Bertrand Georgeot

Quantum chaos, from quantum computers to astrophysics

Quantware group, LPT: D. Braun, R. Fleckinger, K. Frahm , O. Giraud, D. Shepelyansky

Collaborators: I. Garcia-Mata (Buenos Aires), F. Lignières (OMP), J. Martin (Liège), F. Mila (Lausanne), M. Znidaric (Ljubjana)

Support: ANR Jeune Chercheur, European Integrated Project Eurosqip, PEPS-PTI CNRS

Quantware group
Laboratoire de Physique Théorique, IRSAMC, UMR 5152 du CNRS
Université Paul Sabatier, Toulouse

Integrability and chaos

Two limiting cases for Hamiltonian dynamical systems with N degrees of freedom (phase space $2N$-dimensional)

- Integrability: N constants of motion in involution; motion takes place on N-dimensional tori
- Chaos: No constant of motion beyond possibly energy; motion is typically ergodic on the energy surface. Hard chaos: exponential divergence of nearby trajectories.

In practice, most systems have integrable and chaotic parts

Quantum chaos field: What happens in quantum mechanics?

Spectral statistics

Density of states $d(E) = d^{smooth}(E) + d^{osc}(E)$

- $d^{smooth}(E)$ contains geometrical information, independent of the dynamics (Weyl's term)
- $d^{osc}(E)$ depends on the dynamics

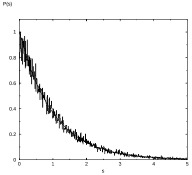
To probe these properties:

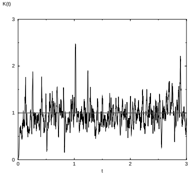
- Nearest-neighbour distribution $P(s)$
- Fourier transform of the two-point correlation function $\langle d^{osc}(E) d^{osc}(E+\epsilon)\rangle_E$, noted $K(t)$ (form factor)
- Variance of the number of levels in a box of size L noted $\Sigma^2(L)$

Conjectures

- Integrable system $\Longrightarrow$ random variables (Berry-Tabor (1977))

$$P(s) = e^{-s} \qquad\qquad K(\tau) = 1$$

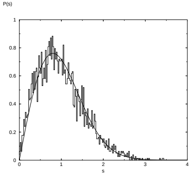


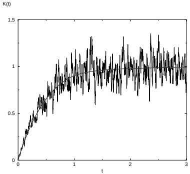


- Chaotic systems $\Longrightarrow$ eigenvalues of random matrices (Bohigas-Giannoni-Schmit (1984))

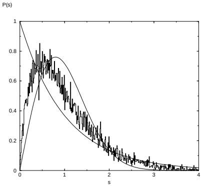
$$P(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

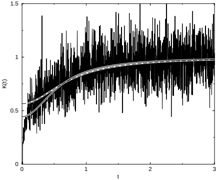
$$K(\tau) = \begin{cases} 2\tau - \tau \ln(1+2\tau) & 0 < \tau < 1 \\ 2 - \tau \ln\left(\frac{2\tau+1}{2\tau-1}\right) & \tau > 1 \end{cases}$$





Intermediate systems





Intermediate statistics (Bogomolny (1999)): $\frac{x_i+x_{i+1}}{2}$ with (x_i) Poissonian

$$P(s) = 4se^{-s}, \qquad K(\tau) = \frac{8+4\pi^2\tau^2}{16+4\pi^2\tau^2}$$

Observed e.g.:

- In 3D Anderson model at metal-insulator transition
- In non-integrable and non-chaotic billiards (e.g. triangular billiards)

Characterized by:

- Level repulsion $P(s)\underset{0}{\sim}0$ and exponential decrease $P(s)\underset{\infty}{\sim}e^{-as}$
- Form factor $0 < K(0) < 1$
- Variance of the number of levels $\Sigma^2(L)_{L\to\infty}^{\sim}K(0)L$

Semiclassical limit

- Quantum mechanics can be approximated by classical mechanics + phases at small $\hbar$ (i.e. $\hbar$ is small compared to quantities of same dimension (=actions) in the system) $\Rightarrow$ semiclassical approximation
- For integrable systems, EBK formulas give semiclassical approximation for energies, wavefunctions in term of individual torus
- 1970's: Gutzwiller, Balian and Bloch: trace formulas to connect quantum observables to a set of classical trajectories (Fourier-like formulas); valid for chaotic systems, but plagued with divergences

Anderson localization

Chain of sites with nearest-neighbour coupling + random onsite disorder =Anderson model (1958) (electrons in a disordered potential)

