Limiting absorption principle for some long range perturbations of Dirac systems at threshold energies

joint work with Nabile Boussaid (Besançon)

Sylvain Golénia

Universität Erlangen-Nürnberg

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The Dirac operator

Anti-commutation relations

Let α_i, for $i \in \{1,2,3,4\}$, be linearly independent self-adjoint linear applications, acting in $\mathbb{C}^{2\nu}$, satisfying the anti-commutation relations:

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{i,j}\,\mathrm{Id}_{\mathbb{C}^{2\nu}},$$

for $i,j = 1,\dots,4$. We set $\beta := \alpha_4$.

For $\nu = 1$, there is no solution.

When $\nu = 2$, one may choose the *Pauli-Dirac* representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathrm{Id}_{\mathbb{C}^\nu} & 0 \\ 0 & -\mathrm{Id}_{\mathbb{C}^\nu} \end{pmatrix}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for $i = 1,2,3$.

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The movement of a relativistic massive charged particles with spin-$1/2$ particle is given by the Dirac equation,

$$\mathrm{i}\hbar \frac{\partial \varphi}{\partial t} = D_m \varphi, \quad \text{in } L^2(\mathbb{R}^3; \mathbb{C}^{2\nu}),$$

where $m > 0$ is the mass, c the speed of light, $\hbar$ the reduced Planck constant, and

$$D_m := c\hbar\, \alpha \cdot P + mc^2 \beta = -\mathrm{i}c\hbar \sum_{k=1}^{3} \alpha_k \partial_k + mc^2 \beta.$$

Here we set $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := \alpha_4$.

We take $c = \hbar = 1$.

We define D_m on $\mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^{2\nu})$. We also denote its closure by D_m.

It is self-adjoint with domain $\mathcal{D}(D_m) = \mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$.

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One has:

$$D_m^2 = (-\Delta_{\mathbb{R}^3} + m^2) \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}},$$

where $L^2(\mathbb{R}^3; \mathbb{C}^{2\nu}) \simeq L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2\nu}$.

Set $\alpha_5 := \alpha_1\alpha_2\alpha_3\alpha_4$. It is unitary.

Moreover, using the anti-commutation relation, we infer

$$\alpha_5 \, D_m \, \alpha_5^{-1} = -D_m$$

Then,

$$\alpha_5 \, \varphi(D_m) \, \alpha_5^{-1} = \varphi(-D_m), \text{ for all } \varphi : \mathbb{R} \to \mathbb{C} \text{ measurable.}$$

Therefore, the spectrum of D_m is given by:

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We have the Hardy inequality:

$$\frac{1}{4}\int_{\mathbb{R}^3}\left|\frac{1}{|x|}f(x)\right|^2 dx \leq \left|\langle f, -\Delta_{\mathbb{R}^3} f\rangle\right| = \|\nabla f\|^2 = \|\sigma \cdot P f\|^2,$$

where $f \in \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^{2\nu})$.

For $j = 1, \ldots, n$, we choose n distinct points x_j of $\mathbb{R}^3$. On $\mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^{2\nu})$, we set:

$$H_\gamma := D_m + \gamma \sum_{j=1}^{n} \frac{1}{|Q - x_j|} \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}} = \alpha \cdot P + m\beta + \gamma \sum_{j=1}^{n} \frac{1}{|Q - x_j|} \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}}.$$

One has:

- $|\gamma| < 1/2$: H_γ is essentially self-adjoint and $\mathcal{D}(H_\gamma) = \mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$.

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The Coulombic potential

Self-adjointness

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where $f \in \mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^{2\nu})$.

For $j = 1, \ldots, n$, we choose n distinct points x_j of $\mathbb{R}^3$. On $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^{2\nu})$, we set:

$$H_\gamma := D_m + \gamma\sum_{j=1}^{n}\frac{1}{|Q-x_i|}\otimes \mathrm{Id}_{\mathbb{C}^{2\nu}} = \alpha\cdot P + m\beta + \gamma\sum_{j=1}^{n}\frac{1}{|Q-x_i|}\otimes \mathrm{Id}_{\mathbb{C}^{2\nu}}.$$

One has:

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We have the Hardy inequality:

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In fact, one has:

- $|\gamma| < \sqrt{\mathbf{3}}/2$: H_γ is essentially self-adjoint and $\mathcal{D}(H_\gamma) = \mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$.

Theorem

There are $\kappa, \delta, C > 0$ *such that the following* limiting absorption principle *holds:*

$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma - \lambda - \mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C.$$

In particular, H_γ *has no eigenvalue in* $\pm m$.

