High order perturbation calculations for colliders physics

Xin Guan

SLAC

Mainly based on 2209.14259, 2209.14953
In collaboration with Xiang Chen, Chuan-Qi He, Zhao Li, Xiao Liu and Yan-Qing Ma

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Outline

I. Introduction

II. High order calculation methods

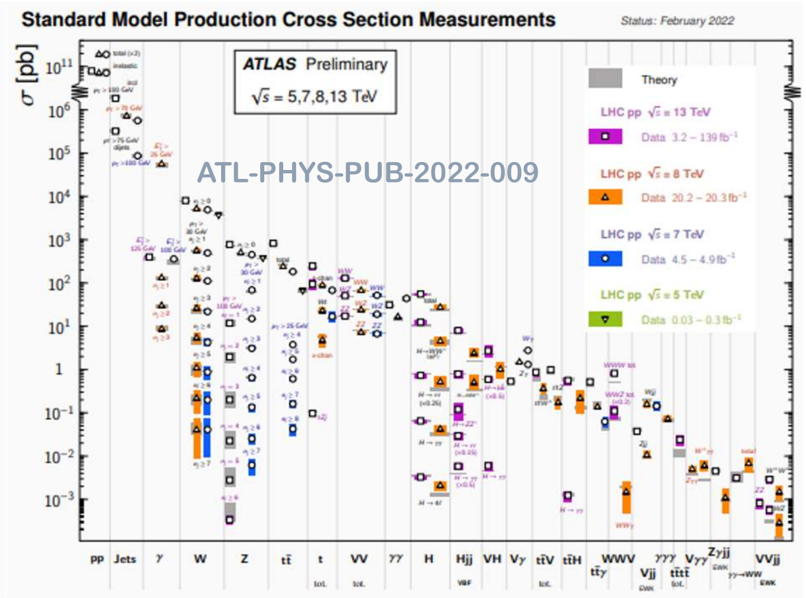
III. Towards e^+e^- colliders

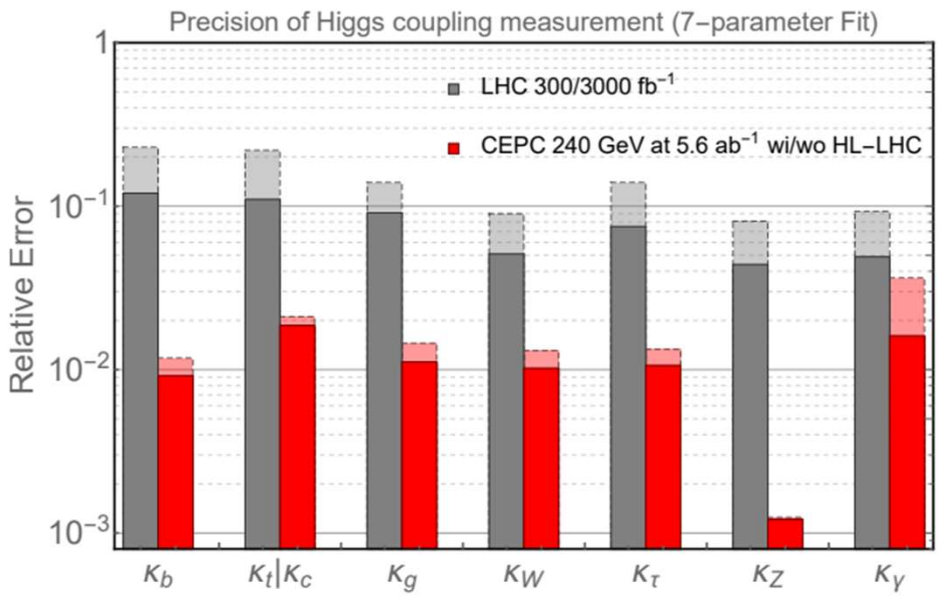
- $e^+e^- \rightarrow t\,\bar{t}$ **at NNNLO QCD**
- $e^+e^- \rightarrow H\,Z$ **at NNLO EW**

V. Summary and outlook

Era of precision physics

➢ High-precision data





- Experiments are the major ways of discovering and validation
- Many observables probed at percent level precision

Era of precision physics

➢ High-precision predictions

$$\sigma = \sigma_{LO}(1 + \alpha\, \sigma^{(1)} + \alpha^2\, \sigma^{(2)} + \alpha^3\, \sigma^{(3)} + \cdots)$$

- **Perturbation calculation is the bridge between theory and experiments**

process		known		
$pp \to H$	σ	$\mathrm{N^3LO_{HEFT}{+}N^2LO_{QCD}}\left(\frac{1}{m_t^6}\right)$		
	$\mathrm{d}\sigma$	$\mathrm{NLO_{EW}}$	$\mathrm{d}\sigma$	N
	$\mathrm{d}\sigma$	$\mathrm{N^2LO_{HEFT}{+}NLO_{QCD}{+}PS}$		+
$pp \to H+j$	$\mathrm{d}\sigma$	$\mathrm{N^2LO_{HEFT}}$	$\mathrm{d}\sigma$	N
	$\mathrm{d}\sigma$	$\mathrm{NLO_{EW}}$		
$pp \to H+2j$	$\mathrm{d}\sigma$	$\mathrm{NLO_{HEFT}{+}LO_{QCD}}$	$\mathrm{d}\sigma$	N
	$\mathrm{d}\sigma$	$\mathrm{N^2LO_{QCD}(VBF^*)}$	$\mathrm{d}\sigma$	N
	$\mathrm{d}\sigma$	$\mathrm{NLO_{EW}(VBF)}$		
$pp \to H+3j$	$\mathrm{d}\sigma$	$\mathrm{NLO_{HEFT}}$	$\mathrm{d}\sigma$	N
	$\mathrm{d}\sigma$	$\mathrm{NLO_{EW}}$		
$pp \to H+V$	$\mathrm{d}\sigma$	$\mathrm{N^2LO_{QCD}}$	$\mathrm{d}\sigma$	N