Hamiltonian $H_0 + V$

$\Rightarrow$ H_0 diagonal matrix with entries $(H_0)_{ij} = \epsilon_i \delta_{i,j}$ describes on-site disorder; $\delta_{i,j}$ are Kronecker symbols, and ϵ_i Gaussian independent random numbers

$\Rightarrow$ V tridiagonal matrix $(H_1)_{ij} = V(\delta_{i,j+1} + \delta_{i+1,j})$ describes the hopping between nearest-neighbors

one-dimensional model, states always localized: Wavefunctions have envelopes of the form $\exp(-x/l)$ where l is the localization length

Classical model $\Rightarrow$ diffusion, no localization
periodic disorder $\Rightarrow$ Bloch waves, balistic transport

For dimension $d \geq 3$, the model presents a transition for a critical value of disorder, between extended and localized states
At the transition point: multifractal states

Model with intermediate statistics

J. Martin, O. Giraud and B. Georgeot, Phys. Rev. E 77, 035201(R) (2008))

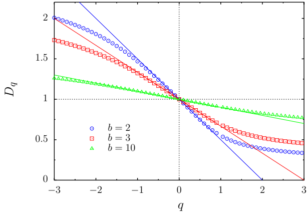
Classical map:

$$\bar{p} = p + \gamma \pmod{1}$$
$$\bar{q} = q + 2\bar{p} \pmod{1},$$

Quantization of this map:

- Simple quantum map with intermediate statistics
- Spectral statistics controlled by parameter γ
- γ irrational $\Rightarrow$ Random Matrix Theory
- $\gamma = a/b$ rational $\Rightarrow$ intermediate statistics

D_q for $\gamma = 1/b$ with $b = 2, 3, 10$.



$\Longrightarrow$ simple model where multifractality and intermediate statistics vary together, controlled by a single parameter γ

$\Longrightarrow$ Enable to probe the links between the two properties

$\Longrightarrow$ Insight into the 3D Anderson transition

Quantum information

- New way to treat information, using quantum mechanical properties of matter
- Field not tied to a specific type of systems: any quantum mechanical device with some specific properties can be chosen $\Longrightarrow$ many different experimental implementations possible
- Applications: quantum cryptography, teleportation...
- General all-purpose quantum device: quantum computer
- Very hard to build experimentally, only very small ones actually realized

Quantum computer

- classical computer: building blocks: bits 0 or 1
- quantum computer: building blocks:qubits = two-level system $|0>$ et $|1>$
  Any state of the form $(\alpha|0\rangle + \beta|1\rangle)$ is allowed, but measurement gives only one value (with probabilities $|\alpha|^2$ and $|\beta|^2$).
- A quantum computer can be thought as a set of n qubits (Hilbert space of dimension $N = 2^n$). General quantum state of the computer: $\sum_{i=0}^{N-1} a_i|i\rangle$ with $\sum_{i=0}^{N-1} |a_i|^2 = 1$ (from $i = |00...0\rangle$ to $i = |11...1\rangle$).
- **Logical operations**: unitary transformations in Hilbert space $\Rightarrow$ reversible computation, no dissipation ($\neq$ classical computation). Only source of irreversibility comes from quantum measurements.
- Quantum information theory $\Rightarrow$ The information contained in a quantum state can be measured in units of qubits

Quantum gates

One acts on the wave function of the quantum computer through unitary transformation. In practice, one uses elementary quantum gates which arelocal and compose them to build the unitary evolution needed.

- Hadamard gate applied to one qubit $|0\rangle \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$; $|1\rangle \rightarrow (|0\rangle - |1\rangle)/\sqrt{2}$;
- controlled not or CNOT applied to two qubits: $|00\rangle \rightarrow |00\rangle$; $|01\rangle \rightarrow |01\rangle$;$|10\rangle \rightarrow |11\rangle$; $|11\rangle \rightarrow |10\rangle$; the second qubit is changed if the first is in the state $|1\rangle$;

$\Longrightarrow$ Universal sets of quantum gates are enough to build any unitary transformations (for example, one-qubit gates +CNOT). Different universal sets are possible, their choice depends on experimental implementations.
$\Longrightarrow$ Quantum algorithms are unitary operations transforming aninitial quantum state into a desired one from which information can be extracted through quantum measurements
$\Longrightarrow$ Complexity of a quantum algorithm measured by number of quantum gates needed; famous example: Shor's quantum algorithm factors a number exponentially faster than any known classical algorithm

