Analytic applications of conformal methods in relativity

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Conformal compactification

Notion introduced by Roger Penrose in the 1960's.

Purpose : on a "generic asymptotically flat spacetime"

- Asymptotic behaviour of fields along outgoing light rays ;
- Peeling of test fields ;
- Definition of generic asymptotically flat spacetimes ;
- Sachs peeling.

General principle

Ingredients :

- "Physical" spacetime : $(\mathcal{M}, g)$ smooth, 4-dimensional, real, Lorentzian manifold.
- "Unphysical" spacetime : $\bar{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$.
  $\bar{\mathcal{M}}$ smooth manifold with boundary $\mathcal{B}$.
  $\mathcal{B} = \mathscr{I}^+ \cup \mathscr{I}^-$, $\mathscr{I}^\pm \simeq \mathbb{R} \times S^2$.
- Conformal factor : $\Omega > 0$ on $\mathcal{M}$, smooth on $\bar{\mathcal{M}}$,
  $\Omega|_{\mathcal{B}} = 0$, $\mathrm{d}\Omega|_{\mathcal{B}} \neq 0$.
- Rescaled metric : $\hat{g} := \Omega^2 g$ is a smooth non degenerate Lorentzian metric on $\bar{\mathcal{M}}$.
- Scri : $\mathscr{I}^\pm$ are smooth null hypersurfaces of $(\bar{\mathcal{M}}, \hat{g})$.

Flat spacetime

Minkowski spacetime : $\mathbb{M} = \mathbb{R}_t \times \mathbb{R}^3$ with the Minkowski metric

$$\eta := \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2 \,,\ \mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \,.$$

$$\begin{aligned} \tau &= \arctan(t+r) + \arctan(t-r)\,, \\ \zeta &= \arctan(t+r) - \arctan(t-r)\,. \end{aligned}$$

$\mathbb{M}$ is described as

$$\{|\tau| + \zeta < \pi \,,\ \zeta > 0\} \times S^2_\omega \,.$$

η takes the new expression

$$\eta = \frac{(1+(t+r)^2)(1+(t-r)^2)}{4} \left(\mathrm{d}\tau^2 - \mathrm{d}\zeta^2\right) - r^2 \mathrm{d}\omega^2 \,.$$

Flat spacetime

We choose the conformal factor

$$\Omega^2 = \frac{4}{(1+(t+r)^2)(1+(t-r)^2)} .$$

We rescale the metric using it :

$$\begin{aligned}\hat{\eta} := \Omega^2 \eta &= \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - (\sin\zeta)^2\,\mathrm{d}\omega^2 \\ &= \mathrm{d}\tau^2 - \sigma^2_{S^3} ,\end{aligned}$$

where $\sigma^2_{S^3}$ is the euclidian metric on the 3-sphere.
Einstein cylinder : $\mathcal{E} = \mathbb{R}_\tau \times S^3$ equipped with $\hat{\eta}$.
$\{\mathbb{M} , \ \hat{\eta}\}$ is a submanifold of $\{\mathcal{E} , \ \hat{\eta}\}$.

Flat spacetime

Boundary of $\mathbb{M}$ in $\mathcal{E}$

- $\mathscr{I}^{\pm} = \{0 < \zeta < \pi\,,\ \tau = \pm(\pi - \zeta)\}$, future and past null infinities ;
- $i^{\pm} = (\tau = \pm\pi, \zeta = 0)$, future and past timelike infinities,
- $i^{0} = (\tau = 0, \zeta = \pi)$, spacelike infinity.

Flat spacetime

Interpretation of the boundary of $\mathbb{M}$ in $\mathcal{E}$.