Les Houches 2015, 1605.04692

process	known	desired
$pp \to H$	$\mathrm{N^3LO_{HTL}}$	
	$\mathrm{NNLO^{(t)}_{QCD}}$	$\mathrm{N^4LO_{HTL}}$ (incl.)
	$\mathrm{N^{(1,1)}LO^{(HTL)}_{QCD\otimes EW}}$	$\mathrm{NNLO^{(b,c)}_{QCD}}$
	$\mathrm{NLO_{QCD}}$	
$pp \to H+j$	$\mathrm{NNLO_{HTL}}$	
	$\mathrm{NLO_{QCD}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$
	$\mathrm{N^{(1,1)}LO_{QCD\otimes EW}}$	
$pp \to H+2j$	$\mathrm{NLO_{HTL} \otimes LO_{QCD}}$	$\mathrm{NNLO_{HTL} \otimes NLO_{QCD} + NLO_{EW}}$
	$\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ (incl.)	$\mathrm{N^3LO^{(VBF^*)}_{QCD}}$
	$\mathrm{NNLO^{(VBF^*)}_{QCD}}$	

Les Houches 2021, 2207.02122

- **Precision calculation is a prosperous field**
- **Higher order perturbative calculation is highly demanded**

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Methodologies for high order calculation

Generate amplitude	⇨	Manipulate amplitude	⇨	Integral reduction	⇨	Evaluate master integral

$$A = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i$$

$$I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} M_j$$

$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}{}^{-\nu_{K+1}} \ldots \mathcal{D}_N{}^{-\nu_N}}{(\mathcal{D}_1+\mathrm{i}0)^{\nu_1} \ldots (\mathcal{D}_K+\mathrm{i}0)^{\nu_K}}$$

$$\mathcal{D}_i = q_i^2 - m_i^2 + i0^+$$

Qgraf
FeynArts

http://cfif.ist.utl.pt/~paulo/qgraf.html
https://feynarts.de/

CalcLoop
(FeynCalc,...)

https://gitlab.com/multiloop-pku/calcloop

Blade
(Air,Fire,Kira,Reduze, LiteRed,FiniteFlow,NeatIBP, ...)

https://gitlab.com/multiloop-pku/blade

AMFlow
(FIESTA, SecDec, Feyntrop, DiffExp, SeaSyde,...)

https://gitlab.com/multiloop-pku/amflow

- **AMFlow: any loop, high precision, full automatic**
- **Blade: improve the efficiency by 1-2 orders**

Current status of integral reduction

➢ IBP is taking center stage

- Integration-by-part identity

Chetyrkin, Tkachov, NPB(1981)

$$\int \prod_{i=1}^{L} \mathrm{d}^D \ell_i \frac{\partial}{\partial \ell_j^\mu} [v^\mu \mathcal{I}(\vec{v})] = 0$$

⇒ Linear relations among Feynman integrals

➢ Laporta algorithm

Laporta, hep-ph/ 0102033

- Gaussian elimination
- Widely used, many public codes

Air, Anastasiou, Lazopoulos, hep-ph/0404258,
Reduze, von Manteuffel, Studerus, 0912.2546, 1201.4330
LiteRed, Lee, 1212.2685, 1310.1145
Fire , Smirnov, et al, 0807.3243, 1302.5885, 1408.2372,1901.07808
Kira, Maierhöfer, et al, 1705.05610, 1812.01491, 2008.06494
Blade, XG, Liu, Ma, Wu, to appear

Current status of integral reduction

➢ Difficulties of IBP method

- Complicated intermediate expression
- Many auxiliary integrals, very sparse
- Resource-consuming due to large scale of linear equations: E.g. Laporta, 1910.01248

  Hundreds GB RAM — Months of runtime using super computer

  E.g. Klappert, et al., 2008.06494 — E.g. Baikov, Chetyrkin and Kühn, 1606.08659

➢ Selected developments

- Finite field: avoid intermediate expression swell
  - Manteuffel, Schabinger, 1406.4513
  - **FiniteFlow**, Peraro, 1905.08019
  - **FireFly**, Klappert and Lange, 1904.00009, 2004.01463
  - **Reconstruction.m**, Belitsky, Smirnov and Yakovlev, 2303.02511
- Syzygy equations: trimming IBP system
  - Gluza, Kajda and Kosower, 1009.0472
  - Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
  - **NeatIBP**, Wu, et al. 2305.08783
- A better choice of basis: UT basis/ D-factorized
  - Abreu, et al., 1812.04586
  - Usovitsch, 2002.08173
  - A. V. Smirnov and V. A. Smirnov, 2002.08042
- Block-triangular form: minimize IBP system (needs input)
  - Liu, Ma, 1801.10523,
  - XG, Liu, Ma, 1912.09294