Quantum entanglement

Qubits can presentcorrelations impossible to obtain classically (cf Bell's theorem)

- Entanglement of a quantum state describes its degree of non-factorizability in products of one-qubit states.
- Example: Einstein-Podolsky-Rosen paradox; measuring one qubit of the state $(|00\rangle + |11\rangle)/\sqrt{2}$ influences the other one, whatever their distance.
- Entanglement can be quantified (although there are competing ways of doing it). It is crucial for, say, quantum teleportation.
- Entanglement is believed to be a key resource in quantum computation, but it is not clearly understood exactly how.
- Result (Jozsa and Linden 2002): for small enough entanglement, the quantum process can be simulated classically efficiently $\Longrightarrow$ entanglement needed for quantum gain

Random states

- Random states appear in various quantum protocols
- Correspond to ensembles of quantum states with a given statistical distribution
- Similar to random numbers in classical information
- Can describe "typical quantum states"

$\Longrightarrow$ Many proposals to generate random states, e.g. with chaotic maps (Emerson et al., 2003)
$\Longrightarrow$ Example: columns of random matrices
Questions:

- Construct different ensembles of random vectors
- Explore through them the properties of quantum resources such as entanglement?
- Obtain through such ensembles new types of resources for quantum information, or evaluate the difficulty of constructing such states.
- Devise optimal ways to construct such random states

What is entanglement? Entanglement and localization

O. Giraud, J. Martin, B. Georgeot (Phys. Rev. A 76, 042333 (2007))

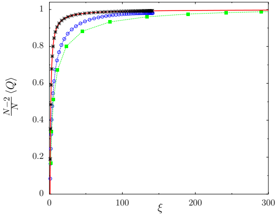
Meyer-Wallach entanglement: $Q = 2\left(1 - \frac{1}{n}\sum_{\alpha=1}^{n} R_\alpha\right)$, where $R_\alpha = \mathrm{tr}\rho_\alpha^2$,

For localized random states

Localization measure: $\xi = \langle \sum_i |\psi_i|^2 / \sum_i |\psi_i|^4 \rangle$; results for n qubits ($N = 2^n$):

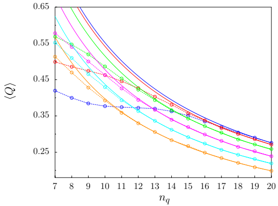
States randomly distributed:
$\langle Q \rangle = \frac{N-2}{N-1}\left(1 - \frac{1}{\xi}\right) \rightarrow 1 - 1/\xi$ for $n \rightarrow \infty$

Application: interacting spins



States distributed on adjacent basis states: $\langle Q \rangle \sim (\log_2(2\xi) + \frac{4}{3})/n \rightarrow 0$ for $n \rightarrow \infty$

Application: Anderson localization



Generalizations

O. Giraud, J. Martin, B. Georgeot (Phys. Rev. A 79, 032308 (2009))

Entanglement in terms of moments $p_q = \sum_{i=1}^{N} |\psi_i|^{2q}$.
Entropy of entanglement of $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$: if $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$,

$$S = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$$

- First order: random states, bipartitions $(\nu, n-\nu)$

$$\langle S \rangle \simeq \nu - \frac{2^\nu - 1}{2\ln 2}\left(1 - \frac{N - 2^\nu}{N-1}(1 - \langle p_2 \rangle)\right)$$

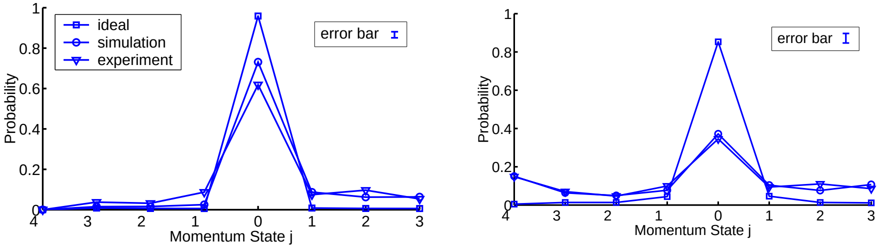
- Second order $\langle S^{(2)} \rangle = N(N-2)(N^2 - 6N + 16)c_{1111} + 4N(N-2)(N-4)c_{211} + 4N(N-2)c_{22}$

$$c_{22} = \frac{\langle p_2^2 \rangle - \langle p_4 \rangle}{N(N-1)}, \quad c_{211} = \frac{\langle p_2 \rangle - \langle p_2^2 \rangle - 2\langle p_3 \rangle + 2\langle p_4 \rangle}{N(N-1)(N-2)},$$

$$c_{1111} = \frac{1 - 6\langle p_2 \rangle + 8\langle p_3 \rangle + 3\langle p_2^2 \rangle - 6\langle p_4 \rangle}{N(N-1)(N-2)(N-3)}$$