Moreover, there is C' *so that*

$$\sup_{|\gamma|\leq\kappa} \int_{\mathbb{R}} \|\langle Q\rangle^{-1} e^{-\mathrm{i}tH_\gamma} E_{\mathcal{I}}(H_\gamma) f\|^2 dt \leq C'\|f\|^2,$$

where $\mathcal{I} = [-m-\delta, -m] \cup [m, m+\delta]$ *and where* $E_{\mathcal{I}}(H_\gamma)$ *denotes the spectral measure of* H_γ.

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Since we are interested in small coupling constants, by perturbation theory, it is enough to consider:

$$H_\gamma^{\mathrm{bd}} := D_m + \gamma v(Q) \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}},$$

with $v : \mathbb{R}^3 \to \mathbb{R}$, smooth with

$$\|v\|_\infty \leq m/2$$

and

$$v(x) = \sum_{j=1}^{n} \frac{1}{|Q - x_j|},$$

for $|x|$ big enough.

To show the limiting absorption principle (LAP)

$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}}-\lambda-\mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C,$$

for some $\kappa > 0$. It is equivalent to show:

$$\sup_{\lambda\in[m,m+\delta],\, \varepsilon>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}}-\lambda-\mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C,$$

Indeed, we have:

$$\alpha_5\,(D_m+\gamma v)\,\alpha_5^{-1} = -D_m+\gamma v.$$

Then, we shall work at energy $[m, m+\delta]$ with v and with $-v$.

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$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}} - \lambda - \mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C,$$

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Since $\beta = \alpha_4$ has the eigenvalues ± 1 and the eigenspaces have the same dimension.

Let P^+ be the orthogonal projection on $\ker(\beta - 1)$. Let $P^- := 1 - P^+$.

By the anti-commutation relation, we get $P^{\pm}\alpha_j P^{\pm} = 0$. We set:

$$\alpha_j^+ := P^+\alpha_j P^- \text{ and } \alpha_j^- := P^-\alpha_j P^+, \text{ for } j \in \{1, 2, 3\}.$$

They are partial isometries: $(\alpha_j^+)^* = \alpha_j^-$, $\alpha_j^+\alpha_j^- = P^+$ and $\alpha_j^-\alpha_j^+ = P^-$, for $j \in \{1, 2, 3\}$.

We set $\mathbb{C}^\nu_\pm := P^\pm\mathbb{C}^{2\nu}$. In the direct sum $\mathbb{C}^\nu_+ \oplus \mathbb{C}^\nu_-$, one can write

$$\beta = \begin{pmatrix} \mathrm{Id}_{\mathbb{C}^\nu} & 0 \\ 0 & -\mathrm{Id}_{\mathbb{C}^\nu} \end{pmatrix} \text{ and } \alpha_j = \begin{pmatrix} 0 & \alpha_j^+ \\ \alpha_j^- & 0 \end{pmatrix}, \text{ for } j \in \{1, 2, 3\}.$$

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LAP at threshold energy

The spin down/up decomposition, on the way to the resolvent equation

We now split the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3; \mathbb{C}^{2\nu})$ with respect to the spin-up and -down part:

$$\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-, \text{ where } \mathscr{H}^\pm := L^2(\mathbb{R}^3; \mathbb{C}^\nu_\pm) \simeq L^2(\mathbb{R}^3; \mathbb{C}^\nu).$$

We rewrite the equation $(D_m + v(Q) - z)\psi = f$ to get:

$$\begin{cases} \alpha^+ \cdot P\psi_- + m\psi_+ + v(Q)\psi_+ - z\psi_+ = f_+, \\ \alpha^- \cdot P\psi_+ - m\psi_- + v(Q)\psi_- - z\psi_- = f_-. \end{cases}$$

then

$$\begin{cases} \left(\alpha^+ \cdot P \dfrac{1}{m - v(Q) + z} \alpha^- \cdot P + v(Q) + m - z\right)\psi_+ = & f_+ + \alpha^+ \cdot P \dfrac{1}{m - v(Q) + z} f_-, \\ \psi_- = & \dfrac{1}{m - v(Q) + z}(\alpha^- \cdot P\psi_+ - f_-.) \end{cases}$$

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In other words,

$$\begin{cases} (\Delta_{m,v,z} + m - z)\,\psi_+ = & f_+ + \alpha^+ \cdot P \dfrac{1}{m - v(Q) + z} f_-, \\ \psi_- = & \dfrac{1}{m - v(Q) + z}\left(\alpha^- \cdot P\psi_+ - f_-.\right) \end{cases}$$

where we defined the operator $\Delta_{m,v,z}$, as being the closure of:

$$\Delta_{m,v,z} := \alpha^+ \cdot P \frac{1}{m - v(Q) + z} \alpha^- \cdot P + v(Q),$$

acting on $\mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}_+^\nu)$.

