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DYNAMICAL BOUNDS FOR THE FIBONACCI HAMILTONIANS

We study the long-time behavior of the solution to the time-dependent Schrödinger equation, $i\partial_t\psi = H\psi$ in the Hilbert space $l^2(\mathbf{Z})$ with an initial state $\psi = \delta_0$, namely, $\psi(t) = \exp(-itH)\delta_0$. The Schrödinger operator is of the form

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n),$$

where

$$V(n) = \lambda\chi_{[1-\varphi^{-1},1)}(n\varphi^{-1} + \theta \ mod \ 1),$$

$\lambda > 0$ is the coupling constant, φ is the golden mean

$$\varphi = \frac{\sqrt{5}+1}{2}$$

and $\theta \in [0,1)$ is the phase.

One studies the quantities

$$P_r(N,t) = \sum_{n \geq N} | < e^{-itH}\delta_0, \delta_n > |^2,$$

$$P_l(N,t) = \sum_{n \leq -N} | < e^{-itH}\delta_0, \delta_n > |^2,$$

where $N \geq 1$, called right and left probabilities, and

$$P(N,t) = P_l(N,t) + P_r(N,t),$$

called outside probabilities. One studies also the time-averaged probabilities (especially when proving dynamical lower bounds).

To control the tails of the wavepacket, for any $0 \leq \alpha < +\infty$ define

$$S^-(\alpha) = -\liminf_{t \to +\infty} \frac{\log P(t^\alpha - 1, t)}{\log t},$$

$$S^+(\alpha) = -\limsup_{t \to +\infty} \frac{\log P(t^\alpha - 1, t)}{\log t}.$$

One has $0 \leq S^+(\alpha) \leq S^-(\alpha) \leq +\infty$. One can define similar growth exponents for the time-averaged probabilities. Interpretation: if $S^-(\alpha) = L < +\infty$, then for any $\delta > 0$, $P(t^\alpha, t) \geq C(\delta) t^{-L-\delta}$. If $S^+(\alpha) < +\infty$, similar bound holds for some sequence of times.

The following critical exponents are of interest:

$$\alpha_l^{\pm} = \sup\{\alpha \geq 0 \ : \ S^{\pm}(\alpha) = 0\},$$

$$\alpha_u^{\pm} = \sup\{\alpha \geq 0 \ : \ S^{\pm}(\alpha) < +\infty\}.$$

Interpretation: rates of propagation of the essential part of the wavepacket and of the fastest (polynomially small) part of the wavepacket. If $\alpha < \alpha_l^-$, then for any $\delta > 0$, $P(t^\alpha, t) \geq C(\delta) t^{-\delta}$. If $\alpha > \alpha_u^+$, then $P(t^\alpha, t)$ goes to 0 faster than any inverse power of t. Similar exponents can be defined for the time-averaged probabilities.

The exponents $\alpha_l,\ \alpha_u$ are closely related with the growth exponents of the moments of the position operator. Namely, let

$$X^p(t) = \sum_n n^p | < e^{-itH}\delta_0, \delta_n > |^2,$$

and

$$\beta^-(p) = \liminf_{t\to+\infty} \frac{\log |X|^p(t)}{\log t},$$

$$\beta^+(p) = \limsup_{t\to+\infty} \frac{\log |X|^p(t)}{\log t}.$$

Then

$$\alpha_l^\pm = \lim_{p\to 0} \beta^\pm(p),\ \alpha_u^\pm = \lim_{p\to+\infty} \beta^\pm(p).$$

Similarly for the time-averaged quantities.

The aim is to obtain tight bounds for $\alpha_l^{\pm}$, $\alpha_u^{\pm}$. In the present talk: $\alpha_u^{\pm}$. Main technical tool is the integration in the complex plane. Let $R(z) = (H - zI)^{-1}$. Assume that H is bounded and $\sigma(H) \subset [-K+1, K-1]$ for some $K > 1$. Then

$$\begin{aligned} &< e^{-itH}\delta_0, \delta_n > \\ &\quad = -(2\pi i)^{-1} \int_\Gamma \exp(-tz) < R(z)\delta_0, \delta_n >, \end{aligned}$$

where Γ is any positively oriented simple closed contour in $\mathbf{C}$ such that $\sigma(H)$ lies inside Γ. It follows that

$$\begin{aligned} P_r(N,t) \leq \quad & C(\exp(-cN) + \\ & \int_{-K}^{K} \sum_{n \geq N} | < R(E + it^{-1})\delta_0, \delta_n > |^2 dE) \end{aligned}$$

and similarly for $P_l(N,t)$. These formula allow to bound from above the probabilities (without time-averaging).

For the time-averaged probabilities

$$\widetilde{P}_r(N,T)$$
$$= 2/T \int_0^{+\infty} \exp(-2t/T) | < e^{-itH}\delta_0, \delta_n > |^2 dt,$$

the well-known Parseval equality holds:

$$\widetilde{P}_r(N,T)$$
$$= (\pi T)^{-1} \int_{\mathbf{R}} \sum_{n \geq N} | < R(E + iT^{-1})\delta_0, \delta_n > |^2 dE,$$

and similarly for $\widetilde{P}_l(N,T)$. These equalities allow to bound from above and from below the time-averaged probabilities.

In the case of discrete one-dimensional operators, one can control the resolvent using the transfer matrices $\Phi(n, z)$. The definition is such that $u : \mathbf{Z} \to \mathbf{C}$ solves

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n)$$

if and only if

$$(u(n+1), u(n))^T = \Phi(n, z)(u(1), u(0))^T$$

for any n.