For multifractal quantum states, second order of the entanglement depends on the multifractal exponents

Experimental realization of a Toulouse quantum algorithm



Quantum map simulation on a NMR quantum computer with three qubits (Cory group, MIT, USA). Localization of the wave function in momentum space is visible, but differs from the ideal result. A numerical simulation with noise and decoherence can reproduce the experimental data. (from M. K. Henry, J. Emerson, R. Martinez, D. Cory, Physical Review A **74**, 062317 (2006)).

Optimized algorithms for small quantum computers

M. Znidaric, O. Giraud and B. Georgeot, Phys. Rev. A v. 77, 032320 (2008)

Complexity of a quantum algorithm measured by the total number of quantum gates in the limit of many qubits
Experiments so far have been able to manipulate only very small number of qubits
$\Longrightarrow$ Need to optimize algorithms for such systems

Additionally, usually, two-qubit gates much harder to realize than one-qubit gates
$\Longrightarrow$ minimize the number of two-qubit gates for small systems
To generate all unitary transformation: exponentially many CNOTs needed.

For few-qubit: exactly how much is exponential?
Unitary transformations of two qubits: 3 CNOTs are needed
Unitary transformations of three qubits: 14 CNOTs lower bound, actual algorithm gives 20. Can one do better with states?
For two qubits $\Rightarrow$ one CNOT is enough

How many CNOTs are needed for a three-qubit state?

For three qubits: $\Rightarrow$ Starting from a separable state (such as $|000\rangle$): one can reach any three-qubit pure state using three CNOTs

Class 0: One needs zero CNOT to transform $|\psi\rangle$ to $|000\rangle$ iff the state is of the product form $|\psi\rangle = |\alpha\beta\gamma\rangle$,

Class 1: One needs one CNOT iff the state is of the form $|\psi\rangle = |\alpha\rangle_1|\chi\rangle_{23}$ (i.e., it is bi-separable), where $|\chi\rangle_{23}$ is any entangled state of the last two qubits

Class 2: One needs two CNOT gates iff the state is of the form $|\psi\rangle = \cos\varphi|\alpha\beta\gamma\rangle + \sin\varphi|\alpha_\perp\beta'\gamma'\rangle$, with $\langle\alpha|\alpha_\perp\rangle = 0$, $|\langle\beta|\beta'\rangle| < 1$ and $|\langle\gamma|\gamma'\rangle| < 1$

Class 3: One needs three CNOT gates iff a state is not in class 0, 1 or 2.

Procedure is explicit

Generalizations:

$\Rightarrow$ Starting from GHZ state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ only two CNOTs are needed to reach any three-qubit pure state
$\Rightarrow$ Consequence: any three-qubit state can be transformed into any other using at most four CNOT gates

Again, procedure is explicit

If only nearest-neighbour CNOTs are allowed:
$\Rightarrow$ still two CNOTs are needed starting from GHZ
$\Rightarrow$ certain states need four CNOT starting from $|000\rangle$

Enables to obtain optimal algorithms to build exactly random states for small quantum computers available experimentally (up to three qubits) (O. Giraud, M. Znidaric, and B. Georgeot, preprint arxiv:0903.4109 (2009))

Quantum simulators

One of the main use of quantum computers: fast simulations of quantum Hamiltonians
Alternative to quantum computers:

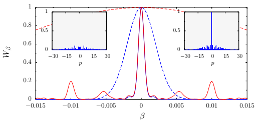
- Bose-Einstein condensate of cold atoms in optical lattice
- When lattice parameters are changed,quantum phase transition from superfluid to Mott insulator (Bose-Hubbard model) (observed in Greiner et al, Nature 2002).
- Adding electric fields and magnetic fields and changing the parameters of the optical lattice $\Rightarrow$ Possibility to simulate many different many-body Hamiltonians, in a controllable way.

