}{(p_1 + p_2)^4} \times \quad (2) \\
 \left\{ \frac{1}{q^4} [\bar{v}(p_2) \gamma_\mu u(p_1)] \left[\bar{u}(p_3) \left(\frac{2p_3^\alpha \gamma^\mu + \gamma^\alpha \not{q} \gamma^\mu}{(p_3 + q)^2 - m_\mu^2} - \frac{2p_4^\alpha \gamma^\mu + \gamma^\mu \not{q} \gamma^\alpha}{(p_4 + q)^2 - m_\mu^2} \right) v(p_4) \right] [\bar{u}(p_5) \gamma_\alpha v(p_6)] \times \right. \\
 &\times [\bar{v}(p_6) \gamma_\beta u(p_5)] \left[\bar{v}(p_4) \left(\frac{2p_3^\beta \gamma^\nu + \gamma^\nu \not{q} \gamma^\beta}{(p_3 + q)^2 - m_\mu^2} - \frac{2p_4^\beta \gamma^\nu + \gamma^\beta \not{q} \gamma^\nu}{(p_4 + q)^2 - m_\mu^2} \right) u(p_3) \right] [\bar{u}(p_1) \gamma_\nu v(p_2)] + \\
 &+ \frac{1}{q_2^4} [\bar{v}(p_2) \gamma_\mu u(p_1)] \left[\bar{u}(p_5) \left(\frac{2p_5^\alpha \gamma^\mu + \gamma^\alpha \not{q}_2 \gamma^\mu}{(p_5 + q_2)^2 - m_\tau^2} - \frac{2p_6^\alpha \gamma^\mu + \gamma^\mu \not{q}_2 \gamma^\alpha}{(p_6 + q_2)^2 - m_\tau^2} \right) v(p_6) \right] [\bar{u}(p_3) \gamma_\alpha v(p_4)] \times \\
 &\times [\bar{v}(p_4) \gamma_\beta u(p_3)] \left[\bar{v}(p_6) \left(\frac{2p_5^\beta \gamma^\nu + \gamma^\nu \not{q}_2 \gamma^\beta}{(p_5 + q_2)^2 - m_\tau^2} - \frac{2p_6^\beta \gamma^\nu + \gamma^\beta \not{q}_2 \gamma^\nu}{(p_6 + q_2)^2 - m_\tau^2} \right) u(p_5) \right] [\bar{u}(p_1) \gamma_\nu v(p_2)] + \\
 &\frac{1}{q^2 q_2^2} [\bar{v}(p_2) \gamma_\mu u(p_1)] \left[\bar{u}(p_3) \left(\frac{2p_3^\alpha \gamma^\mu + \gamma^\alpha \not{q} \gamma^\mu}{(p_3 + q)^2 - m_\mu^2} - \frac{2p_4^\alpha \gamma^\mu + \gamma^\mu \not{q} \gamma^\alpha}{(p_4 + q)^2 - m_\mu^2} \right) v(p_4) \right] [\bar{u}(p_5) \gamma_\alpha v(p_6)] \times \\
 &\times [\bar{v}(p_6) \left(\frac{2p_5^\beta \gamma^\nu + \gamma^\nu \not{q}_2 \gamma^\beta}{(p_5 + q_2)^2 - m_\tau^2} - \frac{2p_6^\beta \gamma^\nu + \gamma^\beta \not{q}_2 \gamma^\nu}{(p_6 + q_2)^2 - m_\tau^2} \right) u(p_5)] [\bar{v}(p_4) \gamma_\beta u(p_3)] [\bar{u}(p_1) \gamma_\nu v(p_2)] + \\
 &\frac{1}{q^2 q_2^2} [\bar{v}(p_2) \gamma_\mu u(p_1)] \left[\bar{u}(p_5) \left(\frac{2p_5^\alpha \gamma^\mu + \gamma^\alpha \not{q}_2 \gamma^\mu}{(p_5 + q_2)^2 - m_\tau^2} - \frac{2p_6^\alpha \gamma^\mu + \gamma^\mu \not{q}_2 \gamma^\alpha}{(p_6 + q_2)^2 - m_\tau^2} \right) v(p_6) \right] [\bar{u}(p_3) \gamma_\alpha v(p_4)] \times \\
 &\times [\bar{v}(p_4) \left(\frac{2p_3^\beta \gamma^\nu + \gamma^\nu \not{q} \gamma^\beta}{(p_3 + q)^2 - m_\mu^2} - \frac{2p_4^\beta \gamma^\nu + \gamma^\beta \not{q} \gamma^\nu}{(p_4 + q)^2 - m_\mu^2} \right) u(p_3)] [\bar{v}(p_6) \gamma_\beta u(p_5)] [\bar{u}(p_1) \gamma_\nu v(p_2)] \left. \right\}
 \end{aligned}$$

Spin summation

Lepton pair interference is switched off here and appear to be numerically unimportant (formally this is gauge dependent)

$$\begin{aligned}
 \sum_{spins} |M_1 + M_2|^2 &= \frac{\alpha^4 (4\pi)^4}{(p_1 + p_2)^4 q^4} \left\{ Tr [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] Tr [(\not{p}_3 + m_\mu) \gamma^\mu (\not{p}_4 - m_\mu) \gamma^\nu] \times \right. \\
 &\quad \times Tr [(\not{p}_5 + m_\tau) \gamma_\alpha (\not{p}_6 - m_\tau) \gamma_\beta] \left(\frac{2\not{p}_3^\alpha}{(p_3 + q)^2 - m_\mu^2} - \frac{2\not{p}_4^\alpha}{(p_4 + q)^2 - m_\mu^2} \right) \left(\frac{2\not{p}_3^\beta}{(p_3 + q)^2 - m_\mu^2} - \frac{2\not{p}_4^\beta}{(p_4 + q)^2 - m_\mu^2} \right) + \\
 &\quad + Tr [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] Tr [(\not{p}_5 + m_\tau) \gamma_\alpha (\not{p}_6 - m_\tau) \gamma_\beta] \times \\
 &\quad \times Tr \left[(\not{p}_3 + m_\mu) \left(\frac{\gamma^\alpha \not{q} \gamma^\mu}{(p_3 + q)^2 - m_\mu^2} - \frac{\gamma^\mu \not{q} \gamma^\alpha}{(p_4 + q)^2 - m_\mu^2} \right) (\not{p}_4 - m_\mu) \left(\frac{\gamma^\nu \not{q} \gamma^\beta}{(p_3 + q)^2 - m_\mu^2} - \frac{\gamma^\beta \not{q} \gamma^\nu}{(p_4 + q)^2 - m_\mu^2} \right) \right] + \\
 &\quad + Tr [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] Tr \left[(\not{p}_5 + m_\tau) \left(\frac{2\not{p}_3}{(p_3 + q)^2 - m_\mu^2} - \frac{2\not{p}_4}{(p_4 + q)^2 - m_\mu^2} \right) (\not{p}_6 - m_\tau) \gamma_\beta \right] \times \\
 &\quad \times Tr \left[(\not{p}_3 + m_\mu) \gamma^\mu (\not{p}_4 - m_\mu) \left(\frac{\gamma^\nu \not{q} \gamma^\beta}{(p_3 + q)^2 - m_\mu^2} - \frac{\gamma^\beta \not{q} \gamma^\nu}{(p_4 + q)^2 - m_\mu^2} \right) \right] + \\
 &\quad + Tr [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] Tr \left[(\not{p}_5 + m_\tau) \gamma_\alpha (\not{p}_6 - m_\tau) \left(\frac{2\not{p}_3}{(p_3 + q)^2 - m_\mu^2} - \frac{2\not{p}_4}{(p_4 + q)^2 - m_\mu^2} \right) \right] \times \\
 &\quad \times Tr \left[(\not{p}_3 + m_\mu) \left(\frac{\gamma^\alpha \not{q} \gamma^\mu}{(p_3 + q)^2 - m_\mu^2} - \frac{\gamma^\mu \not{q} \gamma^\alpha}{(p_4 + q)^2 - m_\mu^2} \right) (\not{p}_4 - m_\mu) \gamma^\nu \right] \left. \right\} \quad (3)
 \end{aligned}$$