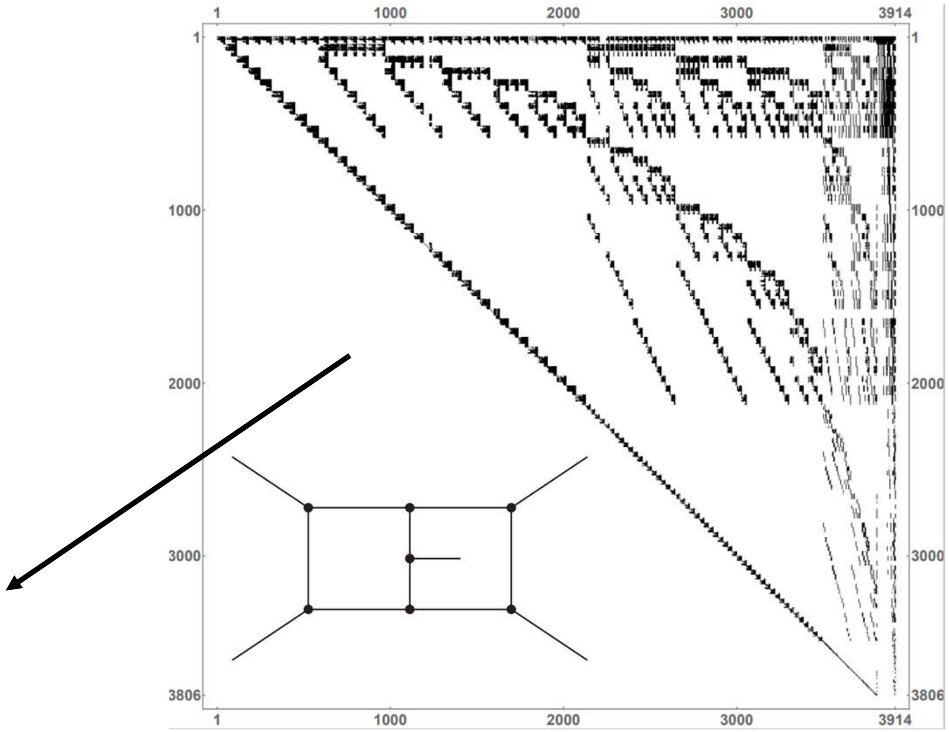
Block-triangular form

➢Improved linear system

Liu, Ma, 1801.10523
XG, Liu, Ma, 1912.09294

- Search algorithm to construct block-triangular system
- Order of magnitudes of equations fewer than plain IBP system
- Be solved strictly block by block

$$
\begin{aligned}
Q_{11}\, I_1 + Q_{12}\, I_2 + Q_{13}\, I_3 + Q_{14}\, I_4 + \ \ldots + Q_{1N}\, I_N &= 0 \\
Q_{21}\, I_1 + Q_{22}\, I_2 + Q_{23}\, I_3 + Q_{24}\, I_4 + \ \ldots + Q_{2N}\, I_N &= 0 \\
Q_{33}\, I_3 + Q_{34}\, I_4 + \ \ldots + Q_{3N}\, I_N &= 0 \\
Q_{43}\, I_3 + Q_{44}\, I_4 + \ \ldots + Q_{4N}\, I_N &= 0 \\
\ldots &
\end{aligned}
$$



Block-triangular form of double-pentagon topology

Search relations among Feynman integrals

- **Decomposition of $Q_i(\vec{s},\epsilon)$** $\sum Q_i(\vec{s},\epsilon)\, I_i(\vec{s},\epsilon) = 0$

$$Q_i(\vec{s},\epsilon) = \sum_{\mu_0=0}^{\epsilon_{max}} \sum_{\mu} \tilde{Q}_i^{\mu_0\mu_1\cdots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r}$$

  - $\tilde{Q}_i^{\mu_0\mu_1\cdots\mu_r}$ are unknowns
  - $\mu_1 + \cdots + \mu_r = d_i$

- **Input from numerical IBP** $I_i(\vec{s}_0,\epsilon_0) = \sum_{j=1}^{n} C_{ij}(\vec{s}_0,\epsilon_0) M_j(\vec{s}_0,\epsilon_0)$

$$\Rightarrow \sum_{\mu_0,\mu} \sum_{j=1}^{n} \tilde{Q}_i^{\mu_0\cdots\mu_r} \epsilon_0^{\mu_0} s_{1,0}^{\mu_1} \ldots s_{r,0}^{\mu_r} C_{ij}(\vec{s}_0,\epsilon_0) M_j(\vec{s}_0,\epsilon_0) = 0$$

  - $\vec{s}_0, \epsilon_0$ and $C_{ij}(\vec{s}_0,\epsilon_0)$ are numbers under finite field

- **Linear equations:** $\sum_{\mu_0,\mu} \tilde{Q}_i^{\mu_0\cdots\mu_r} \epsilon_0^{\mu_0} s_{1,0}^{\mu_1} \ldots s_{r,0}^{\mu_r} C_{ij}(\vec{s}_0,\epsilon_0) = 0$

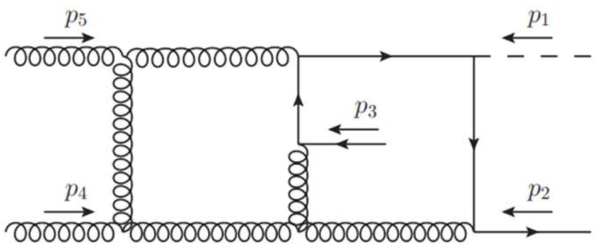
  - With enough constraints $\Rightarrow \tilde{Q}_i^{\mu_0\cdots\mu_r}$
  - With finite field technique, equations can be efficiently solved
  - Relations among $G \equiv \{I_1, I_2, \ldots I_N\}$ under finite field can be determined

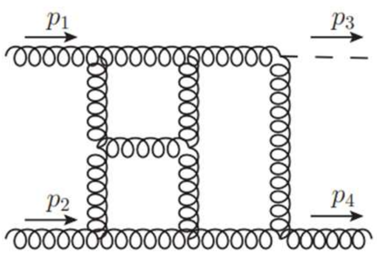
Blade

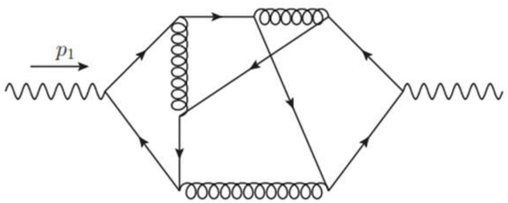
➢Efficient

XG, Liu, Ma, Wu, to appear

- **Fully automated, block-triangular form improved integral reduction package**
- **Typically improve the IBP reduction efficiency by 1-2 orders**