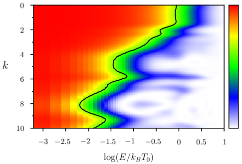
$\rightarrow$ "quantum analog computer": not universal, but easier to use than a general-purpose quantum computer
$\rightarrow$ Other physical implementations possible
$\rightarrow$ Cold noninteracting atoms: can simulate dynamics of complex one-body Hamiltonians

Loschmidt cooling by time reversal of atomic matter waves

J. Martin, B. Georgeot and D. L. Shepelyansky, Phys. Rev. Lett. v. 100, 044106 (2008)

⇒ Time reversal: fundamental question of statistical mechanics (Boltzmann-Loschmidt controversy); time reversal of spin systems, acoustic and electromagnetic waves already performed
⇒ Experimental scheme to realize approximate time reversal of matter waves for ultracold atoms in optical lattices in a regime of quantum chaos.
⇒ A significant fraction of the atoms return back to their original state, being at the same time cooled down by several orders of magnitude.
⇒ The proposed scheme can be implemented with existing experimental setups.



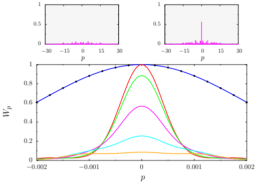


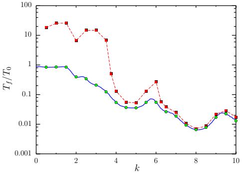
Initial states probability and final return probability as a function of quasimomentum (left). Return probability distribution as a function of the atom energy and of the kick strength. Black curve: final temperature of the return Loschmidt peak (right)

Time reversal of Bose-Einstein condensates

J. Martin, B. Georgeot and D. L. Shepelyansky, Phys. Rev. Lett. v. 101, 074102 (2008)

⇒ Bose-Einstein condensates (BEC): interaction between atoms
⇒ Simulations using Gross-Pitaevskii equation → preceding scheme can be implemented for weak nonlinearity
⇒ accuracy of time reversal decreases with the increase of atom interactions inside BEC, until it is completely lost.
⇒ Surprisingly, quantum chaos helps to restore time reversibility.
⇒ These predictions can be tested with existing experimental setups.





Initial and final return probability distributions for various values of the nonlinearity parameter g (left)
Loschmidt cooling of time reversed BEC characterized by the ratio of final to initial temperatures, with $g = 0; 0.5; 10$ (right)

Quantum chaos in rotating stars

F. Lignières and B.G., Phys. Rev. E **78**, 016215 (2008) and A & A **500**, 1173 (2009)

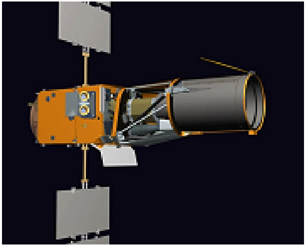
→ Acoustic oscillation spectra are among the most important information obtained from the stars

→ Slowly rotating stars (e.g. the sun): almost spherical, asymptotic theory built on EBK quantization of modes (integrable system) ⇒ well understood

→ Rapidly rotating stars: not well understood, no theory to interpret data

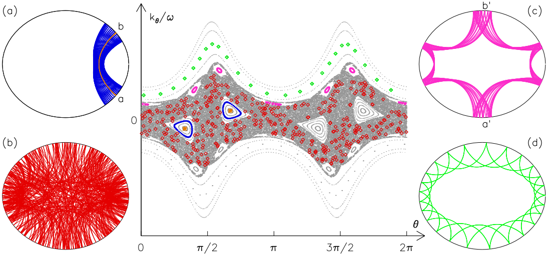
→ Observations: very recent space missions Corot, Kepler

⇒ unprecedented precision



Acoustic rays dynamics

- Short-wavelength limit of acoustic wave equation
  $\Longrightarrow$ system described by classical dynamical system ("rays")
- Poincaré Surface of Section close to the boundary of the star $\Rightarrow$ enables to visualize phase space



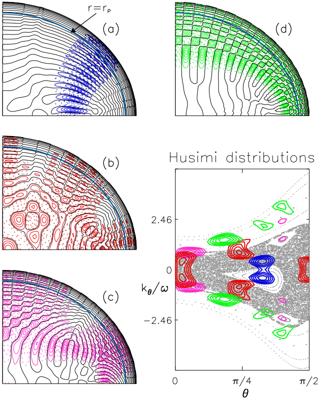
a) and c) 2- and 6-period island, b) chaotic d) whispering gallery

- Phase space displays integrable and chaotic zones (mixed systems).