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Improvement. Hard pair emission kernel

$$\begin{aligned}
 & \sum_{\text{spins}} |M_1 + M_2|^2 - \sum_{\text{spins}} |M_1 + M_2|_{\text{soft}}^2 = \frac{\alpha^4 (4\pi)^4}{(p_1 + p_2)^4} \frac{1}{q^4} \text{Tr} [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] \times \quad (4) \\
 & \times \left\{ \frac{16}{(2(p_3 q) + q^2)^2} \left[4g^{\mu\nu} m_\mu^2 \frac{q^2}{2} \left(\frac{q^2}{2} + m_\tau^2 \right) + \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2} \cdot p_3 q + \text{Tr} [\not{p}_4 \gamma^\mu (\not{p}_5 - \not{p}_6) \gamma^\nu] \frac{q^2}{2} \cdot p_3 (p_5 - p_6) - \right. \right. \\
 & - \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_5 \gamma^\nu] \left(2(p_3 p_6)^2 + m_\mu^2 \frac{q^2}{2} \right) - \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_6 \gamma^\nu] \left(2(p_3 p_5)^2 + m_\mu^2 \frac{q^2}{2} \right) \Big] + \\
 & + \frac{16}{(2(p_4 q) + q^2)^2} \left[4g^{\mu\nu} m_\mu^2 \frac{q^2}{2} \left(\frac{q^2}{2} + m_\tau^2 \right) + \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2} \cdot p_4 q + \text{Tr} [\not{p}_3 \gamma^\mu (\not{p}_5 - \not{p}_6) \gamma^\nu] \frac{q^2}{2} \cdot p_4 (p_5 - p_6) - \right. \\
 & - \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_5 \gamma^\nu] \left(2(p_4 p_6)^2 + m_\mu^2 \frac{q^2}{2} \right) - \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_6 \gamma^\nu] \left(2(p_4 p_5)^2 + m_\mu^2 \frac{q^2}{2} \right) \Big] + \\
 & + \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[-2 \text{Tr} [\not{p}_5 \gamma^\mu \not{p}_6 \gamma^\nu] (m_\mu^2 m_\tau^2 + p_3 p_6 \cdot p_4 p_5 + p_3 p_5 \cdot p_4 p_6 - p_3 p_4 \cdot p_5 p_6) - \right. \\
 & - (\text{Tr} [\not{p}_5 \gamma^\mu \not{p}_5 \gamma^\nu] + \text{Tr} [\not{p}_6 \gamma^\mu \not{p}_6 \gamma^\nu]) \left(m_\mu^2 \frac{q^2}{2} + m_\tau^2 \frac{q^2}{2} \right) + \text{Tr} [\not{q}_2 \gamma^\mu \not{q}_1 \gamma^\nu] \left(p_3 p_4 \frac{q^2}{2} \right) - \\
 & - \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_3 \gamma^\nu] p_4 q \frac{q^2}{2} - \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_4 \gamma^\nu] p_3 q \frac{q^2}{2} + \\
 & + 2 \text{Tr} [(\not{p}_3 + \not{p}_4 + \not{p}_6) \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_5 \cdot p_4 p_5) + 2 \text{Tr} [(\not{p}_3 + \not{p}_4 + \not{p}_5) \gamma^\mu \not{p}_5 \gamma^\nu] (p_3 p_6 \cdot p_4 p_6) + \\
 & + 4g^{\mu\nu} \left(2m_\tau^2 (p_3 p_5 \cdot p_4 p_5 + p_3 p_6 \cdot p_4 p_6) - 2p_5 p_6 (p_3 p_6 \cdot p_4 p_5 + p_3 p_5 \cdot p_4 p_6) - 2m_\tau^2 \frac{q^2}{2} \left(\frac{q_\tau^2}{2} + p_3 p_4 \right) \right) \Big] + \\
 & + 16m_\tau^2 \left(\frac{1}{(2(p_3 q) + q^2)} + \frac{1}{(2(p_4 q) + q^2)} \right) \times \\
 & \times \left\{ \left(\frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{q}_1 \gamma^\nu] p_4 q}{2p_4 q + q^2} + \frac{\text{Tr} [\not{p}_4 \gamma^\mu \not{q}_1 \gamma^\nu] p_3 q}{2p_3 q + q^2} - \frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2}}{2(p_3 q) + q^2} - \frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2}}{2(p_4 q) + q^2} \right) \right\}
 \end{aligned}$$

Installation of matrix element

PHOTOS operates matrix element that is ratio of full matrix element to Born level matrix element with soft pair emission assumed, so, in order to be combined with matrix element (4), matrix element $|M_{\text{PHOTOS}}|^2$ has to be replaced by $|M_{\text{PHOTOS}}|^2 \times F$, where F is hard emission correction factor

$$F = \frac{\text{Tr} [(p_1 + m_e) \gamma^\mu (p_2 - m_e) \gamma^\nu] \text{Tr} [(p_3 + m_l) \gamma_\mu (p_4 - m_l) \gamma_\nu]}{\text{Tr} [(\not{p}_1 + m_e) \gamma^\mu (\not{p}_2 - m_e) \gamma^\nu] \text{Tr} [(\not{P}'_3 + m_\mu) \gamma_\mu (\not{P}'_4 - m_\mu) \gamma_\nu]}, \quad (5)$$

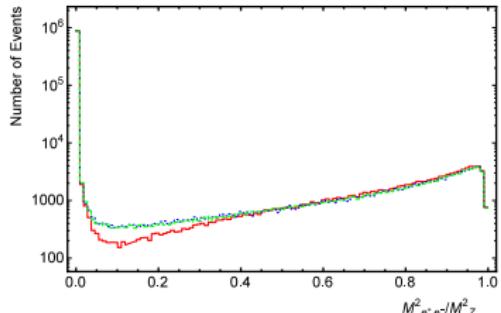
where p_1, p_2 are 4-momentums of incoming electron-positron pair, P_3, P_4 are 4-momentums of born level outgoing lepton pair, p_3, p_4 are 4-momentums of outgoing lepton pair after modification by PHOTOS.

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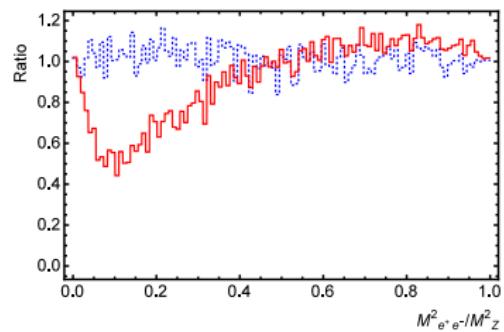
Extra pair
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Extra pair
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PHOTOS

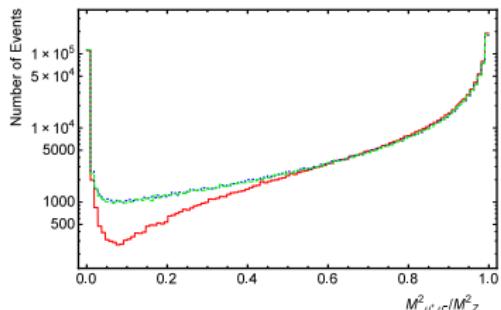
Extra pair
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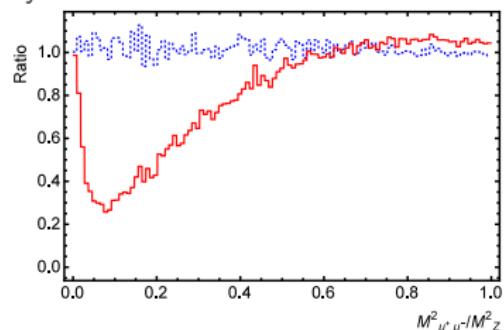
a) Normalized to M_Z^2 spectrum of electron pair mass squared.



b) Normalized to M_Z^2 ratio of PHOTOS generated spectrum to the one generated by KORALW.



a) Normalized to M_Z^2 spectrum of muon pair mass squared.

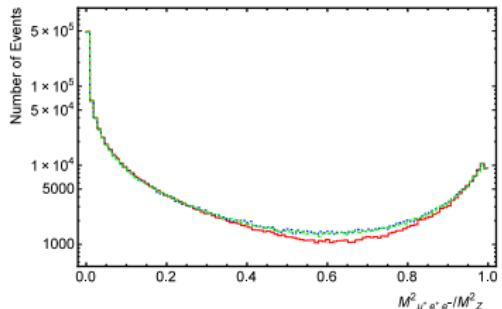


b) Normalized to M_Z^2 ratio of PHOTOS generated spectrum to the one generated by KORALW.

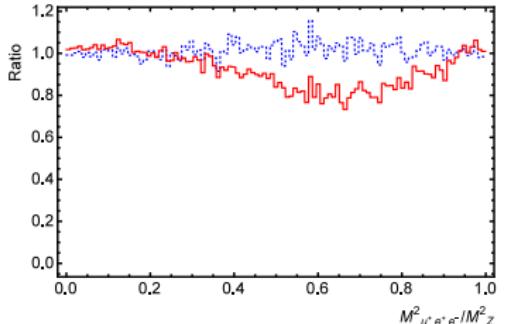
Figure 3: Normalized to M_Z^2 ratio of PHOTOS generated spectra in the channel $Z \rightarrow \mu^+ \mu^- e^+ e^-$ to the one generated by KORALW. Results generated by PHOTOS are obtained from samples of equal number of $Z \rightarrow e^+ e^-$ and $Z \rightarrow \mu^+ \mu^-$ decays. Solid red line represents output by PHOTOS. Dotted blue line represents output by PHOTOS with matrix element (4). Dashed green line represents output by KORALW.