- Null lines : $\gamma_{u,\omega_0}(r) = (t = r + u, r, \omega = \omega_0)$
  end up on $\mathscr{I}^{\pm}$ as $t \to \pm\infty$.
- Timelike lines : $\beta_{\lambda,u,\omega_0}(r) = (t = \lambda r + u, r, \omega = \omega_0)$ with
  $\lambda > 1$
  end up at $i^{\pm}$ as $t \to \pm\infty$.
- Spacelike lines : $\beta_{\lambda,u,\omega_0}(r) = (t = \lambda r + u, r, \omega = \omega_0)$ with
  $0 < \lambda < 1$
  end up at i^0 as $t \to \pm\infty$.

Applications in flat spacetime

Conformally invariant equation : e.g. $(\Box_g + \frac{1}{6}\mathrm{Scal}_g)f = 0$

$$\Box_\eta f = 0 \Leftrightarrow \Box_{\hat{\eta}} \hat{f} + \hat{f} = 0$$

where $\hat{f} = \Omega^{-1} f$: rescaled field.

Trace properties for $\hat{f}$ at the boundary of $\mathbb{M}$

$\Updownarrow$

Asymptotic properties for f at infinity

Applications in flat spacetime

Smooth data for $\hat{f}$: $\hat{f}|_{\tau=0}\,,\ \partial_\tau \hat{f}|_{\tau=0} \in \mathcal{C}^\infty(S^3)$
Unique global solution $\hat{f} \in \mathcal{C}^\infty(\mathbb{R} \times S^3)$

- Pointwise decay of f : for r, ω constant, $t \to \infty$,

$$\Omega \simeq 1/t^2\,.$$

$$\Rightarrow f(t, r_0, \omega_0) \simeq 1/t^2\,.$$

- Decay of f along outgoing light rays : for u, ω constant, $t = r + u$, $r \to \infty$,

$$\Omega \simeq 1/t\,.$$

$$\Rightarrow f(t, t - u_0, \omega_0) \simeq 1/t\,.$$

Valid for which solutions?

Away from i^0, we have

$$\hat{f}|_{\tau=0} = \frac{1+r^2}{2} f|_{t=0}$$

$$\partial_\tau \hat{f}|_{\tau=0} = \frac{(1+r^2)^2}{4} \partial_t f|_{t=0}$$

Smoothness of data for $\hat{f}$

$\Downarrow$

Fall-off assumptions on data for f

$$f(0,r,\omega_0) \simeq 1/r^2\,,\ \partial_t f(0,r,\omega_0) \simeq 1/r^4$$

and so on. $\mathcal{C}^k$ data less stringent.

More applications?

Behaviour along outgoing radial light rays :

Basic question : asymptotic profiles

$$r\,f(t = r + u, r, \omega) \to \phi^+_{u,\omega}\,,\, r \to +\infty\,.$$

First generalization : scattering

$$r\,f(t = r + u, r, \omega) \to \phi^+_{u,\omega}$$

globally in a Hilbert space on $\mathbb{R}_u \times S^2_\omega$.

Second generalization : peeling

$$F(u, R, \omega) := \frac{1}{R} f(t = \frac{1}{R} + u, \frac{1}{R}, \omega)$$

extends as a smooth function on $\mathbb{R}_u \times [0, +\infty[_R \times S^2_\omega$.

Yes.

Scattering and peeling in flat spacetime can both be proved using conformal techniques, provided we work with conformally invariant equations.
Peeling : Penrose 1965.
Scattering : Friedlander 1980, 2000, Baez, Segal, Zhou 1990.

Application to scattering

Framework : asymptotically simple spacetimes
Spacetimes $(\mathcal{M}, g)$ diffeomorphic to Minkowski space ;
$\exists \Omega$ s.t. $(\bar{\mathcal{M}}, \hat{g})$ has smooth $\mathscr{I}^{\pm}$ and $i^{\pm}$.

Advantage of the conformal approach :

- All that matters is the asymptotic structure, time dependence is no problem.

Limitations :

- "nice" asymptotic structure (Is it really a limitation? Yes for now.) ;
- conformally invariant equations (Is it really a limitation? Yes for now.).