Comparison with acoustic modes

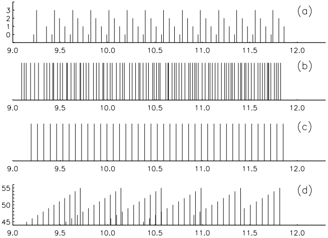
$\Rightarrow$ Percival, Berry-Robnik:
modes should asymptotically be associated with different phase space regions
$\Rightarrow$ Numerically computed modes fulfill this conjecture

a) and c) 2- and 6-period island
b) chaotic modes
d) whispering gallery modes



Consequences for spectra

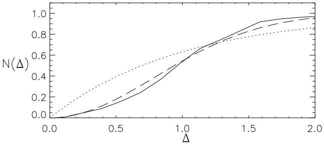
Spectrum is divided into
well-defined subspectra
a) and c) 2- and 6-period island
b) chaotic modes
d) whispering gallery modes



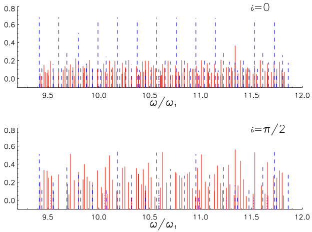
Integrable 2-period island
modes: regular spectrum

$$\omega_{n\ell} = n\delta_n + \ell\delta_\ell + \alpha$$

chaotic modes: Random Matrix Theory spectrum



visibility
From the pole (top) and equator
dashed blue: 2-period island
red: chaotic modes



PageRank vector and google search

O. Giraud, B. Georgeot and D. L. Shepelyansky, Phys. Rev. E **80**, 026107 (2009)

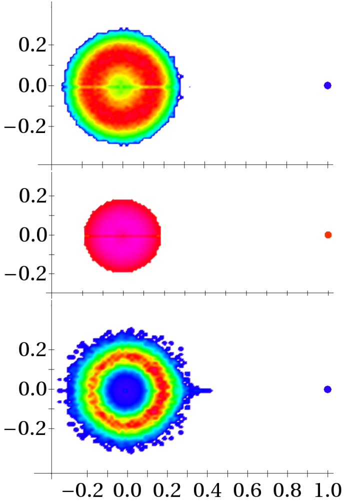
- The Google PageRank algorithm gives the PageRank vector, with amplitudes p_i, with $0 \leq p_i \leq 1$
- All webpages can then be ordered according to their PageRank value
- The PageRank value of a webpage can be understood as the average time a random surfer will spend there
- It ranks websites according to the number of links pointing to them which come from high-PageRank sites.
- The PageRank vector is the eigenvector associated with the largest eigenvalue of the google matrix, built from the network.

$\Rightarrow$ It is important for usefulness of this strategy that the PageRank vector is not evenly spread, but sharply peaked around some preferred webpages

$\Rightarrow$ The PageRank vector should be localized to be useful

$\Rightarrow$ What are the localization properties of eigenvectors of the google matrix?

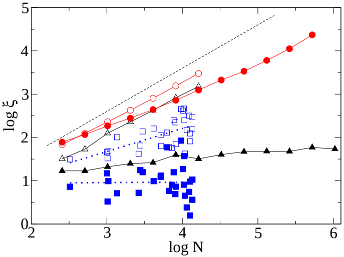
Results



Albert-Barabasi model with $q = 0.1$ (top), $q = 0.7$ (middle), and WWW data (bottom).

- Albert-Barabasi model for $p = 0.2$ and $q = 0.1$: PageRank vector localized
- Delocalization transition in the bulk
- Similar behaviour in real Web networks
- A-B model for $p = 0.2$ and $q = 0.7$: PageRank vector much less localized

Scaling laws



A-B model with $q = 0.1$ (triangles), $q = 0.7$ (circles), and WWW data (squares). PageRank (full), bulk (empty); dashed line: slope -1

- A-B model for $p = 0.2$ and $q = 0.1$, and actual WWW networks: PageRank localization length does not grow with system size $\Rightarrow$ confirms localization
- Albert-Barabasi model for $p = 0.2$ and $q = 0.7$: PageRank localization length grows with system size $\Rightarrow$ confirms delocalization
- For such networks, computation of PageRank vector is not useful anymore
- Other networks: the Internet, citations, etc

Conclusion

- Quantum chaos: many tools and techniques to study complex quantum systems
- Can be applied to many different physical contexts, even outside quantum mechanics or even without wave equation
- Other works in the group: effects of imperfections on Shor factorization quantum algorithm (K. Frahm, D. Shepelyansky), precise quantum measurements through decoherence (D. Braun), ratchet effect in nanostructures (D. Shepelyansky), intermediate statistics in Lax matrices (O. Giraud), etc...