Extra pair
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PHOTOS

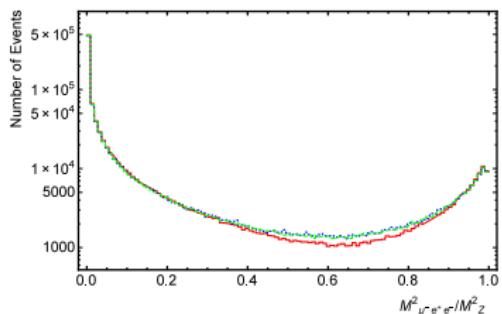
Extra pair
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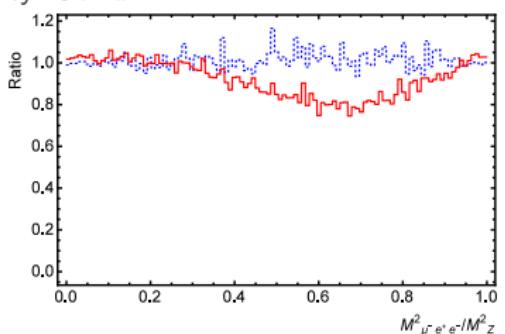
a) Normalized to M_Z^2 spectrum of electron pair mass squared.



b) Normalized to M_Z^2 ratio of PHOTOS generated spectrum to the one generated by KORALW.



a) Normalized to M_Z^2 spectrum of muon pair mass squared.



b) Normalized to M_Z^2 ratio of PHOTOS generated spectrum to the one generated by KORALW.

Figure 4: Normalized to M_Z^2 ratio of PHOTOS generated spectra in the channel $Z \rightarrow \mu^+ \mu^- e^+ e^-$ to the one generated by KORALW. Results generated by PHOTOS are obtained from samples of equal number of $Z \rightarrow e^+ e^-$ and $Z \rightarrow \mu^+ \mu^-$ decays. Solid red line represents output by PHOTOS. Dotted blue line represents output by PHOTOS with matrix element (4). Dashed green line represents output by KORALW.

Organization of cancellations

I analyze matrix element (4) and separate tensors $H_i^{\mu\nu}$ ($i = 1..7$) that are suppressed for extra pair with small invariant mass:

$$\sum_{\text{spins}} |M_1 + M_2|^2 - \sum_{\text{spins}} |M_1 + M_2|_{\text{soft}}^2 = \frac{\alpha^4 (4\pi)^4}{(\rho_1 + \rho_2)^4} \frac{1}{q^4} \text{Tr} [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] \times \sum_{i=1}^7 H_i^{\mu\nu}, \quad (6)$$

where

$$H_1^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[\text{Tr} [\not{q}_2 \gamma^\mu \not{q}_1 \gamma^\nu] \left(p_3 p_4 \frac{q^2}{2} \right) - 4g^{\mu\nu} \left(2m_\tau^2 \frac{q^2}{2} 2p_3 p_4 \right) \right] \quad (7)$$

$$H_2^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)^2} \left[\text{Tr} [\not{p}_4 \gamma^\mu (\not{p}_5 - \not{p}_6) \gamma^\nu] \frac{q^2}{2} \cdot p_3 (p_5 - p_6) \right] + \quad (8)$$

$$+ \frac{16}{(2(p_4 q) + q^2)^2} \left[\text{Tr} [\not{p}_3 \gamma^\mu (\not{p}_5 - \not{p}_6) \gamma^\nu] \frac{q^2}{2} \cdot p_4 (p_5 - p_6) \right],$$

$$H_3^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)^2} \left[\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2} \cdot p_3 q \right] - \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_3 \gamma^\nu] p_4 q \frac{q^2}{2} \right] + \quad (9)$$

$$+ \frac{16}{(2(p_4 q) + q^2)^2} \left[\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2} \cdot p_4 q \right] - \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[\text{Tr} [\not{p}_4 \gamma^\mu \not{p}_4 \gamma^\nu] p_3 q \frac{q^2}{2} \right] +$$

$$- 16m_\tau^2 \left(\frac{1}{(2(p_3 q) + q^2)} + \frac{1}{(2(p_4 q) + q^2)} \right) \left(\frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2}}{2(p_3 q) + q^2} + \frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu] \frac{q^2}{2}}{2(p_4 q) + q^2} \right),$$

Extra pair
emissions with
PHOTOS

Extra pair
emissions with
TAUOLA

Organization of cancellations

$$H_4^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)^2} \left[-2 \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_5 \gamma^\nu] (p_3 p_6)^2 - 2 \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_5)^2 \right] + \quad (10)$$

$$+ \frac{16}{(2(p_4 q) + q^2)^2} \left[-2 \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_5 \gamma^\nu] (p_4 p_6)^2 - 2 \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_6 \gamma^\nu] (p_4 p_5)^2 \right] +$$

$$+ \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[2 \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_5 \cdot p_4 p_5) + 2 \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_5 \cdot p_4 p_5) + \right. \\ \left. + 2 \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_5 \gamma^\nu] (p_3 p_6 \cdot p_4 p_6) + 2 \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_5 \gamma^\nu] (p_3 p_6 \cdot p_4 p_6) \right] +$$

$$+ 16 m_\tau^2 \left(\frac{1}{(2(p_3 q) + q^2)} + \frac{1}{(2(p_4 q) + q^2)} \right) \left(\frac{\text{Tr} [\not{p}_3 \gamma^\mu \not{q} \gamma^\nu] p_4 q}{2 p_4 q + q^2} + \frac{\text{Tr} [\not{p}_4 \gamma^\mu \not{q} \gamma^\nu] p_3 q}{2 p_3 q + q^2} \right),$$

$$H_5^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)^2} \left[4 g^{\mu\nu} m_\mu^2 \frac{q^2}{2} \left(\frac{q^2}{2} + m_\tau^2 \right) - \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_5 \gamma^\nu] m_\mu^2 \frac{q^2}{2} - \text{Tr} [\not{p}_4 \gamma^\mu \not{p}_6 \gamma^\nu] m_\mu^2 \frac{q^2}{2} \right] + \quad (11)$$

$$+ \frac{16}{(2(p_4 q) + q^2)^2} \left[4 g^{\mu\nu} m_\mu^2 \frac{q^2}{2} \left(\frac{q^2}{2} + m_\tau^2 \right) - \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_5 \gamma^\nu] m_\mu^2 \frac{q^2}{2} - \text{Tr} [\not{p}_3 \gamma^\mu \not{p}_6 \gamma^\nu] m_\mu^2 \frac{q^2}{2} \right] +$$

$$+ \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[-2 \text{Tr} [\not{p}_5 \gamma^\mu \not{p}_6 \gamma^\nu] (m_\mu^2 m_\tau^2) - 4 g^{\mu\nu} \left(2 m_\tau^2 \frac{q^2}{2} m_\mu^2 \right) - \right.$$

$$\left. - (\text{Tr} [\not{p}_5 \gamma^\mu \not{p}_5 \gamma^\nu] + \text{Tr} [\not{p}_6 \gamma^\mu \not{p}_6 \gamma^\nu]) \left(m_\mu^2 \frac{q^2}{2} + m_\tau^2 m_\mu^2 \right) \right],$$

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emissions with
PHOTOS

Extra pair
emissions with
TAUOLA

$$H_6^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[2 \text{Tr} [\not{p}_6 \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_5 \cdot p_4 p_5) + 2 \text{Tr} [\not{p}_5 \gamma^\mu \not{p}_5 \gamma^\nu] (p_3 p_6 \cdot p_4 p_6) - 2 \text{Tr} [\not{p}_5 \gamma^\mu \not{p}_6 \gamma^\nu] (p_3 p_6 \cdot p_4 p_5 + p_3 p_5 \cdot p_4 p_6) + 4g^{\mu\nu} (2m_\tau^2 (p_3 p_5 \cdot p_4 p_5 + p_3 p_6 \cdot p_4 p_6) - 2p_5 p_6 (p_3 p_6 \cdot p_4 p_5 + p_3 p_5 \cdot p_4 p_6)) \right] \quad (12)$$

$$H_7^{\mu\nu} = \frac{16}{(2(p_3 q) + q^2)(2(p_4 q) + q^2)} \left[2 \text{Tr} [\not{p}_5 \gamma^\mu \not{p}_6 \gamma^\nu] p_3 p_4 \cdot p_5 p_6 - (\text{Tr} [\not{p}_5 \gamma^\mu \not{p}_5 \gamma^\nu] + \text{Tr} [\not{p}_6 \gamma^\mu \not{p}_6 \gamma^\nu]) m_\tau^2 \cdot p_3 p_4 \right]. \quad (13)$$