At least formally, we get $(H_1^{\mathrm{bd}} - z)^{-1} =$

$$\begin{pmatrix} (\Delta_{m,v,z} + m - z)^{-1} & (\Delta_{m,v,z} + m - z)^{-1}\alpha^{+} \cdot P\dfrac{1}{m - v(Q) + z} \\ \dfrac{1}{m - v(Q) + z}\alpha^{-} \cdot P(\Delta_{m,v,z} + m - z)^{-1} & \dfrac{1}{m - v(Q) + z}\alpha^{-} \cdot P(\Delta_{m,v,z} + m - z)^{-1}\alpha^{+} \cdot P\dfrac{1}{m - v(Q) + z} - \dfrac{1}{m - v(Q) + z} \end{pmatrix}.$$

Problem: Does $(\Delta_{m,v,z} + m - z)^{-1}$ even exist for $\Im z \neq 0$?

Using $\|v\|_\infty \leq m/2$, one shows $\mathcal{D}(\Delta_{m,v,z}) = \mathcal{D}\big((\Delta_{m,v,z})^*\big) = \mathscr{H}^2(\mathbb{R}^3; \mathbb{C}^\nu_+)$.

Take now $f \in \mathscr{H}^2(\mathbb{R}^3; \mathbb{C}^\nu_+)$. Since

$$\Im\langle f, \Delta_{m,v,z} f\rangle = \langle \alpha^- \cdot P f, \frac{-\Im(z)}{\big(m - v(Q) + \Re(z)\big)^2 + \Im(z)^2} \alpha^- \cdot P f\rangle,$$

is of the sign of $-\Im(z)$.

The numerical range theorem ensures that the spectrum of $\Delta_{m,v,z}$ is contained in the lower/upper half-plane which does not contain z.

In other words:

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Strategy:

1. Reduce the problem to show:

$$\sup_{\Re(z)\in[m,m+\delta],\, \Im(z)>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(\Delta_{m,\gamma v,z}+m-z)^{-1}\langle Q\rangle^{-1}\| \leq C, \tag{1}$$

   for some $\kappa > 0$.

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1. Reduce the problem to show:

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   for some $\kappa > 0$.
2. Prove (1).

We put $\langle Q\rangle^{-1}$ on the right and on the left of spin up/down decomposition of $(H_\gamma^{\mathrm{bd}} - z)^{-1}$.

For instance, we need to control uniformly in $\Re z \in [m, m+\delta]$ and $\Im z \neq 0$:

$$\langle Q\rangle^{-1}(\Delta_{m,v,z} + m - z)^{-1}\,\alpha^+ \cdot P\frac{1}{m - v(Q) + z}\langle Q\rangle^{-1}.$$

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$$\underbrace{\langle Q\rangle^{-1}(\Delta_{m,v,z} + m - z)^{-1}\langle Q\rangle^{-1}}_{\text{bounded from LAP for } \Delta_{m,v,z}} \underbrace{\langle Q\rangle\, \alpha^+ \cdot P\langle Q\rangle^{-1}}_{\text{unbounded}} \underbrace{\frac{1}{m - v(Q) + z}}_{\text{bounded by } \|v\|_\infty \leq m/2} .$$

Idea: There is $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ such that

$$\sup_{\Re z \in [m, m+\delta], \Im z > 0, |\gamma| \leq \kappa} \left\| \langle Q\rangle^{-1}\varphi(\alpha\cdot P)(D_m + \gamma V(Q) - z)^{-1}\varphi(\alpha\cdot P)\langle Q\rangle^{-1} \right\| < \infty,$$

is equivalent to

$$\sup_{\Re z \in [m, m+\delta], \Im z > 0, |\gamma| \leq \kappa} \left\| \langle Q\rangle^{-1}(D_m + \gamma V(Q) - z)^{-1}\langle Q\rangle^{-1} \right\| < \infty.$$

LAP at threshold energy

Remark on the first point

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For instance, we need to control uniformly in $\Re z \in [m, m+\delta]$ and $\Im z \neq 0$:

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Problem:

- In $(\Delta_{m,v,z} + m - z)^{-1}$, the operator depends on the spectral parameter.
- It is more convenient to work for a spectral estimate above $[0,\delta]$, instead of $[m, m+\delta]$.