Description of the method

Consider a conformally invariant equation, e.g. Dirac's equation
$(\mathcal{M}, g)$ asymptotically simple space-time, Ω conformal factor,
$(\bar{\mathcal{M}}, \hat{g})$ rescaled spacetime.
ϕ Dirac field on $(\mathcal{M}, g)$, $\hat{\phi}$ rescaled Dirac field.

Step 1. Trace operators $T^{\pm}$ on $\mathscr{I}^{\pm}$

- Smooth data with compact support at $t = 0$, trace on $\mathscr{I}^{\pm}$ for $\hat{\phi}$ trivial ;
- Energy estimates both ways between $\{t = 0\}$ and $\mathscr{I}^{\pm}$;
- Entail $T^{\pm}$ extend to minimum regularity (L^2), are injective with closed range.

Step 2. Surjectivity of trace operators (range dense in L^2)

Scattering operator : $S := T^{+}(T^{-})^{-1}$.

Remarks

1. Scattering theory with wave operators and comparison dynamics of asymptotic profile type.

2. Step 2 is the difficult one : solution of a Goursat problem.
Method due to Hörmander (JFA, 1990)

- based on energy estimates ;
- reduction to the Cauchy problem.

Additional difficulty : data on a null hypersurface are "lighter".
Data must be added when reducing to the Cauchy problem (not at random!).

State of the art

Conformal scattering.

Friedlander 1980, 2000, ideas put forward in the flat case.
Baez, Segal, Zhou 1990's, explicit constructions in the flat case.
Mason, Nicolas 2004, generic geometries, Dirac, Maxwell, wave.

Goursat problem.

Penrose 1963, flat space integral formulae ;
Friedlander 1975, wave equation on curved spacetimes ;
Hörmander 1990, simple approach ;
Nicolas 2006, extension of Hörmander 1990 to low regularity.

Perspectives.

Joudioux Extension of Penrose's results to general curved spacetimes ; conformal scattering for non linear equations (curved case) ;
Projects Conformal scattering for black hole spacetimes ; for non conformally invariant equations.

Application to peeling : some history

Sachs peeling. Fall-off of components of the curvature along null directions.
Peeling of test fields. Penrose 1965, regularity of rescaled fields at null infinity.
Peeling as a conjecture. Penrose 1965? He says no...
Question : on a given asymptotically simple spacetime, what is the class of initial data for a given field equation which guarantees smoothness across $\mathscr{I}$ of the rescaled field?
Penrose's approach suggests that he thinks : the same as in flat spacetime, provided the spacetime is "generic".
Big controversy for more than 40 years. Counterexamples (wrong!).
In the simplest non flat case (Schwarzschild metric), question unsolved.

Peeling in flat spacetime

$\mathcal{C}^\infty$ case. Trivial with conformal techniques (that's the point).
Intermediate regularity. $\mathcal{C}^k$ is a bad choice. H^k is good.
Trivial by conformal techniques + vector field methods.
Method (wave equation) : $\Box_{\hat{\eta}}\psi + \psi = 0$ on $\mathcal{E}$

- Energy equality between $\{\tau = 0\}$ and $\mathscr{I}^+$;
- Commute ∂_τ^k in the equation

$$\|\partial_\tau^k \psi|_{\mathscr{I}^+}\|^2_{H^k(\mathscr{I}^+)} = \|\partial_\tau \psi(\tau = 0)\|^2_{H^k(S^3)} + \|\psi(\tau = 0)\|^2_{H^{k+1}(S^3)} .$$

$\Rightarrow$ new definition of the peeling at a given order.

The Schwarzschild case

L. Mason, J.-P. Nicolas, 2009.
We lose the essential structure : smooth i^0.
Similar method :

- Conformal compactification, the simplest ;
- Choice of energy (Morawetz vector field adapted, Dafemos-Rodnianski) ;
- Choice of vector field to increase regularity (null radial outgoing).

Penrose's intuition confirmed at all orders.

Perspectives

Extension to generic asymptotically flat spacetimes.
Other equations (spinorial, non linear).