Organization of cancelations

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PHOTOS

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emissions with
TAUOLA

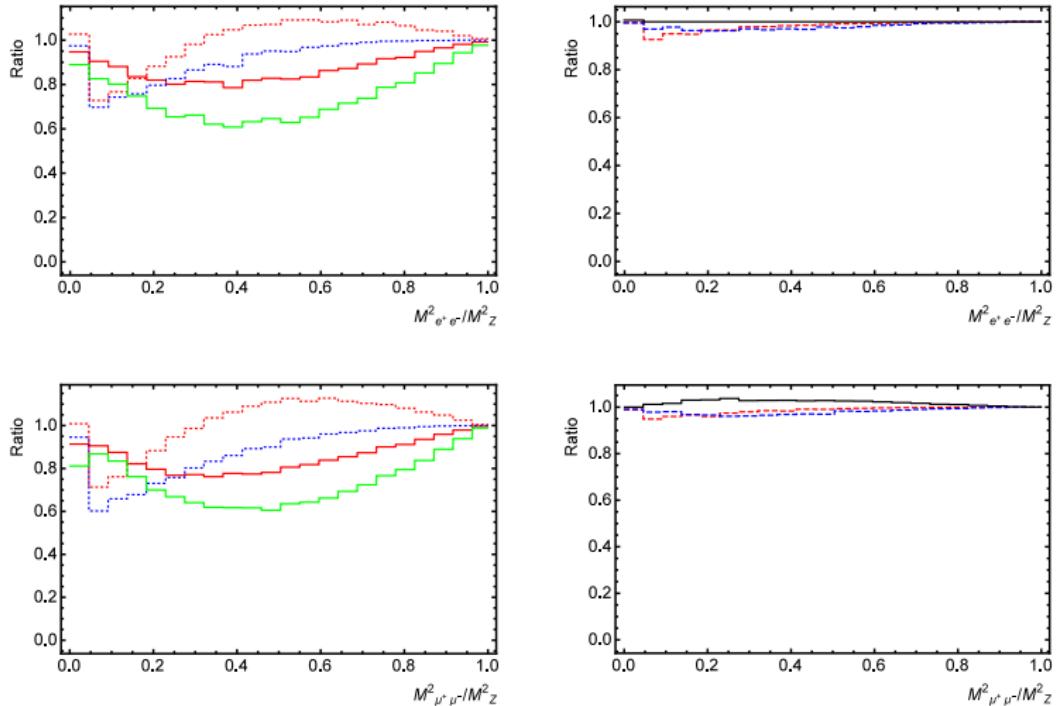


Figure 5: Normalized to M_Z^2 ratio of PHOTOS generated spectra (see axis labels) in the channel $Z \rightarrow \mu^+\mu^-e^+e^-$ to the ones, that are generated by PHOTOS with matrix element (4). Results are obtained from samples of equal number of $Z \rightarrow e^+e^-$ and $Z \rightarrow \mu^+\mu^-$ decays. Green solid line represents absence of the tensor expression $H_1^{\mu,\nu}$ in the complete matrix element (4). Blue dotted line – tensor $H_2^{\mu,\nu}$, solid red line – tensor $H_3^{\mu,\nu}$, red dotted line – tensor $H_4^{\mu,\nu}$, solid black line – tensor $H_5^{\mu,\nu}$, red dashed line – tensor $H_6^{\mu,\nu}$, blue dashed line – tensor $H_7^{\mu,\nu}$.

Simplification

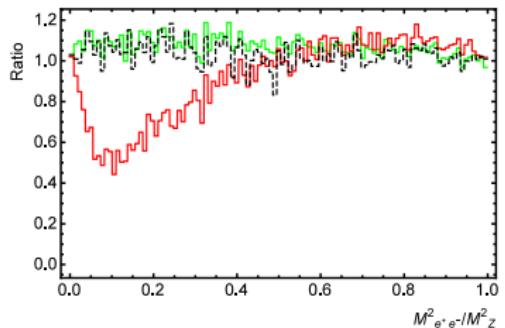
Simulation of
fermion pair
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decays

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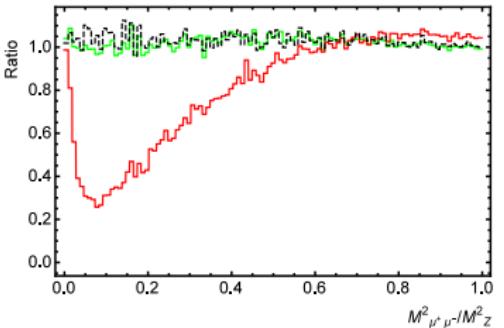
Extra pair
emissions with
PHOTOS

Extra pair
emissions with
TAUOLA

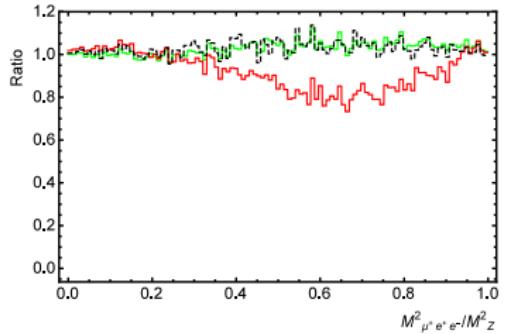
$$|M|_{approx}^2 = \frac{\alpha^4 (4\pi)^4}{(p_1 + p_2)^4} \frac{1}{q^4} Tr [(\not{p}_1 + m_e) \gamma_\mu (\not{p}_2 - m_e) \gamma_\nu] \times \quad (14)$$
$$\times (H_2^{\mu\nu} + H_3^{\mu\nu} + H_4^{\mu\nu} - H_6^{\mu\nu} - H_7^{\mu\nu}),$$



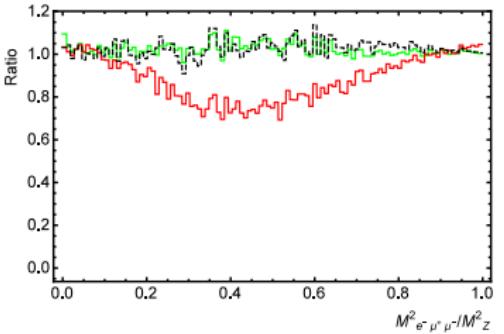
a) e^+e^- .



b) $\mu^+\mu^-$.

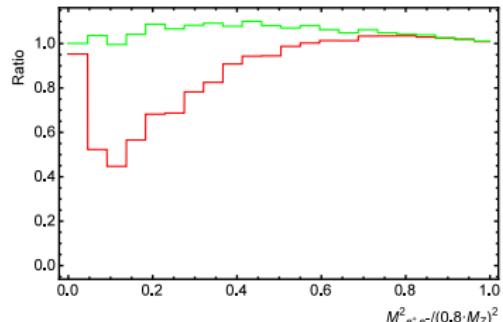


c) $\mu^+e^+e^-$.

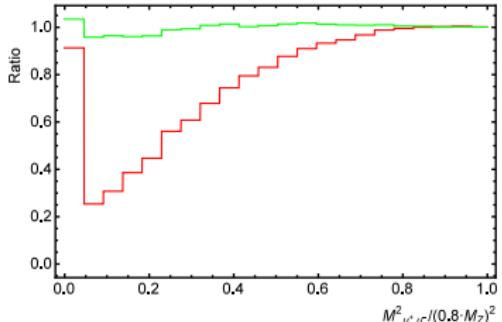


d) $e^-\mu^+\mu^-$.

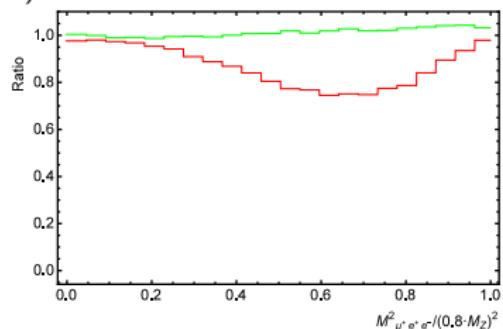
Figure 6: Normalized to M_Z^2 ratio of PHOTOS generated spectra in the channel $Z \rightarrow \mu^+\mu^-e^+e^-$ to the one generated by KORALW. Results generated by PHOTOS are obtained from samples of equal number of $Z \rightarrow e^+e^-$ and $Z \rightarrow \mu^+\mu^-$ decays. Solid red line represents data that correspond to PHOTOS. Black dashed line and solid green line represent data that correspond to PHOTOS with matrix element (4) and simplified matrix element (14) correspondingly.



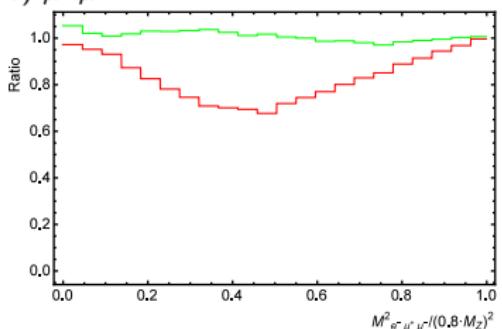
a) e^+e^- .



b) $\mu^+\mu^-$.