➢ Download

https://gitlab.com/multiloop-pku/blade

Auxiliary mass flow: FIs to vacuum integrals

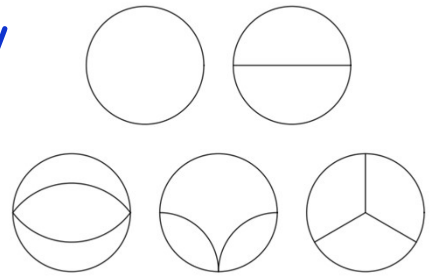
➢Auxiliary mass make boundary simple

Liu, Ma, Wang, 1711.09572
Liu, Ma, 2107.01864

- Get equal-mass vacuum integrals when η goes to infinity

$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}}$$

$\eta \to \infty$



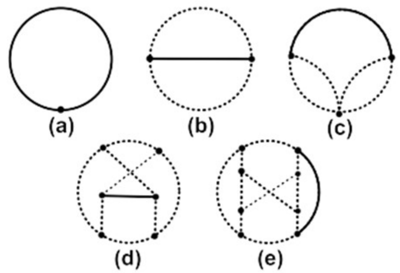
- Analytic continuation from infinity to zero

$$\frac{\partial}{\partial\eta}\vec{I}(\eta) = A(\eta)\vec{I}(\eta)$$

$$I(D;\{\nu_\alpha\};0) \equiv \lim_{\eta\to 0^+} I(D;\{\nu_\alpha\};\eta)$$

➢ Single-mass vacuum is preferable

- Fewer master integrals, simpler



Auxiliary mass flow: vacuum to vacuum

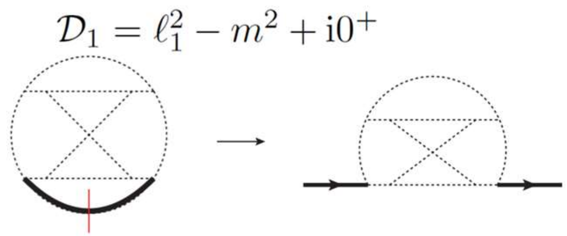
Liu, Yan-Qing Ma, 2201.11637

➢Reduce single-mass vacuum integrals

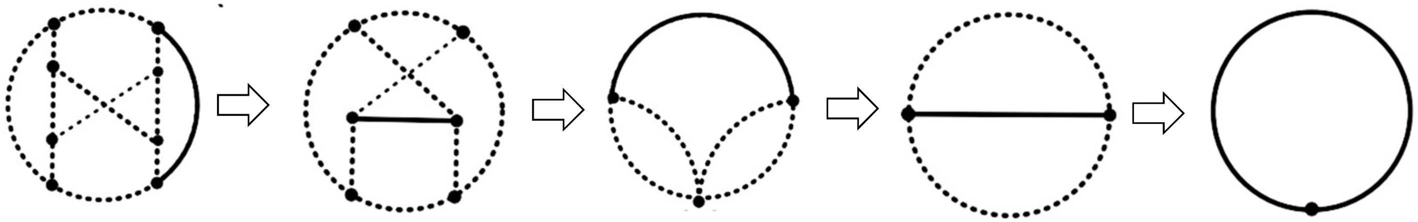
$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}{}^{-\nu_{K+1}} \ldots \mathcal{D}_N{}^{-\nu_N}}{(l_1^2 - 1 + \mathrm{i}0)^{\nu_1} (\mathcal{D}_2 + \mathrm{i}0)^{\nu_2} \ldots (\mathcal{D}_K + \mathrm{i}0)^{\nu_K}}$$

$$\hat{I}(\vec{\nu'}, l_1^2) = \int \prod_{i=2}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}{}^{-\nu_{K+1}} \ldots \mathcal{D}_N{}^{-\nu_N}}{(\mathcal{D}_2 + \mathrm{i}0)^{\nu_2} \ldots (\mathcal{D}_K + \mathrm{i}0)^{\nu_K}}$$

$$I(\vec{\nu}) = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{(-l_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(l_1^2 - 1 + \mathrm{i}0)^{\nu_1}} \hat{I}(\vec{\nu'}, -1) = \frac{\Gamma(\nu - LD/2)\,\Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1} \Gamma(\nu_1) \Gamma(D/2)} \hat{I}(\vec{\nu'}, -1)$$



- L loop vacuum integral is reduced to (L-1) loop p-integrals
- AMFlow: reduce (L-1) loop p-integrals to (L-1) loop vacuum integrals



- **No input, any loops, any space-time dimension**

Numeric differential equation

➢ From one phase point to the whole physical region

- Systematic and efficient
- Easy to obtain high precision

Caffo, Czyz, Remiddi, hep-ph/0203256
Lee, V. A. Smironov, 1709.07525
Bonciani, Degrassi, Giardino, Grober, 1812.02698
Frellesvig, Hidding, Maestri, Moriello, Salvatori, 1911.06308
Cheng, Czakon and Niggetiedt, 2109.01917
Fael, Lange, Schonwald and Steinhauser, 2202.05276
DiffExp: Hidding, 2006.05510
SeaSyde: Armadillo, Bonciani, Devoto, Rana, Vicini, 2205.03345
Liu, Ma, 2201.11669

