A step towards a semi-classical analysis

on the Heisenberg group

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OUTLINE OF THE TALK

1. The Heisenberg Group.

2. A Class of Symbols on the Heisenberg Group.

3. An Algebra of Pseudodifferential Operators, action in $\mathcal{S}$.

4. Action on Sobolev Spaces.

I. The Heisenberg Group

1. Definition

The Heisenberg Group $\mathbf{H}^d$ is defined as $\mathbf{R}^{2d+1}$ with the group law

$$(x,y,s)\cdot(x',y',s') = \big(x+x',y+y',s+s'-2(x|y')+2(y|x')\big)$$
$$(x,y,s),(x',y',s') \in \mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R} = \mathbf{R}^{d+1}$$

where $(x|y)$ denotes the Euclidin scalar product in $\mathbf{R}^d$.

It is the simplest non commutative Lie group and as such has been widely studied (Complex Analysis, Harmonic Analysis, PDE's, Quantum Mechanic).

d=1: $\mathbf{H}^d = \mathbf{H}$ and

$$(x,y,s)\cdot(x',y',s') = (x+x',y+y',s+s'-2(xy'-yx')).$$

- Lie Algebra of left-invariant vetor fields

A tangent vector field X to $\mathbf{H}$ is said to be left-invariant if

$$\forall h \in \mathbf{H}, \;\; \tau_h(Xf) = X\left(\tau_h(f)\right)$$

where τ_h is the left-translation defined by

$$\forall w \in \mathbf{H}, \;\; \tau_h(f)(w) = f(h \cdot w)$$

The vector space of left-invariant vector fields is generated by

$$X = \partial_x + 2\,y\,\partial_s, \quad Y = \partial_y - 2\,x\,\partial_s, \;\; S = \partial_s = \frac{1}{4}[Y, X].$$

- The Laplace-Kohn Operator

$$\Delta_{\mathbf{H}^d} = X^2 + Y^2$$

• The Haar measure

$$\mathrm{dw} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \quad (\text{Lebesgue} \;\; \text{measure}).$$

• $L^2(\mathbf{H})$ space

$$f \in L^2(\mathbf{H}, \mathbf{C}) \iff \int_{\mathbf{H}} |f(w)|^2\, dw < +\infty.$$

• Sobolev spaces with exponent $k \in \mathbf{N}$

$$f \in H^k(\mathbf{H}, \mathbf{C}) \iff T_1 \circ \cdots \circ T_k f \in L^2(\mathbf{H}, \mathbf{C})$$

for any choice of $T_1, \cdots, T_k$ among X and Y.

• The complex setting

$$w \in \mathbf{H}, \quad w = (x, y, s) = (z, s), \quad z = x + iy \in \mathbf{C}$$

- The product law

$$(z, s) \cdot (z', s') = \big(z + z', s + s' + 2\,\mathcal{I}m(z\,\overline{z'})\big)\,.$$

- The left-invariant vectors fields Lie algebra is generated by

$$Z = \partial_z + i\overline{z}\partial_s, \quad \overline{Z} = \partial_{\overline{z}} - iz\partial_s \quad \textbf{with} \quad S = \partial_s = \frac{1}{2i}[\overline{Z}, Z].$$

- The Laplace-Kohn Operator writes

$$\Delta_{\mathbf{H}} = 2(Z\overline{Z} + \overline{Z}Z)$$

2. The Fourier transform

The Fourier transform on $\mathbf{H}$ is defined via the representations of $\mathbf{H}$. All irreducible representations are unitarily equivalent to one of the two representations :

- the Bargman representation
- the Schrödinger representation

• The Schrödinger representation

The family of unitary operators (v_w^λ), $\lambda \in \mathbf{R}^*$, $w = (x, y, s) \in \mathbf{H}$ defined for $f \in L^2(\mathbf{R})$ by

$$v_w^\lambda f(\xi) = \mathrm{e}^{i\lambda(s-2x\cdot y+2y\cdot\xi)} f(\xi - 2x), \quad \forall \xi \in \mathbf{R}$$

gives a representation of $\mathbf{H}$ on $L^2(\mathbf{R})$.