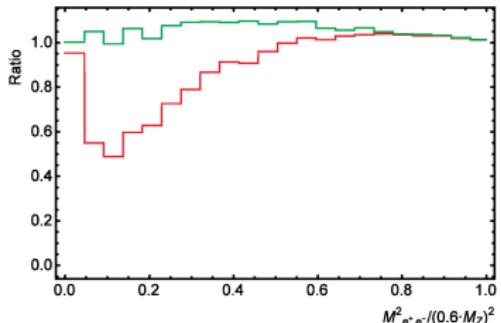


c) $\mu^+e^+e^-$.

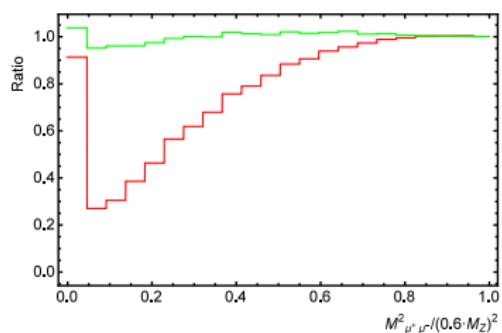


d) $e^-\mu^+\mu^-$.

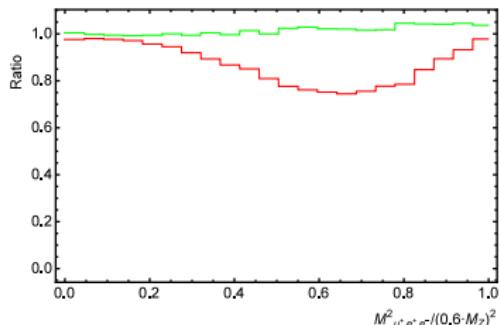
Figure 7: Normalized to M_Z^2 ratio of PHOTOS generated spectra in the channel $Z \rightarrow \mu^+\mu^-e^+e^-$ to the one generated by PHOTOS with matrix element (4). Results generated by PHOTOS are obtained from samples of equal number of $Z \rightarrow e^+e^-$ and $Z \rightarrow \mu^+\mu^-$ decays. Solid red line represents data that correspond to PHOTOS. Solid green line represent data that correspond to PHOTOS with simplified matrix element (14).



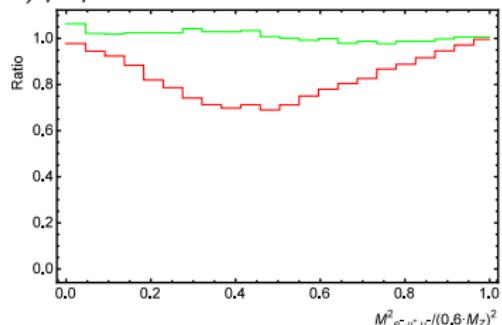
a) $e^+ e^-$.



b) $\mu^+ \mu^-$.



c) $\mu^+ e^+ e^-$.



d) $e^- \mu^+ \mu^-$.

Figure 8: Normalized to M_Z^2 ratio of PHOTOS generated spectra in the channel $Z \rightarrow \mu^+ \mu^- e^+ e^-$ to the one generated by PHOTOS with matrix element (4). Results generated by PHOTOS are obtained from samples of equal number of $Z \rightarrow e^+ e^-$ and $Z \rightarrow \mu^+ \mu^-$ decays. Solid red line represents data that correspond to PHOTOS. Solid green line represent data that correspond to PHOTOS with simplified matrix element (14).

Achievement

Simulation of
fermion pair
bremsstrahlung in
decays

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Extra pair
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- ▶ Complete matrix element (4) $e^+e^- \rightarrow Z \rightarrow l^+l^- + (L^+L^-)$ is installed into PHOTOS and works perfectly.
- ▶ Simplification of matrix element (4) is installed into PHOTOS and works as good as complete one.
- ▶ Matrix element kernel, as of Fig. 3, works better than matrix element of published version of PHOTOS. Difference is up to factor of 3.5 for some parts of the spectra.
- ▶ Kernel, as of Fig. 3, has a potential of being used universally.

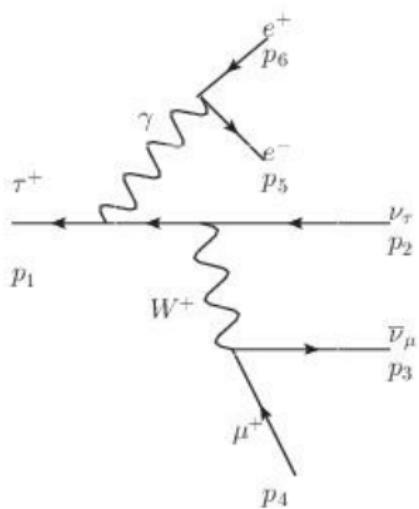
- ▶ TAUOLA¹ is a MC Generator dedicated to generating tau-lepton decays.
- ▶ Phase space generation is precise and similar to one by PHOTOS.
- ▶ Matrix element for $\tau \rightarrow \mu\nu\bar{\nu}$ is complete down to $O(\alpha^2)$.
- ▶ TAUOLA-PHOTOS comparison in the channel $\tau \rightarrow \mu\nu\bar{\nu}$ will provide source of tests for new PHOTOS kernels.

¹M. Chrzaszcz, T. Przedzinski, Z. Was and J. Zaremba, Comput. Phys. Commun. 232, 220 (2018) doi:10.1016/j.cpc.2018.05.017 [arXiv:1609.04617 [hep-ph]].

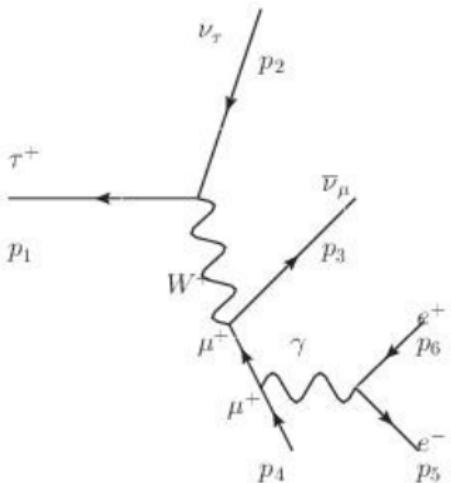
Extra pair emission for tau decay

Simulation of
fermion pair
bremsstrahlung in
decays

S. Antropov



I



II

Figure 9: Feynman graphs corresponding to real pair emissions for $\tau^+ \rightarrow \mu^+ \nu \bar{\nu} e^+ e^-$ decay process.

Extra pair
emissions with
PHOTOS

Extra pair
emissions with
TAUOLA

Amplitude for tau decay with e^+e^- pair emission

$$\sum_{\text{spins}} |M_1 + M_2|^2 = \frac{e^4}{\left((p_1 - p_2)^2 - M_W^2 \right)^2} \frac{e^4}{64 \cdot \sin^4 \theta_W} \frac{q^4}{q^4} \text{Tr} [(\not{p}_5 + m_e) \gamma_\alpha (\not{p}_6 - m_e) \gamma_\beta] \times \quad (15)$$