➢ A general procedure

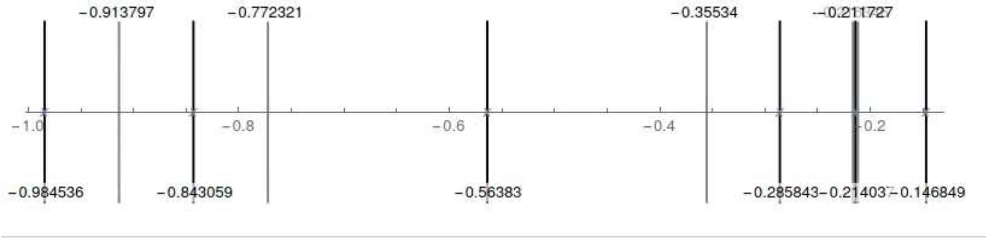
- Construct differential equation w.r.t kinematic invariants (e.g., Blade)

$$\frac{d\, I_i(\epsilon,x)}{d\, x} = \sum_j A_{ij}\, I_j(\epsilon,x)$$

- Obtain boundary conditions (e.g., AMFlow)
- Generalized series expansion around singularities

$$I(\epsilon,x) = \sum_{\mu,k,n} c_{\mu,k,n}(\epsilon)\, x^{\mu(\epsilon)} \log^k x\ \ x^n$$

- Taylor expansion around regular point to cover the whole region



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$e^+e^- \to t\,\bar{t}$ production

➢ Pin down properties of top-quark

- Mass uncertainty can be reduced to 50 MeV — ILD Concept group, 2003.01116
- Electro-weak coupling with uncertainties smaller than 0.3%

➢ $\mathrm{N^3LO_{QCD}}$ correction is indispensable

- ~2% theoretic uncertainty at NNLO for $\sqrt{s} = 500$GeV — Gao, Zhu, 1408.5150, 1410.3165; Chen, Dekkers, Heisler, Bernreuther, Si, 1610.07897
- Partial results for total cross section at NNNLO exist

  Padé approximation(exclude singlet contribution) — Hoang, Mateu, Zebarjad, 0807.4173; Kiyo, Maier, Maierhofer, Marquard, 0907.2120

  Near threshold production — Beneke, Kiyo, Maier, Piclum, 1605.03010

  Massive form factor at three loops — Fael, Lange, Schonwald, Steinhauser, 2202.05276, 2207.00027

- No differential observables is available at $\mathrm{N^3}$LO

Technique details

➢ Feynman amplitude

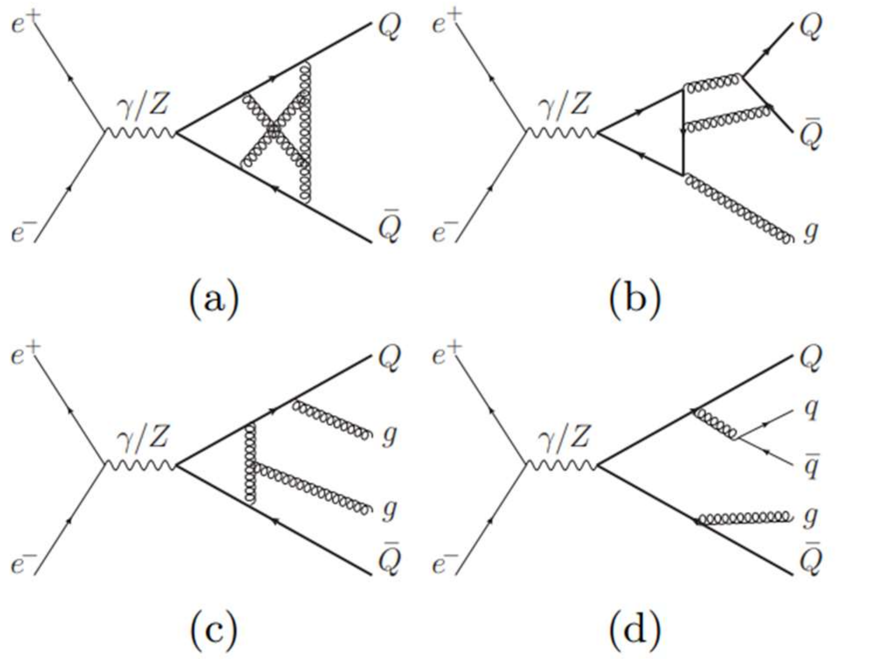
Chen, XG, He, Liu, Ma. 2209.14259

qgraf: P. Nogueira, J.Comput.Phys. 105 (1993) 279-289

- Reverse unitarity: from phase space integration to loop integration

$$\delta(p^2 - m^2) \propto \frac{1}{p^2 - m^2 - i0} - \frac{1}{p^2 - m^2 + i0}$$

- Exclude four top-quark final state
- 475 integral families(4062 master integrals)



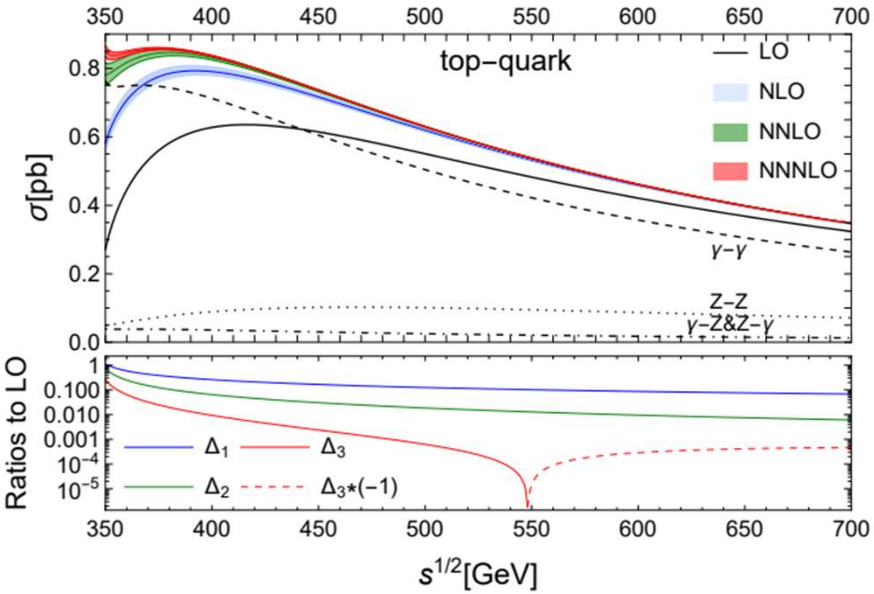
Representative Feynman diagrams at NNNLO