$$v_w^\lambda \circ v_{w'}^\lambda = v_{w\cdot w'}^\lambda, \quad (v_w^\lambda)^{-1} = v_{w^{-1}}^\lambda = (v_w^\lambda)^*.$$

• The Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathbf{H}} v_w^\lambda f(w) dw.$$

maps $L^1(\mathbf{H})$ on operators uniformly bounded with respect to λ
maps $L^2(\mathbf{H})$ on families of HS_λ, the space of families of Hilbert-Schmidt operators such that

$$\int_{\mathbf{R}} \|\mathcal{F}(f)(\lambda)\|_{HS}^2 \, d\lambda < \infty.$$

• Inversion formula

$$f(w) = \pi^{-2} \int_{\mathbf{R}} \mathrm{tr} \left(v_{w^{-1}}^\lambda \mathcal{F}(f)(\lambda) \right) |\lambda| d\lambda,$$

• Fourier-Plancherel formula

$$\|f\|_{L^2(\mathbf{H}^d)}^2 = \pi^{-2} \int_{\mathbf{R}} \|\mathcal{F}(f)(\lambda)\|_{HS(\mathcal{H}_\lambda)}^2 |\lambda| d\lambda.$$

• Fourier transform and left-invariant vector fields

$$\mathcal{F}(Zf)(\lambda) = \mathcal{F}(f)(\lambda) \circ Q_\lambda, \quad \mathcal{F}(\overline{Z}f)(\lambda) = \mathcal{F}(f)(\lambda) \circ \overline{Q}_\lambda$$

where

$$T_\lambda Q_\lambda T_\lambda^* = \sqrt{|\lambda|}\,(\partial_\xi - \xi) \quad \text{and} \quad T_\lambda \overline{Q}_\lambda T_\lambda^* = \sqrt{|\lambda|}\,(\partial_\xi + \xi) \ \text{ if } \ \lambda > 0$$
$$T_\lambda Q_\lambda T_\lambda^* = \sqrt{|\lambda|}\ (\partial_\xi + \xi) \quad \text{and} \quad T_\lambda \overline{Q}_\lambda T_\lambda^* = \sqrt{|\lambda|}\ (\partial_\xi - \xi) \ \text{ if } \ \lambda < 0$$

and T_λ is the unitary operator defined by

$$T_\lambda f(\xi) = |\lambda|^{-1/4} f(|\lambda|^{-1/2}\xi), \ \ \xi \in \mathbf{R}^d, \ \ f \in L^2(\mathbf{R}).$$

In particular

$$\mathcal{F}(-\Delta_{\mathbf{H}} f)(\lambda) = \mathcal{F}(f)(\lambda) \circ D_\lambda \ \text{ with } \ D_\lambda = 4|\lambda| T_\lambda^* (-\Delta_\xi + \xi^2) T_\lambda.$$

The derivation corresponds to composition on the right-hand side of the Fourier transform

3. Sobolev spaces with real-valued exponents

One defines Sobolev spaces with real-valued exponents

$$H^s(\mathbf{H}^d) = \{f \in L^2(\mathbf{H}),\ \ \mathcal{F}(f)(\lambda) \circ D_\lambda^{\frac{s}{2}} \in HS_\lambda\},\ \ s \in \mathbf{R}\,.$$

Similarly, one can define via the Fourier transform Besov spaces.