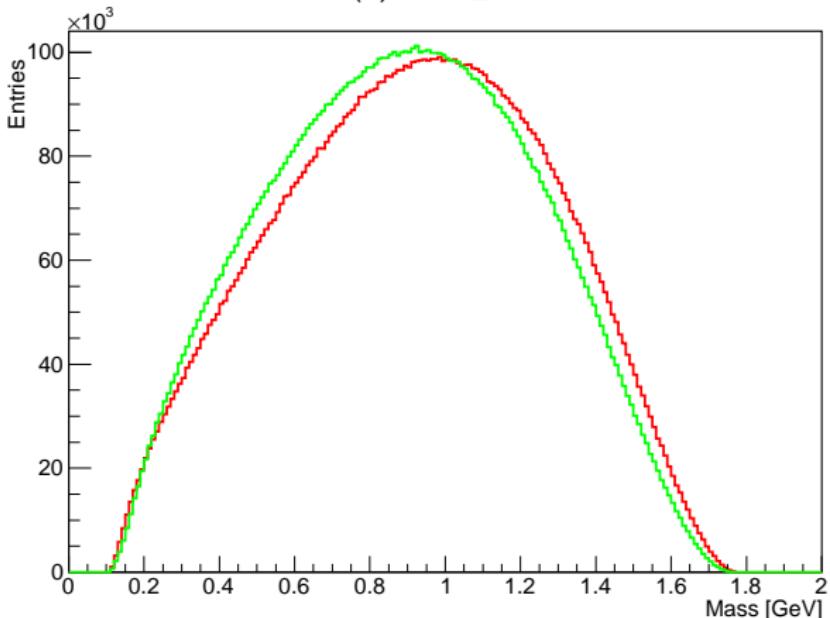
$$\begin{aligned} & \times \left\{ \text{Tr} [(\not{p}_1 - m_\tau) \gamma^\mu (1 - \gamma^5) (\not{p}_2 - m_\nu) (1 + \gamma^5) \gamma_\nu] \times \right. \\ & \times \text{Tr} \left[(\not{p}_3 + m_{\bar{\nu}}) \gamma_\mu (1 - \gamma^5) \frac{\not{q}\gamma^\alpha + 2p_4^\alpha}{(q + p_4)^2 - m_\mu^2} (\not{p}_4 - m_\mu) \frac{\gamma^\beta \not{q} + 2p_4^\beta}{(q + p_4)^2 - m_\mu^2} (1 + \gamma^5) \gamma^\nu \right] + \\ & + \text{Tr} \left[(\not{p}_1 - m_\tau) \gamma^\mu (1 - \gamma^5) (\not{p}_2 - m_\nu) (1 + \gamma^5) \gamma_\nu \frac{\not{q}\gamma^\beta + 2p_1^\beta}{(p_1 - q)^2 - m_\tau^2} \right] \times \\ & \times \text{Tr} \left[(\not{p}_3 + m_{\bar{\nu}}) \gamma_\mu (1 - \gamma^5) \frac{\not{q}\gamma^\alpha + 2p_4^\alpha}{(q + p_4)^2 - m_\mu^2} (\not{p}_4 - m_\mu) (1 + \gamma^5) \gamma^\nu \right] + \\ & + \text{Tr} \left[(\not{p}_1 - m_\tau) \frac{\gamma^\alpha \not{q} + 2p_1^\alpha}{(p_1 - q)^2 - m_\tau^2} \gamma^\mu (1 - \gamma^5) (\not{p}_2 - m_\nu) (1 + \gamma^5) \gamma_\nu \right] \times \\ & \times \text{Tr} \left[(\not{p}_3 + m_{\bar{\nu}}) \gamma_\mu (1 - \gamma^5) (\not{p}_4 - m_\mu) \frac{\gamma^\beta \not{q} + 2p_4^\beta}{(q + p_4)^2 - m_\mu^2} (1 + \gamma^5) \gamma^\nu \right] + \\ & + \text{Tr} \left[(\not{p}_1 - m_\tau) \frac{\gamma^\alpha \not{q} + 2p_1^\alpha}{(p_1 - q)^2 - m_\tau^2} \gamma^\mu (1 - \gamma^5) (\not{p}_2 - m_\nu) (1 + \gamma^5) \gamma_\nu \frac{\gamma^\alpha \not{q} + 2p_1^\alpha}{(p_1 - q)^2 - m_\tau^2} \right] \times \\ & \left. \times \text{Tr} [(\not{p}_3 + m_{\bar{\nu}}) \gamma_\mu (1 - \gamma^5) (\not{p}_4 - m_\mu) (1 + \gamma^5) \gamma^\nu] \right\} \end{aligned}$$

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$$\begin{aligned}
 \sum_{spins} |M_1 + M_2|_{soft}^2 &= \frac{e^4}{\left((p_1 - p_2)^2 - M_W^2 \right)^2 64 \cdot \sin^4 \theta_W} \times \quad (16) \\
 &\times \text{Tr} [(\not{p}_1 - m_\tau) \gamma^\mu (1 - \gamma^5) (\not{p}_2 - m_\nu) (1 + \gamma^5) \gamma_\nu] \times \\
 &\times \text{Tr} [(\not{p}_3 + m_{\bar{\nu}}) \gamma_\mu (1 - \gamma^5) (\not{p}_4 - m_\mu) (1 + \gamma^5) \gamma^\nu] \times \\
 &\times \frac{e^4}{q^4} \text{Tr} [(\not{p}_5 + m_e) \gamma_\alpha (\not{p}_6 - m_e) \gamma_\beta] \times \\
 &\times \left(\frac{p_4}{qp_4 + \frac{q^2}{2}} - \frac{p_1}{qp_1 - \frac{q^2}{2}} \right)_\alpha \left(\frac{p_4}{qp_4 + \frac{q^2}{2}} - \frac{p_1}{qp_1 - \frac{q^2}{2}} \right)_\beta
 \end{aligned}$$

Mass(1) of nu_mu~ mu-



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Figure 5: Invariant mass of $\mu^- \nu_\mu$ ¹. Red line corresponds to the data from $\tau^- \rightarrow \mu^- \nu\bar{\nu}$ channel, while green line corresponds to the data from $\tau^- \rightarrow \mu^- \nu\bar{\nu} e^+ e^-$ channel.

¹All TAUOLA modifications are provided by dr inz. Jakub Zaremba, data are prepared by him as well.

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PHOTOS

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Conclusions

- ▶ Full agreement between PHOTOS and KORALW is reached in the channel $Z \rightarrow 4f$ both for exact matrix element installed in PHOTOS (4) and by it's further simplification.
- ▶ This agreement is reached without terms in matrix element that correspond to interference between lepton pairs.
- ▶ Construction of emission kernel for individual lepton is rather straightforward.
- ▶ First TAUOLA tests are done and look promising. Preparation of tests for $\tau^- \rightarrow \pi^- \nu_\tau e^+ e^-$ is rather straightforward¹.
- ▶ Kernel to be installed into PHOTOS may be universal.

¹A. Guevara, G. Lopez Castro and P. Roig, Phys. Rev. D 88, no. 3, 033007 (2013)
doi:10.1103/PhysRevD.88.033007 [arXiv:1306.1732 [hep-ph]].

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PHOTOS

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Thank you for your attention!

BACKUP. Math of phase space

Cross section for a decay with 3-body final state is an integral of matrix element squared $|M|^2 \equiv |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2$ over 3-body phase space $dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)$

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$$\int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu) = \int |M|^2 \frac{d^3 p_\nu}{(2\pi)^3 2p_\nu^0} \frac{d^3 p_{\bar{\nu}}}{(2\pi)^3 2p_{\bar{\nu}}^0} \frac{d^3 p_\mu}{(2\pi)^3 2p_\mu^0} (2\pi)^4 \delta^4(p_\tau - p_\nu - p_{\bar{\nu}} - p_\mu) = \quad (17)$$

$$= \frac{1}{2^{11}\pi^5} \int_{m_\mu^2}^{(m_\tau - m_\mu)^2} dM_{\bar{\nu}\mu}^2 \int_{-1}^1 d\cos\theta_\nu \int_0^{2\pi} d\varphi_\nu \left(1 - \frac{M_{\bar{\nu}\mu}^2}{m_\tau^2}\right) \int_{-1}^1 d\cos\theta_{\bar{\nu}} \int_0^{2\pi} d\varphi_{\bar{\nu}} \left(1 - \frac{m_\mu^2}{M_{\bar{\nu}\mu}^2}\right) |M|^2, \quad (18)$$

where $p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu$ are four-momentums of τ^- , ν , $\bar{\nu}_\mu$, μ^- correspondingly; $d\cos\theta_\nu d\varphi_\nu$ is the solid angle element of p_ν in the rest frame of τ^- , $d\cos\theta_{\bar{\nu}} d\varphi_{\bar{\nu}}$ is the solid angle element of $p_{\bar{\nu}}$ in the rest frame of $(p_{\bar{\nu}} + p_\mu)$; $M_{\bar{\nu}\mu}^2 = (p_{\bar{\nu}} + p_\mu)^2$; m_μ is mass of μ^- and m_τ is mass of τ^- .

In the following tests we focus exclusively on soft pair emissions.

We proceed with writing a cross section for the 5-body decay

$\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau e^+ e^-$ assuming matrix element

$|M|^2 \equiv |M(p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu)|^2$ can be factorized

like $|M|^2 = |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2 \times |M_F(p_{e-}, p_{e+})|^2$:

$$\begin{aligned} & \int |M|^2 dLips_5(p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu) = \\ &= \int |M|^2 \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} \frac{d^3 p_\nu}{(2\pi)^3 2p_\nu^0} \frac{d^3 p_{\bar{\nu}}}{(2\pi)^3 2p_{\bar{\nu}}^0} \frac{d^3 p_\mu}{(2\pi)^3 2p_\mu^0} (2\pi)^4 \delta^4(p_\tau - p_\nu - p_{\bar{\nu}} - p_\mu - p_{e-} - p_{e+}) = \\ &= \int |M_F|^2 \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} d^4 R \delta^4(R - p_\tau + p_{e-} + p_{e+}) \times \end{aligned} \quad (19)$$

$$\times \int |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2 \frac{d^3 p_\nu}{(2\pi)^3 2p_\nu^0} \frac{d^3 p_{\bar{\nu}}}{(2\pi)^3 2p_{\bar{\nu}}^0} \frac{d^3 p_\mu}{(2\pi)^3 2p_\mu^0} (2\pi)^4 \delta^4(R - p_\nu - p_{\bar{\nu}} - p_\mu), \quad (20)$$

where $p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu$ are four-momentums of $\tau^-, e^-, e^+, \nu, \bar{\nu}_\mu, \mu^-$ correspondingly.