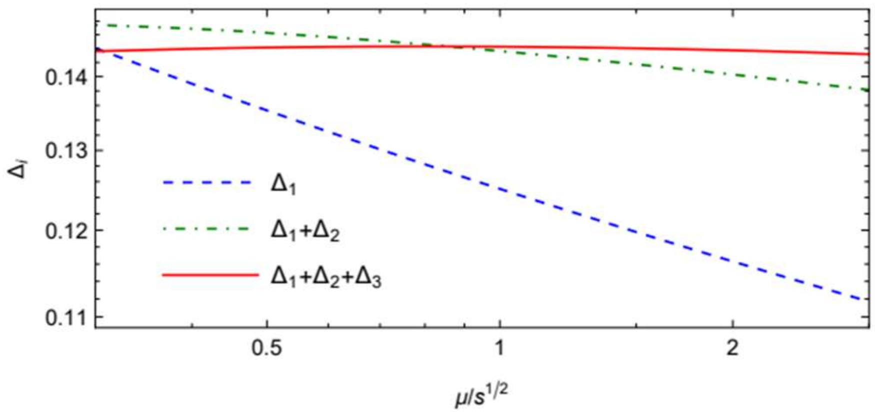
➢ KKS scheme for γ_5

- $\{\gamma^\mu, \gamma_5\} = 0$
- Take average of reading points

Kreimer, Phys. Lett. B 237 (1990) 59–62
Korner, Kreimer, Schilcher, Z. Phys. C 54 (1992) 503–512
Kreimer, hep-ph/9401354
Chen, 2304.13814

Total cross section





$$\sigma_{\text{NNNLO}} = \sigma_{\text{LO}}(1 + \Delta_1 + \Delta_2 + \Delta_3)$$

- **Around 0.1% corrections at 500GeV**
- **The scale dependence has been diminished to 0.06%**
- **Resummation is needed near threshold, e.g., $\sqrt{s} < 370$ GeV**

Meets the requirement of future colliders

Invariant mass distribution

➢ Differential distribution from delta function

- Insert the delta function in the integrand of total cross section

$$\delta\left(\left(p_Q + p_{\bar{Q}}\right)^2 - M_{Q\bar{Q}}^2\right)$$

- Transform the delta function to standard propagators

$$\propto \frac{1}{\left(p_Q + p_{\bar{Q}}\right)^2 - M_{Q\bar{Q}}^2 - i0} - \frac{1}{\left(p_Q + p_{\bar{Q}}\right)^2 - M_{Q\bar{Q}}^2 + i0}$$

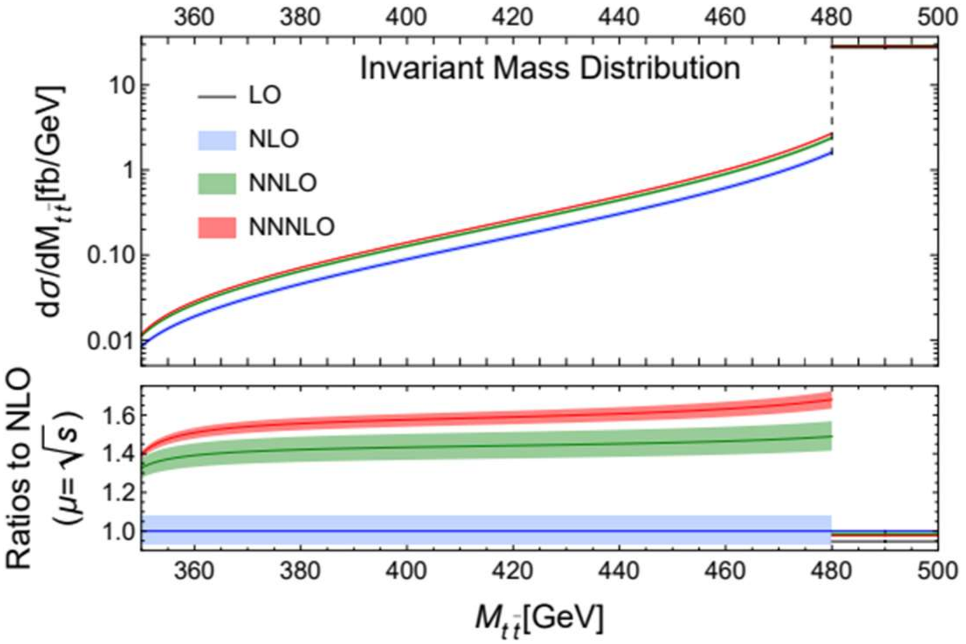
Apply previous outlined techniques

➢ Boundary condition from total cross section

- Solve the differential equation w.r.t $M_{Q\bar{Q}}^2$ with unknow boundary conditions
- Integrating over $M_{Q\bar{Q}}^2$ → Feynman integrals of total cross section(known)

⟹ Determine both unknows and the series expansion via matching

Invariant mass distribution



- Soft radiation at the right most region, prompting 20 GeV bin average
- -1% corrections to the lower order result for the bin
- 9% to 13% corrections for $M_{t\bar{t}} \in$ [370, 480] GeV
- Around 5% scale uncertainty across the majority of the distribution region, Represents a considerable impact on phenomenological study