4. Littlewood-Paley theory

Once given a dyadic partition of the unity

$$1 = \psi(\xi) + \sum_{p \geq 1} \phi(2^{-2p}\xi)$$

one decomposes a function $f \in L^2(\mathbf{H})$ as $f = \Sigma_{p \geq -1} \Delta_p f$ with

$$\mathcal{F}(\Delta_{-1} f) = \mathcal{F}(f) \circ \psi(D_\lambda),\ \ \mathcal{F}(\Delta_p f) = \mathcal{F}(f) \circ \phi(2^{-2p} D_\lambda).$$

(Bahouri, Chemin, Gallagher, Gérard, Lemarié, Xu)

II. A Class of Symbols on the Heisenberg Group

1- The idea...

Clasical case : The operator of symbol a is defined via the inverse Fourier formula by

$$\mathrm{Op}(a)f(x) = (2\pi)^{-1} \int_{\mathbf{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) a(x,\xi) d\xi, \quad x \in \mathbf{R}\,.$$

Heisenberg case : One wants to define an operator A by

$$Af(w) = \pi^{-2} \int_{\mathbf{R}} \mathrm{tr}\left(v^{\lambda}_{w^{-1}} \mathcal{F}(f)(\lambda) A_\lambda(w) \right) |\lambda| d\lambda, \quad w \in \mathbf{H}.$$

Question: How to chose the family of operators $A_\lambda(w)$ to get an algebra of operators which contains the differential operators and the elements of which act continously on Sobolev spaces?

2- Analysis of two examples

- **The operator Z**

$$Zf(w) = \pi^{-2} \int_{\mathbf{R}} \operatorname{tr}\left(v^{\lambda}_{w^{-1}} \mathcal{F}(f)(\lambda) Q_\lambda\right) |\lambda| d\lambda,$$

with

$$Q_\lambda = T_\lambda \sqrt{|\lambda|}(\partial_\xi - \xi) T^*_\lambda \quad \text{if} \quad \lambda > 0$$
$$Q_\lambda = T_\lambda \sqrt{|\lambda|}(\partial_\xi + \xi) T^*_\lambda \quad \text{if} \quad \lambda < 0$$

$\Longrightarrow$ Chose

$$A_\lambda(w) = T_\lambda \operatorname{op}^w \left(a\left(w, \operatorname{sgn}(\lambda)\sqrt{|\lambda|}\xi, \sqrt{|\lambda|}\eta\right)\right) T^*_\lambda$$

where $(\xi, \eta) \mapsto a(w, \xi, \eta)$ belongs to a class of symbols associated with the Harmonic oscillator (with Weyl-Hörmander quantification).

Then, the symbol of Z is $a(w, \xi, \eta) = i\eta - \xi$.

• The operator S

We have $\mathcal{F}(Sf)(\lambda) = i\lambda\mathcal{F}(f)(\lambda)$ whence

$$Sf(w) = \pi^{-2} \int_{\mathbf{R}} \mathrm{tr}\left(v^{\lambda}_{w^{-1}}\mathcal{F}(f)(\lambda)\right) i\lambda \, |\lambda| d\lambda$$

$\Longrightarrow$ The symbols have to depend also on λ

Then, the symbol of S is $i\lambda$.

3- The class of symbols

Let $\mu \in \mathbf{R}$. We say that $a \in S_{\mathbf{H}}(\mu)$ if and only if

- $a \in \mathcal{C}^\infty(\mathbf{H} \times \mathbf{R}^3)$
- for all $k, n \in \mathbf{N}$, $\rho \in \mathbf{N}$, there exists a constant $C_{n,k} > 0$ such that

$$\|\partial_\lambda^k \partial_{(\xi,\eta)}^\beta a(\cdot, \lambda, \xi, \eta)\|_{C^\rho(\mathbf{H})} \leq C_{n,k} \left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu - |\beta|}{2}} (1+|\lambda|)^{-k}$$

for $|\beta| \leq n$ and $(w, \lambda, \xi, \eta) \in \mathbf{H} \times \mathbf{R}^3$.

Examples: $i\eta - \xi \in S_{\mathbf{H}}(1)$ and $i\lambda \in S_{\mathbf{H}}(2)$.