One can prove the following:

$$P_r(N,t) \le C(\exp(-cN) + \tag{1}$$
$$t^4 \int_{-K}^{K} \left(\max_{0\le n\le N-1} ||\Phi(n, E+it^{-1})||^2 \right)^{-1} dE),$$

and similarly for $P_l(N,t)$. For the time-averaged outside probabilities, the following lower bounds hold:

$$\widetilde{P}_r(N,T) \le C(\exp(-cN) + \tag{2}$$
$$T^3 \int_{-K}^{K} \left(\max_{0\le n\le N-1} ||\Phi(n, E+iT^{-1})||^2 \right)^{-1} dE),$$

and similarly for $\widetilde{P}_l(N,T)$. On the other hand,

$$\begin{aligned} \widetilde{P}(N,T) = \quad & \widetilde{P}_r(N,T) + \widetilde{P}_l(N,T) \\ & \ge CT^{-1} \int_{\mathbf{R}} ||\Phi(N, E+iT^{-1})||^{-2}. \end{aligned} \tag{3}$$

Thus, the problem is to control the norm of the transfer matrices $||\Phi(N,z)||$ for large N and complex z close to $\sigma(H)$ (which give the main contribution to the integrals over E). The inequalities (1) and (2) serve to prove that $P(t^\alpha,t)$ or $\widetilde{P}(T^\alpha,T)$ goes fast to 0 if α is large enough. The inequality (3) allows to prove that $\widetilde{P}(T^\alpha,T)$ is not too small if α is small enough. Thus, one can prove upper bounds for $\alpha_u^\pm, \tilde{\alpha}_u^\pm$ and lower bounds for $\tilde{\alpha}_u^\pm$.

For Fibonacci hamiltonian, the key analytical tool are the traces of the transfer matrices:

$$x_k(z) = \frac{1}{2}\mathrm{Tr}\Phi_{\theta=0}(F_k, z),$$

where F_k are the Fibonacci numbers. Since $|x_k(z)| \leq ||\Phi_{\theta=0}(F_k, z)||$, lower bounds on the traces imply lower bounds on the norms of the transfer matrices. On the other hand, if $|x_k(z)| \leq 1 + \delta, \delta \in (0, 1)$ for some large k, then the norms $||\Phi(N, z)||$ remain bounded polynomially in N for $N \leq F_k$ for any θ.

Consider the sets

$$\sigma_k^\delta = \{z \in \mathbf{C} \ : \ |x_k(z)| \le 1 + \delta\}.$$

The set σ_k^δ has exactly F_k connected components, each of them being a topological disk symmetric about the real axis.

If z is outside of the two consecutive sets σ_k^δ, σ_{k+1}^δ, then the traces $x_{k+m}(z)$ grow very fast with m. This allows to prove that $P(F_{k+\sqrt{k}}, t)$ (resp. $\widetilde{P}(F_{k+\sqrt{k}}, T)$) are very small provided t^{-1} (resp. T^{-1}) is larger than the size of ALL connected components of σ_k^δ, σ_{k+1}^δ. In this manner one proves upper bounds for $\alpha_u^\pm$, $\widetilde{\alpha}_u^\pm$.

On the other hand, if T^{-1} is small enough with respect to SOME component of σ_k^δ (one takes the largest one), then there is some interval I such for $E \in I$ the norms $||\Phi(N, E + iT^{-1})||$ remain polynomially bounded in N for $N \leq F_k$. This interval gives at least polynomially small contribution to the integral in 3. In this manner one proves lower bound for $\widetilde{\alpha}_u^\pm$.

The crucial moment is the size of the LARGEST connected component of the set σ_k^δ for large k. Define the two functions of the coupling parameter λ:

$$S_l(\lambda) = \frac{1}{2}((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}),$$

$$S_u(\lambda) = 2\lambda + 22.$$

One can show that the size of the largest connected component of σ_k^δ lies between $r = C_1 S_u(\lambda)^{-k/2}$ and $R = C_2 S_l(\lambda)^{-k/2}$ (in the sense that it contains a disk of radius r and is inside a disk of radius R).

This gives the following result for the upper dynamical exponents:

Theorem 1 *Let $\lambda \geq 8$. The bounds hold:*

$$\alpha_u^{\pm} \leq \frac{2 \log \phi}{\log S_l(\lambda)},$$

$$\frac{2 \log \phi}{\log S_u(\lambda)} \leq \tilde{\alpha}_u^{\pm} \leq \frac{2 \log \phi}{\log S_l(\lambda)}.$$

In particular,

$$\lim_{\lambda \to \infty} \tilde{\alpha}_u^{\pm} \cdot \log \lambda = 2 \log \phi.$$

It is interesting to compare with the previous lower bound for $\widetilde{\alpha}_u^\pm$ obtained in [Damanik, Embree, Gorodetski, T.]:

$$\widetilde{\alpha}_u^\pm \geq \text{dim}_B^\pm(\sigma(H_\lambda))$$

It was proven in [DEGT] that

$$\lim_{\lambda\to\infty} \text{dim}_B^\pm(\sigma(H_\lambda)) \cdot \log\lambda = b \log\phi,$$

where $b = 1.83\ldots$. Comparing this result with the statement of the Theorem, one sees that

$$\widetilde{\alpha}_u^\pm > \text{dim}_B^\pm(\sigma(H_\lambda))$$

for large values of λ. This is the first example where such strict inequality holds.