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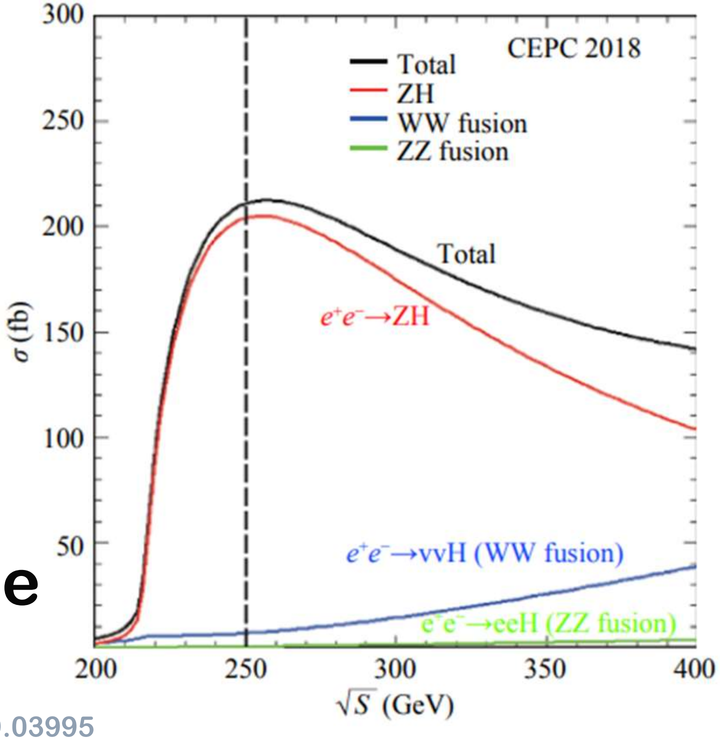
Higgs production at lepton colliders

- **Precise measurement of Higgs**
  - Higgs potential
  - EW spontaneous symmetry breaking
  - Probe to new physics
- **H Z is the dominant channel**
  - Typical energy: 240 - 250GeV
  - Experimental uncertainty: 0.51% An et al: 1810.09037
- **NNLO EW correction is indispensable**
  - 1% uncertainty for NLO EW-QCD mixed correction
    Gong, Li, Xu, Yang, Zhao, 1609.03955; Sun, Feng, Jia, Sang, 1609.03995
  - Fermionic contributions at two loop EW is available



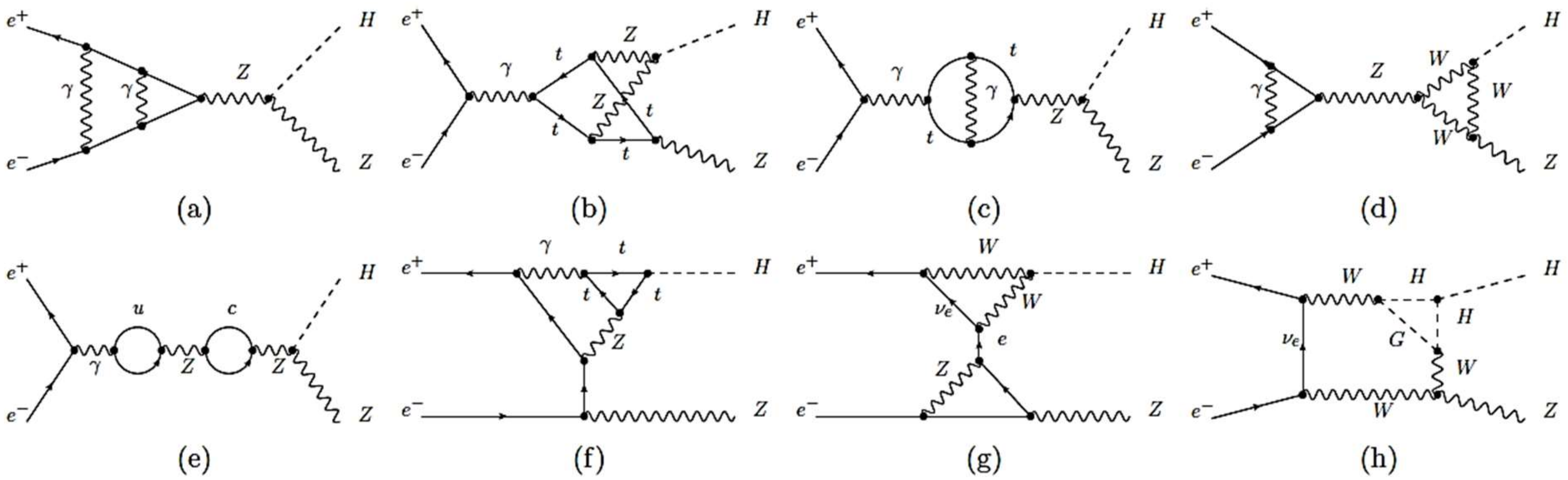
An et al: 1810.09037

Freitas, Song, 2209.07612; Freitas, Song, Xie, 2305.16547

Feynman amplitude

- **25377 Feynman diagrams (qgraf and FeynArts)**
- **372 integral families (7675 master integrals)**
- **KKS scheme for γ_5**

qgraf: P. Nogueira, J.Comput.Phys. 105 (1993) 279-289
FeynArts: T. Hahn hep-ph/0012260