III. The Algebra of Pseudodifferential Operators

1. Definition

With $a \in S_{\mathbf{H}}(\mu)$ one associates the operator

$$A_\lambda(w) = T_\lambda \, \mathrm{op}^w \left(a(w, \lambda, \mathrm{sgn}(\lambda)\sqrt{|\lambda|}\xi, \sqrt{|\lambda|}\eta \right) T_\lambda^*.$$

Then, one defines the operator $\mathrm{Op}(a)$ by

$$\mathrm{Op}(a) f(w) = \pi^{-2} \int_{\mathbf{R}} \mathrm{tr} \left(v_{w^{-1}}^\lambda \mathcal{F}(f)(\lambda) A_\lambda(w) \right) |\lambda| d\lambda.$$

Theorem: If $a \in S_{\mathbf{H}}(\mu)$, then $\mathrm{Op}(a)$ maps continuously $\mathcal{S}(\mathbf{H})$ into itself. Moreover, $\mathrm{Op}(a)^*$ and $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ are pseudodifferential operators of order μ and $\mu + \nu$ respectively if $a \in S_{\mathbf{H}}(\mu)$ and $b \in S_{\mathbf{H}}(\nu)$.

2. Idea of the proof of the continuity on the Schwartz class

Prove $f \in \mathcal{S}(\mathbf{H}) \implies \mathrm{Op}(a)f \in L^\infty(\mathbf{H})$ i.e. find C_0 such that

$$I := \left| \int_{\mathbf{R}} \mathrm{tr}\left(v^{\lambda}_{w^{-1}} \mathcal{F}(f)(\lambda) A_\lambda(w) \right) |\lambda| d\lambda \right| < C_0. \tag{1}$$

We have by Cauchy-Schwartz

$$I \leq \left(\int_{\mathbf{R}} \| v^{\lambda}_{w^{-1}} \mathcal{F}(f)(\lambda) (\mathrm{Id} + D_\lambda)^k \|^2_{HS} |\lambda| d\lambda \right)^{\frac{1}{2}} \times \left(\int_{\mathbf{R}} \| (\mathrm{Id} + D_\lambda)^{-k} A_\lambda(w) \|^2_{HS} |\lambda| d\lambda \right)^{\frac{1}{2}}.$$

We consider successively each term.

• First term:

$f \in \mathcal{S}(\mathbf{H}^d) \Longrightarrow$

$$\begin{aligned} \|v^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|_{HS} &\leq \|\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|_{HS} \\ &\leq C\,\|\mathcal{F}((\mathrm{Id}-\Delta_{\mathbf{H}})^k f)(\lambda)\|_{HS} \end{aligned}$$

and by the Plancherel formula

$$\int_{\mathbf{R}} \|\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|^2_{HS}|\lambda|d\lambda \leq C_1\,\|(\mathrm{Id}-\Delta_{\mathbf{H}})^k f\|^2_{L^2(\mathbf{H})}.$$

• Second term: A standard estimate gives

$$\begin{aligned} \|(\mathrm{Id}+D_\lambda)^{-k}A_\lambda(w)\|_{HS} &\leq \|(\mathrm{Id}+D_\lambda)^{-\frac{\mu}{2}}A_\lambda(w)\|_{\mathcal{L}(L^2(\mathbf{R}))}\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\|_{HS} \\ &\leq C\,\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\|_{HS} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{R}} \|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\|_{HS}^2|\lambda|d\lambda &= \sum_{m\in\mathbf{N}} \int_{\mathbf{R}} (1+|\lambda|(2m+1))^{\frac{\mu}{2}-k}\,|\lambda|d\lambda \\ &\leq C\left(\sum_{m\in\mathbf{N}} \frac{1}{1+m^2}\right)\int_{\mathbf{R}}(1+|\beta|)^{\frac{\mu}{2}-k+1}d\beta \end{aligned}$$

with the change of variables $\beta = \lambda(2m+1)$.