We recall:

$$\Delta_{m,v,z} := \alpha^{+} \cdot P \frac{1}{m - v(Q) + z} \alpha^{-} \cdot P + v(Q),$$

with domain $\mathscr{H}^2(\mathbb{R}^3; \mathbb{C}^{\nu}_{+})$.

We perform the shift, $z \mapsto z + m$ and study the operator

$\Delta_{2m,\gamma v,\xi}$, uniformly in $(\gamma, \xi) \in \mathcal{E} = \mathcal{E}(\kappa, \delta) := [-\kappa, \kappa] \times [0, \delta] \times (0, 1]$.

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In other words, we will show there are $\delta, \kappa, C > 0$ such that

$$\sup_{\Re z \geq 0, \Im z > 0, (\gamma, \xi) \in \mathcal{E}} \left\| |Q|^{-1} (\Delta_{2m,\gamma v,\xi} - z)^{-1} |Q|^{-1} \right\| \leq C,$$

where $\mathcal{E} = \mathcal{E}(\kappa, \delta) := [-\kappa, \kappa] \times [0, \delta] \times (0, 1]$.

and then take $\xi = z$.

Take H, A self-adjoint operators and $c \geq 0$ so that:

$$[H, \mathrm{i}A] - cH > 0,$$

where the symbol $>$ means non-negative and injective.

With further hypothesis, one finds B closed, densely defined and injective such that:

$$\sup_{\Re(z)\geq 0, \Im(z)>0} \|B(H-z)^{-1}B^*\| < \infty.$$

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Positive commutator estimates

The non-self-adjoint case

In our setting, we deal with $\Delta_{m,v,\xi}$ with is non-self-adjoint.

Take A self-adjoint, H non-self-adjoint and $c \geq 0$ so that

$$[\Re(H), \mathrm{i}A] - c\Re(H) > 0,$$

and

$$\Im(H) \geq 0 \text{ and } [\Im(H), \mathrm{i}A] \geq 0.$$

Problem: $\Delta_{m,v,\xi}$ depends on the external parameter ξ and we need estimates uniform in ξ.

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Take A self-adjoint and $H(\xi)$ a family of non-self-adjoint operators so that

$$[\Re(H(\xi)), \mathrm{i}A] - c\Re(H(\xi)) \geq S > 0,$$

and

$$\Im(H(\xi)) \geq 0 \text{ and } [\Im(H(\xi)), \mathrm{i}A] \geq 0,$$

with S a self-adjoint operator independent of ξ.

With further hypothesis, one finds B closed, densely defined and injective and C independent of ξ such that:

$$\sup_{\Re(z)\geq 0, \Im(z)>0} \|B(H-z)^{-1}B^*\| \leq C.$$

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We consider the generator of dilation given by:

$$A = \frac{1}{2}(P \cdot Q + Q \cdot P) \otimes \mathrm{Id}_{\mathbb{C}^\nu_+} \text{ on } L^2(\mathbb{R}^3; \mathbb{C}^\nu_+).$$

Then we have: $[\Re(\Delta_{2m,\gamma v,\xi}), \mathrm{i}A] - \Re(\Delta_{2m,\gamma v,\xi}) =$

$$\begin{aligned} &= \alpha^+ \cdot P \frac{2m - \gamma v + \Re(\xi)}{(2m - \gamma v + \Re(\xi))^2 + \Im(\xi)^2} \alpha^- \cdot P \\ &\quad - \gamma\, \alpha^+ \cdot P \left(\frac{Q \cdot \nabla v(Q)\big((2m - \gamma v + \Re(\xi))^2 - \Im(\xi)^2\big)}{\big((2m - \gamma v + \Re(\xi))^2 + \Im(\xi)^2\big)^2} \right) \alpha^- \cdot P - \gamma\, Q \cdot \nabla v(Q) - \gamma v(Q). \end{aligned}$$

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But we have:

$$\geq -c_0\, \Delta_{\mathbb{R}^3} \otimes \mathrm{Id}_{\mathbb{C}^+_{\nu}} + -\gamma \frac{c_1}{|Q|}$$

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With some care, we can show there are $\delta, \kappa, C > 0$ such that

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Then we take $\xi = z$ and deduce there are $\kappa, \delta, C > 0$ such that

$$\sup_{|\lambda| \in [m, m+\delta],\ \varepsilon > 0, |\gamma| \leq \kappa} \| \langle Q \rangle^{-1} (D_m + \gamma / |Q| - \lambda - \mathrm{i}\varepsilon)^{-1} \langle Q \rangle^{-1} \| \leq C.$$

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