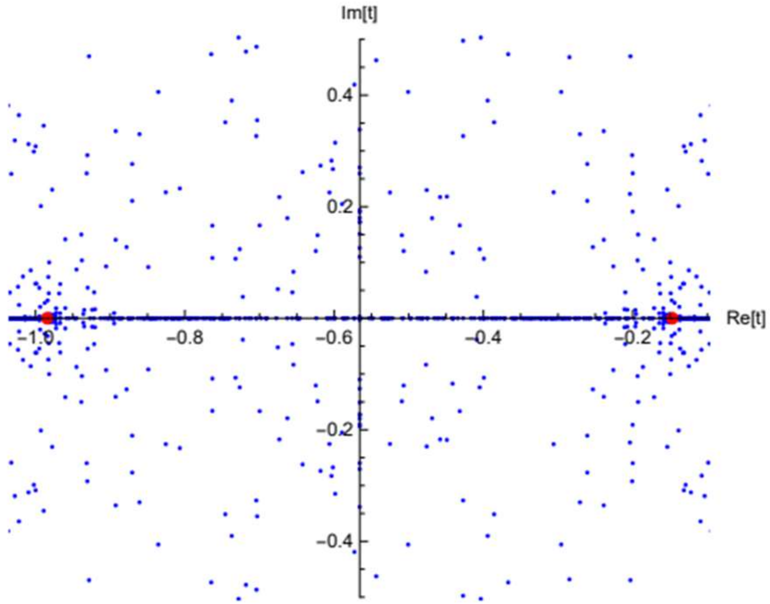
Evaluation of master integrals

➢ Thousands of singularities

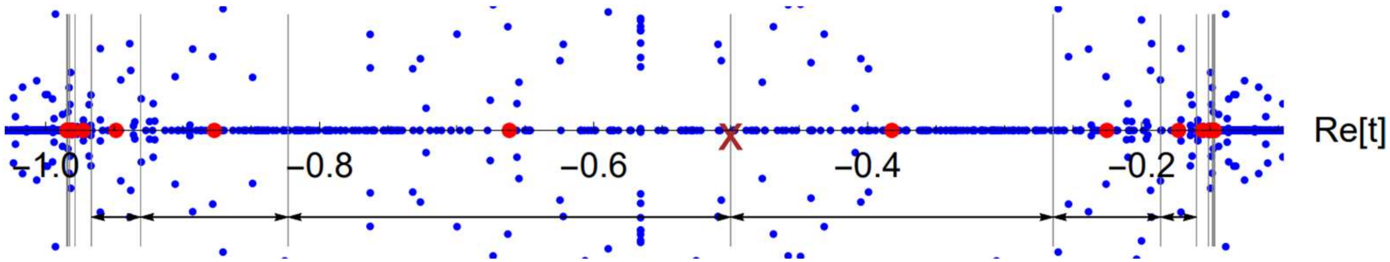
- Singularities for all master integrals
- Most of singularities are non-physical:
  'bad' master integrals or in other Riemann sheets
- Affect the numeric stability

➢ Values in the whole physical region

- Increase working precision to bypass non-physical singularities
- Trial and error: more than 20 pieces,
  each is a represented as a series expansion



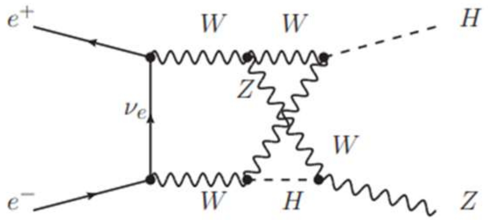
Singularities distribution



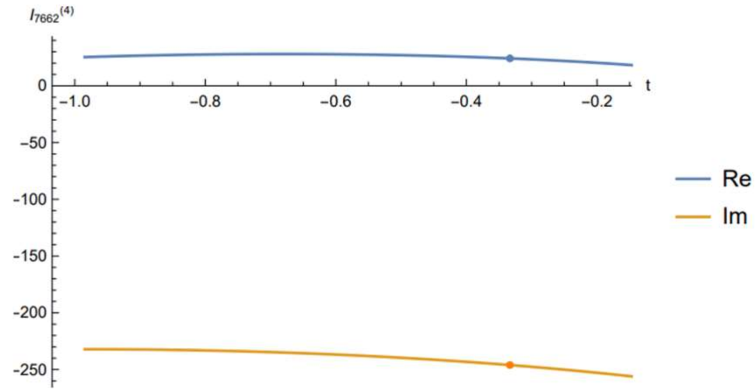
Line segmentations

Numeric results

➢ Master integrals



e.g. one of the most complicated family



Plots of the corner integral, for $m_H^2 = \frac{12}{23}$, $m_Z^2 = \frac{23}{83}$, $m_W^2 = \frac{14}{65}$, $s = \frac{83}{43}$, $m_t = 1$ as a function of t, at order ϵ^4. The solid point represents values computed with AMFlow at $t = -\frac{1}{3}$, which provides highly nontrivial self-consistency check of results.

➢ Bare Squared amplitude

- **Piecewise function of t : high precision, efficient evaluations**

$e.g., \frac{t}{m_t^2} = -\frac{5}{7}$:

$$\mathcal{A}^{(2)} = \alpha^4\left(7.5548083\,\epsilon^{-4} - 319.62821\,\epsilon^{-3} + 1154.8893\,\epsilon^{-2} + 26990.603\,\epsilon^{-1} + 156089.03 + \mathcal{O}(\epsilon)\right)$$

- **Remain both UV and IR divergence (underway)**

Outline

I. Introduction

II. High order calculation methods

III. Towards e^+e^- colliders

- $e^+e^- \rightarrow t\,\bar{t}$ at NNNLO QCD
- $e^+e^- \rightarrow H\,Z$ at NNLO EW

V. Summary and outlook

Summary and outlook

➢ Status

- We compute the NNNLO QCD corrections to the total cross section and the invariant mass distribution of $t\,\bar{t}$, which are valuable for phenomenological study
- We compute the two loop corrections to $H\,Z$ production, the complete results are still underway.

➢ Next

- Our strategic approach is versatile to be applied to other processes, including differential observables
- A fully automated package for high order (beyond NLO) calculation?

Thank you!