Choose k large enough

$$\int_{\mathbf{R}} \|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}A_\lambda(w)\|_{HS}^2|\lambda|d\lambda \leq C_2$$

- Conclusion of the proof:

$$I \leq \sqrt{C_1\, C_2}\, \|(\mathrm{Id} - \Delta_{\mathbf{H}})^k f\|_{L^2(\mathbf{H})}.$$

- Dealing with derivatives and multiplication by w: One uses

$$[Z,\, \mathrm{Op}(a)] = \mathrm{Op}(b_1),\;\; [\overline{Z}\,,\, \mathrm{Op}(a)] = \mathrm{Op}(b_2),$$
$$[z\,,\, \mathrm{Op}(a)] = \mathrm{Op}(c_1),\;\; [\overline{z}\,,\, \mathrm{Op}(a)] = \mathrm{Op}(c_2),$$
$$[is, \mathrm{Op}(a)] = -\mathrm{Op}(\partial_\lambda a),$$

with

$$b_1 = Za + \lambda(\partial_\eta a + i\partial_\xi a),\;\; b_2 = \overline{Z}a + \lambda(\partial_\eta a - i\partial_\xi a) \in S_{\mathbf{H}}(\mu+1),$$
$$c_1 = \frac{1}{2}\left(\frac{1}{i}\partial_\eta a - \partial_\xi a\right),\;\; c_2 = \frac{1}{2}\left(\frac{1}{i}\partial_\eta a + \partial_\xi a\right) \in S_{\mathbf{H}}(\mu - 1),$$
$$\partial_\lambda a \in S_{\mathbf{H}}(\mu).$$

IV. The Action on Sobolev Spaces

Theorem : There exists a constant C such that if a is a symbol of order μ, then

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(H^s(\mathbf{H}),H^{s-\mu}(\mathbf{H}))} \leq C\ \|a\|_{n,S_{\mathbf{H}}(\mu)}.$$

Schedule of the proof:

- Reduction to the case of a fixed regularity index $0 < s < 1$ by use of the Laplace-Kohn operator.
- Case of special symbols called reduced (treated by Littlewood-Paley theory) .
- Approximation of any symbol by reduced symbols (via Fourier series).

(Proof inspired by the proof of Coifman-Meyer in the classical case)

1- The method: the classical case

- Classical case: Consider a dyadic partition of the unity

$$1 = \psi(\xi) + \sum_{p\in\mathbf{N}} \phi(2^{-p}\xi).$$

$$\text{Then } \; a(x,\xi) = a(x,\xi)\psi(\xi) + \sum_{p\in\mathbf{N}} a(x,\xi)\phi(2^{-p}\xi)$$

Use Fourier series to write

$$b_{-1}(x,\xi) := a(x,\xi)\psi(\xi) = \sum_{k\in\mathbf{N}} b_k^{-1}(x)\mathrm{e}^{ik\cdot\xi}\tilde{\psi}(\xi), \quad \psi\tilde{\psi} = 1$$

$$b_p(x,\xi) := a(x,2^p\xi)\phi(\xi) = \sum_{k\in\mathbf{N}} b_k^p(x)\mathrm{e}^{ik\cdot\xi}\tilde{\phi}(\xi), \quad \phi\tilde{\phi} = 1.$$

Then $a(x,\xi) = \Sigma_k t_k(x,\xi)$,

$$t_k = b_k^{-1}(x)\tilde{\psi}(\xi)\mathrm{e}^{ik\xi} + \sum_{p\in\mathbf{N}} b_k^p(x)\mathrm{e}^{i2^{-p}k\xi}\tilde{\phi}(2^{-p}\xi).$$

The boundedness of $\mathrm{Op}(t_k)$ comes from standard theory and the convergence of the t_k series is due to symbol estimates.