In the following test we put $|M_F|^2 \equiv 1$. Since factorized part of matrix element squared $|M_F|^2$ does not depend on p_{e-} , p_{e+} anymore, for a soft pair emission we can drop e^+ and e^- from the conditions of momentum-energy conservation, thus integral

$d^4R\delta^4(R - p_\tau + p_{e-} + p_{e+})$ gives us $R = p_\tau$ causing :

$$\begin{aligned} \int |M|^2 dLips_5(p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu) &\approx \int \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} \times \\ &\times \int |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2 \frac{d^3 p_\nu}{(2\pi)^3 2p_\nu^0} \frac{d^3 p_{\bar{\nu}}}{(2\pi)^3 2p_{\bar{\nu}}^0} \frac{d^3 p_\mu}{(2\pi)^3 2p_\mu^0} (2\pi)^4 \delta^4(p_\tau - p_\nu - p_{\bar{\nu}} - p_\mu) = \\ &= \int \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} \int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu) = \\ &= \int |M|^2 \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} dM_{\bar{\nu}\mu}^2 \left[d\cos\theta_\nu d\varphi_\nu \left(1 - \frac{M_{\bar{\nu}\mu}^2}{m_\tau^2}\right) \right]_{\bar{p}_\tau=0} \left[d\cos\theta_{\bar{\nu}} d\varphi_{\bar{\nu}} \left(1 - \frac{m_\mu^2}{M_{\bar{\nu}\mu}^2}\right) \right]_{\bar{p}_{\bar{\nu}}+\bar{p}_\mu=0} = \\ &= \frac{1}{28\pi^6} \int \left[d\cos\theta_{e-} d\varphi_{e-} \frac{|\bar{p}_{e-}|^2 d|\bar{p}_{e-}|}{\sqrt{|\bar{p}_{e-}|^2 + m_e^2}} \right]_{\bar{p}_\tau=0} \left[d\cos\theta_{e+} d\varphi_{e+} \frac{|\bar{p}_{e+}|^2 d|\bar{p}_{e+}|}{\sqrt{|\bar{p}_{e+}|^2 + m_e^2}} \right]_{\bar{p}_\tau=0} \times \quad (21) \end{aligned}$$

$$\times \frac{1}{2^{11}\pi^5} \int |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2 dM_{\bar{\nu}\mu}^2 \left[d\cos\theta_\nu d\varphi_\nu \left(1 - \frac{M_{\bar{\nu}\mu}^2}{m_\tau^2}\right) \right]_{\bar{p}_\tau=0} \left[d\cos\theta_{\bar{\nu}} d\varphi_{\bar{\nu}} \left(1 - \frac{m_\mu^2}{M_{\bar{\nu}\mu}^2}\right) \right]_{\bar{p}_{\bar{\nu}}+\bar{p}_\mu=0}, \quad (22)$$

where \bar{p}_{e-} , \bar{p}_{e+} are three-momenta of e^- , e^+ correspondingly;

subscript $\bar{p}_\tau = 0$ or $\bar{p}_{\bar{\nu}} + \bar{p}_\mu = 0$ means that the variables into square brackets are in τ^- or in $\bar{\nu}_\mu - \mu^-$ correspondingly;

$d\cos\theta_{e-} d\varphi_{e-}$ is the solid angle element of p_{e-} , $d\cos\theta_{e+} d\varphi_{e+}$ is the solid angle element of p_{e+} , $d\cos\theta_\nu d\varphi_\nu$ is the solid angle element of p_ν , $d\cos\theta_{\bar{\nu}} d\varphi_{\bar{\nu}}$ is the solid angle element of $p_{\bar{\nu}}$;
 $M_{\bar{\nu}\mu}^2 = (p_{\bar{\nu}} + p_\mu)^2$.

Next we introduce a cutoff parameter Δ : $p_{e+}^0 < \Delta$, $p_{e-}^0 < \Delta$. Such conditions match the conditions in TAUOLA code for the test. Finally we have

$$\begin{aligned}
 & \int |M|^2 dLips_5(p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu) \approx \\
 & \approx \frac{1}{2^8 \pi^6} \int_0^{\Delta \gg m_e} 4\pi \frac{|\bar{p}_{e-}|^2 d|\bar{p}_{e-}|}{\sqrt{|\bar{p}_{e-}|^2 + m_e^2}} \int_0^\Delta 4\pi \frac{|\bar{p}_{e+}|^2 d|\bar{p}_{e+}|}{\sqrt{|\bar{p}_{e+}|^2 + m_e^2}} \times \int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu) \approx \\
 & \approx \frac{1}{2^4 \pi^4} \left(\frac{\Delta^2}{2} \right)^2 \times \int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu) = \frac{\Delta^4}{2^6 \pi^4} \int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu).
 \end{aligned} \tag{23}$$

One more way to do the same test, i.e. $|M_F|^2 \equiv 1$, is to write cross section for the 5-body decay in terms of invariant mass variables

$$\int |M|^2 dLips_5(p_\tau) = \frac{1}{2^{11}\pi^5} \int dM_{\bar{\nu}\mu ee}^2 \int d\Omega_\nu \left(1 - \frac{M_{\bar{\nu}\mu ee}^2}{m_\tau^2}\right) \int d\Omega_{\bar{\nu}} \left(1 - \frac{M_{\mu ee}^2}{M_{\bar{\nu}\mu ee}^2}\right) |M(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu)|^2 \times \quad (24)$$

$$\times \frac{1}{2^{12}\pi^6} \int d\Omega_\mu \int d\Omega_e \int dM_{ee}^2 |M_F|^2 \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \int dM_{\mu ee}^2 \frac{\sqrt{(M_{\mu ee}^2 - M_{ee}^2 - m_\mu^2)^2 - 4M_{ee}^2 m_\mu^2}}{M_{\mu ee}^2}, \quad (25)$$

where $M_{\bar{\nu}\mu ee}^2 = (p_{\bar{\nu}} + p_\mu + p_{e-} + p_{e+})^2$,

$M_{\mu ee}^2 = (p_\mu + p_{e-} + p_{e+})^2$, $M_{ee}^2 = (p_{e-} + p_{e+})^2$; $d\Omega_\nu$ is the solid angle element of p_ν in the rest frame of τ^- , $d\Omega_{\bar{\nu}}$ is the solid angle element of $p_{\bar{\nu}}$ in the rest frame of $(p_{e-} + p_{e+} + p_{\bar{\nu}} + p_\mu)$, $d\Omega_\mu$ is the solid angle element of p_μ in the rest frame of $(p_{e-} + p_{e+} + p_\mu)$, $d\Omega_e$ is the solid angle element of p_{e-} in the rest frame of $(p_{e-} + p_{e+})$.

Considering soft pair emission, we can approximate $M_{\bar{\nu}\mu ee}^2 \approx M_{\bar{\nu}\mu}^2$, $M_{\mu ee}^2 = m_\mu^2$, thus first part of cross section (24) coincide with cross section (18) for 3-body decay $\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau$. Soft pair emission factor writes

$$B_f(\Delta) = \frac{1}{2^{12}\pi^6} \int d\Omega_\mu \int d\Omega_e \int dM_{ee}^2 |M_F|^2 \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \int dM_{\mu ee}^2 \frac{\sqrt{(M_{\mu ee}^2 - M_{ee}^2 - m_\mu^2)^2 - 4M_{ee}^2 m_\mu^2}}{M_{\mu ee}^2} = \\ = \left\{ |M_F|^2 = 1 \right\} = \frac{1}{2^8\pi^4} \int_{4m_e^2}^{\Delta^2} dM_{ee}^2 \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \int_{(m_\mu + M_{ee})^2}^{(m_\mu + \Delta)^2} dM_{\mu ee}^2 \frac{\sqrt{(M_{\mu ee}^2 - M_{ee}^2 - m_\mu^2)^2 - 4M_{ee}^2 m_\mu^2}}{M_{\mu ee}^2}, \quad (26)$$

where Δ is cutoff parameter. Here cutoff affects invariant mass parameters of TAUOLA, therefore cutoff conditions could be invoked both in the fortran and in C part of the code. Crosscheck of application of these cutoff conditions pushed us to conclusion to move into double precision.