• Adaptation to Heisenberg case

- Dyadic unity partition in (y, η)
- Expansion in Fourier series

• Main difficulties

- Reduced symbols are more difficult to treat:

$$op^w(\varphi(2^{-2p}|\lambda|(\xi^2+\eta^2)) \neq \varphi(2^{-2p}|\lambda|(\xi^2-\partial_\xi^2))$$

One uses Mehler's formula to see how different these two operators really are.

- The symbols depend on λ and one need another microlocalisation in λ.

2- Decomposition of the symbol

Dyadic partition of the unity

$$\begin{aligned} a(w,\lambda,\xi,\eta) &= a(w,\lambda,y,\eta)\psi\left(\xi^2+\eta^2\right)+\sum_{p\geq 0} a(w,\lambda,\xi,\eta)\phi\left(2^{-2p}(\xi^2+\eta^2)\right) \\ &= b_{-1}(w,\lambda,\xi,\eta)+\sum_{p\geq 0} b_p(w,\lambda,2^{-p}\xi,2^{-p}\eta) \end{aligned}$$

with

$$\psi\left(\xi^2+\eta^2\right)+\sum_{p\geq 0}\phi\left(2^{-2p}(\xi^2+\eta^2)\right)=1.$$

Fourier series expansion

$$b_p(w,\lambda,\xi,\eta)=\sum_{k\in\mathbf{Z}^2} \mathrm{e}^{ik\cdot(\xi,\eta)} b_p^k(w,\lambda)\widetilde{\phi}(\xi^2+\eta^2),$$

Dyadic decomposition in λ

$$b_p^k(w,\lambda)=\sum b_{p,-1}^{k,j}(w)\mathrm{e}^{ij\lambda}\tilde{\psi}(\lambda)+\sum_{r\in\mathbf{N},j\in\mathbf{Z}} b_{p,r}^{k,j}(w)\mathrm{e}^{ij2^{-2r}\lambda}\tilde{\phi}(2^{-2r}\lambda).$$

At the end:

$$a(w,\lambda,\xi,\eta) = \sum_{k,j} t_{k,j}(w,\lambda,\xi,\eta)$$

$$t_{k,j}(w,\lambda,\xi,\eta) = \sum_{p} b^{k,j}_{p,-1}(w)\psi^j(\lambda)\Phi^k_p(\xi,\eta) + \sum_{p,r} b^{k,j}_{p,r}(w)\phi^j(2^{-2r}\lambda)\Phi^k_p(\xi,\eta),$$

$$\text{with}\ \ \psi^j(\lambda) = \mathrm{e}^{ij\lambda}\psi(\lambda),\ \ \phi^j(\lambda) = \mathrm{e}^{ij\lambda}\phi(\lambda),$$

$$\Phi^k_{-1}(\xi,\eta) = \mathrm{e}^{ik\cdot(\xi,\eta)}\psi\left(\xi^2+\eta^2\right),\ \ \Phi^k_p(\xi,\eta) = \mathrm{e}^{ik\cdot(\xi,\eta)}\phi\left(2^{-2p}(\xi^2+\eta^2)\right),\ \ p \in \mathbf{N}\,.$$

3- Analysis of reduced symbols

• Convergence of the $t_{k,j}$ series

It comes from symbolic estimates as in the classical case.

• Action of $\mathrm{Op}(t_{k,j})$ on Sobolev spaces

It comes from the use of Mehler's formula and Littlewood-Paley theory on the Heisenberg group.

CONCLUSION

• Applications

- Wave front
- Microlocal defect measures
- Phase-space and semi-classical analysis

• Extensions

- Stratified Lie groups

• Related works

- Stein, Taylor and the Weyl correspondence.
- Greinier et all. (80's): Laguerre calculus on the Heigroup.
- Ruzhansky (2007): symbolic calculus on compact Lie groups.
- Van Erp (2008): symbolic calculus on the Heisenberg group related to contact manifolds.