In the following test finally we use physical factorized part of matrix element squared

$$|M_F|^2 = 2e^4 \frac{4p_{e-}^\alpha p_{e+}^\beta - q^2 g^{\alpha\beta}}{q^4} \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\alpha \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\beta.$$

it represents soft pair emission from the initial and from the final state. Soft pair emission factor in this case essential depend on four momentum of muon p_μ (without loose of generality it could be electron). This dependence reduces usability of soft pair emission factor, since this factor should be included into formula for three particles decay, which we considered fully independent before:

$$\int |M|^2 dLips_5(p_\tau, p_{e-}, p_{e+}, p_\nu, p_{\bar{\nu}}, p_\mu) \approx \int |M|^2 dLips_3(p_\tau, p_\nu, p_{\bar{\nu}}, p_\mu) B_f(p_4).$$

Following calculations are very similar to the ones, that are discussed in work². Soft pair emission factor writes

$$\begin{aligned} B_f(p_4) &= \int |M_F|^2 \frac{d^3 p_{e-}}{(2\pi)^3 2p_{e-}^0} \frac{d^3 p_{e+}}{(2\pi)^3 2p_{e+}^0} d^4 R \delta^4(R - p_\tau + p_{e-} + p_{e+}) = \\ &= \frac{2(4\pi\alpha)^2}{(2\pi)^6} \int \frac{4p_{e-}^\alpha p_{e+}^\beta - q^2 g^{\alpha\beta}}{q^4} \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\alpha \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\beta \frac{d^3 p_{e-}}{2p_{e-}^0} \frac{d^3 p_{e+}}{2p_{e+}^0} d^4 q dM_{ee}^2 \times \\ &\quad \times \delta(q^2 - M_{ee}^2) \Theta(q^0) \delta^4(q - p_{e+} - p_{e-}) d^4 R dM_{\mu\nu\bar{\nu}}^2 \delta(R^2 - M_{\mu\nu\bar{\nu}}^2) \Theta(R^0) \delta^4 \delta^4(R - p_\tau + q). \end{aligned}$$

In order to perform following integration we work temporarily in $(p_{e+} + p_{e-})$ at rest frame

$$\frac{d^3 p_{e-}}{2p_{e-}^0} \frac{d^3 p_{e+}}{2p_{e+}^0} d^4 q \delta^4(q - p_{e+} - p_{e-}) = \int d^4 q d\cos\theta_1 d\varphi_1 \frac{1}{8} \sqrt{1 - \frac{4m_e^2}{q^2}},$$
$$\int d^4 q d\cos\theta_1 d\varphi_1 \frac{4p_{e-}^\alpha p_{e+}^\beta - q^2 g^{\alpha\beta}}{q^4} \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\alpha \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)_\beta =$$
$$= -\frac{8\pi}{3} \int \frac{d^4 q}{q^2} \left(1 + \frac{2m_e^2}{q^2} \right) \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau} \right)^2.$$

Note that at this point of calculation Lorentz-invariance of function to integrate is restored and we are free to choose any other rest frame.

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Having integration of delta-functions performed

$$\int d^4 q dM_{ee}^2 \delta(q^2 - M_{ee}^2) \Theta(q^0) = \int \frac{d^3 q}{2q^0},$$

$$\int d^4 R dM_{\mu\nu\bar{\nu}}^2 \delta(R^2 - M_{\mu\nu\bar{\nu}}^2) \Theta(R^0) = \int \frac{d^3 R}{2R^0}.$$

we continue

$$\begin{aligned} B_f(p_4) &= -\frac{\alpha^2}{2\pi^4} \frac{8\pi}{3 \cdot 8} \int \frac{dM_{ee}^2}{M_{ee}^2} \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \left(1 + \frac{2m_e^2}{M_{ee}^2}\right) \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2 \frac{d^3 q}{2q^0} \frac{d^3 R}{2R^0} dM_{\mu\nu\bar{\nu}}^2 \delta^4(R - p_1 + q) = \\ &= -\frac{\alpha^2}{6\pi^3} \int \frac{dM_{ee}^2}{M_{ee}^2} \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \left(1 + \frac{2m_e^2}{M_{ee}^2}\right) \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2 dM_{\mu\nu\bar{\nu}}^2 \frac{d^3 q}{2q^0} \delta(M_{\mu\nu\bar{\nu}}^2 - (p_1 - q)^2) = \\ &= -\frac{\alpha^2}{6\pi^3} \int \frac{dM_{ee}^2}{M_{ee}^2} \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \left(1 + \frac{2m_e^2}{M_{ee}^2}\right) \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2 dM_{\mu\nu\bar{\nu}}^2 d\cos\theta_2 d\varphi_2 \frac{\lambda^{1/2}(m_\tau^2, M_{ee}^2, M_{\mu\nu\bar{\nu}}^2)}{8m_\tau^2}, \end{aligned}$$

where $d\cos\theta_2 d\varphi_2$ is solid angle of momentum of the pair $p_{e-} + p_{e+}$ in the rest frame of τ^- .

Next step of calculation is to write four momenta p_μ and p_τ in rest frame of τ^- . For our choice of variables four momenta of muon is not fully independent variable. Space orientation of three momenta \bar{q} (specifically angle θ_2) for given invariant mass $M_{\mu\bar{\nu}\nu}$ affects momentum p_μ . Assuming soft pair emission we ignore such dependence, so expression $\left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2$ can be integrated

$$\begin{aligned} \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2 &= \frac{p_\mu^2}{(qp_\mu)^2} + \frac{p_\tau^2}{(qp_\tau)^2} - \frac{2p_\mu p_\tau}{qp_\mu \cdot qp_\tau} = \\ &= \frac{m_\mu^2}{(p_\mu^0 q^0 - |\bar{p}_\mu| |\bar{q}| \cos\theta_2)^2} + \frac{m_\tau^2}{(m_\tau q^0)^2} - \frac{2m_\tau p_\mu^0}{(p_\mu^0 q^0 - |\bar{p}_\mu| |\bar{q}| \cos\theta_2) \cdot m_\tau q^0}, \\ \int_{-1}^{+1} \left(\frac{p_\mu}{qp_\mu} - \frac{p_\tau}{qp_\tau}\right)^2 d\cos\theta_2 &= \frac{2m_\mu^2}{(p_\mu^0 q^0)^2 - (|\bar{p}_\mu| |\bar{q}|)^2} + \frac{2}{(q^0)^2} + \frac{2p_\mu^0}{q^0 (|\bar{p}_\mu| |\bar{q}|)} \ln \frac{p_\mu^0 q^0 - |\bar{p}_\mu| |\bar{q}|}{p_\mu^0 q^0 + |\bar{p}_\mu| |\bar{q}|}, \end{aligned}$$

leading to

$$\begin{aligned} B_f(p_4) &= -\frac{\alpha^2}{6\pi^3} \frac{2\pi \cdot 2}{8m_\tau^2} \int \frac{dM_{ee}^2}{M_{ee}^2} \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \left(1 + \frac{2m_e^2}{M_{ee}^2}\right) dM_{\mu\nu\bar{\nu}}^2 \times \\ &\times \left(\frac{m_\mu^2}{(p_\mu^0 q^0)^2 - (|\bar{p}_\mu| |\bar{q}|)^2} + \frac{1}{(q^0)^2} + \frac{p_\mu^0}{q^0 (|\bar{p}_\mu| |\bar{q}|)} \ln \frac{p_\mu^0 q^0 - |\bar{p}_\mu| |\bar{q}|}{p_\mu^0 q^0 + |\bar{p}_\mu| |\bar{q}|} \right) \lambda^{1/2}. \end{aligned}$$

We change here integration variable from square of invariant mass $M_{\mu\nu\bar{\nu}}^2$ to energy of the pair E_{ee} at rest frame of

$$E_{ee} = q^0 = \frac{m_\tau^2 + M_{ee}^2 - M_{\mu\nu\bar{\nu}}^2}{2m_\tau},$$

$$dE_{ee} = -\frac{dM_{\mu\nu\bar{\nu}}^2}{2m_\tau},$$

$$\lambda^{1/2}(m_\tau^2, M_{ee}^2, M_{\mu\nu\bar{\nu}}^2) = 2m_\tau \sqrt{E_{ee}^2 - M_{ee}^2}.$$

Next one important thing to make analytical integration possible is to assume $p_\mu^0 \approx |\vec{p}_\mu|$. It is weak assumption since muon mass is not something small to neglect. This assumption works much better when muon is replaced by electron.

Soft pair emission factor writes

$$B_f(p_4) = \frac{\alpha^2}{3\pi^2} \frac{(2m_\tau)^2}{4m_\tau^2} \int_{\frac{4m_e^2}{M_{ee}^2}}^{\Delta} \frac{dM_{ee}^2}{M_{ee}^2} \sqrt{1 - \frac{4m_e^2}{M_{ee}^2}} \left(1 + \frac{2m_e^2}{M_{ee}^2}\right) \times \\ \times \int_{M_{ee}}^{\Delta} dE_{ee} \left(\frac{m_\mu^2 \sqrt{E_{ee}^2 - M_{ee}^2}}{(p_4^0)^2 M_{ee}^2} + \frac{\sqrt{E_{ee}^2 - M_{ee}^2}}{E_{ee}^2} + \left[\frac{1}{E_{ee}} \ln \frac{E_{ee} - \sqrt{E_{ee}^2 - M_{ee}^2}}{E_{ee} + \sqrt{E_{ee}^2 - M_{ee}^2}} \right] \right).$$

Part of the function to integrate which is in square brackets coincide with the one from formula (8) in paper¹

¹S. Jadach, M. Skrzypek and B. F. L. Ward, Phys. Rev. D 49, 1178 (1994).
doi:10.1103/PhysRevD.49.1178

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