The partonic structure of the electron at the next-to-leading logarithmic accuracy in QED

in collaboration with
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Introduction

- Not unreasonable to assume that the future of high-energy physics will involve an e^+e^- collider.
- Presence in the matrix elements of logarithmic terms such as $\log^k(E/m_e)$ which are numerically large and thus prevent the perturbative series from being well behaved.
- Their physical origin well understood (quasi-collinear emissions) $\rightarrow$ a large class of them is process-independent and can be accounted in a universal manner $\rightarrow$ structure-function approach
- QED Parton Distribution Functions (PDFs) known at leading-logarithmic (LL) accuracy i.e. resummation of the $(\alpha \log(E/m_e))^k$ terms. The goal of this work is to extend this formalism at NLL i.e. $\alpha(\alpha \log(E/m_e))^k$ terms.
- Crucial difference w.r.t. hadronic PDFs: QED PDFs are entirely calculable with perturbative techniques!

Outline

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QED factorisation formula

Idea: collect all of the logarithmic terms in some universal factors (PDF Γ), resummed by means of the DGLAP evolution equations.
The *particle*-level cross section is written as:

$$d\sigma_{e^+e^-}(p_{e^+}, p_{e^-}) = \sum_{ij=e^\pm,\gamma} \int dz_+ dz_- \, \Gamma_{i/e^+}(z_+, \mu^2, m^2) \, \Gamma_{j/e^-}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_{e^+}, z_- p_{e^-}, \mu^2) + \mathcal{O}\left(\frac{m}{E}\right) \quad (1)$$

where the short-distance *parton*-level cross section $d\hat{\sigma}_{ij}$ is calculated with massless momenta, and the collinear divergences are subtracted in a chosen factorisation scheme at the scale μ^2 (here we will work in $\overline{\mathrm{MS}}$).

Eq. (1) as $\begin{cases} \text{definition of } d\hat{\sigma}_{ij}\text{, given } \Gamma_{i/k} \\ \text{definition of } \Gamma_{i/k}\text{, after having computed } d\sigma_{e^+e^-} \text{ and } d\hat{\sigma}_{ij} \end{cases}$

Initial conditions for the PDFs of the electron

By explicit computation at NLO [Frixione (2019)], one founds Γ up to $\mathcal{O}(\alpha)$:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

This result is interpreted as the initial condition at a scale $\mu_0 \sim m$.

$$\Gamma_i^{[0]}(z,\mu_0^2) = \delta_{ie^-}\delta(1-z)$$

$$\Gamma_{e^-}^{[1]}(z,\mu_0^2) = \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z)(= 0 \text{ in } \overline{\text{MS}})$$

$$\Gamma_{\gamma}^{[1]}(z,\mu_0^2) = \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z)(= 0 \text{ in } \overline{\text{MS}})$$

$$\Gamma_{e^+}^{[1]}(z,\mu_0^2) = 0$$

Electron PDF, apart from factors, is the same as the initial condition for b-quark fragmentation function obtained in [Mele and Nason (1991)].

Outline

Evolution operator formalism

DGLAP evolution equation in z-space:

$$\frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}[\mathbb{P} \otimes \Gamma](z,\mu^2) \tag{2}$$

We will work in Mellin space, by defining the Mellin transform as:

$$M[f] \equiv f_N = \int_0^1 dz\, z^{N-1} f(z) \tag{3}$$

Convenient to introduce $\Gamma_{\rm S} = \Gamma_{e^-} + \Gamma_{e^+}$ and $\Gamma_{\rm NS} = \Gamma_{e^-} - \Gamma_{e^+}$ and solve a linear and a matrix equation:

$$\frac{\partial}{\partial \log \mu^2} \begin{pmatrix} \Gamma_{\rm S} \\ \Gamma_\gamma \end{pmatrix} = \frac{\alpha}{2\pi} \mathbb{P}_{\rm S} \otimes \begin{pmatrix} \Gamma_{\rm S} \\ \Gamma_\gamma \end{pmatrix} \tag{4}$$

$$\frac{\partial \Gamma_{\rm NS}}{\partial \log \mu^2} = \frac{\alpha}{2\pi} P_{\rm NS} \otimes \Gamma_{\rm NS} \tag{5}$$

By introducing the evolution operator:

$$\Gamma_N(\mu^2) = \mathbb{E}_N(\mu^2, \mu_0^2)\,\Gamma_{0,N}, \quad \mathbb{E}_N(\mu_0^2, \mu_0^2) = \mathbb{I} \tag{6}$$

Since α is running, we introduce a variable t as usually done in QCD [Furmanski and Petronzio (1982)], defined as:

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha(\mu)}{2\pi} \log \frac{\mu^2}{\mu_0^2} + \mathcal{O}(\alpha^2) \quad \rightarrow \quad \alpha(\mu) = \alpha(\mu_0) e^{2\pi b_0 t} \tag{7}$$

i.e. LL terms: t^k and NLL terms: $(\alpha/(2\pi))\, t^k$.
We then obtain a differential equation in t for the evolution operator $\mathbb{E}$:

$$\frac{\partial \mathbb{E}_N(t)}{\partial t} = \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi}\left(\mathbb{P}_N^{[1]} - \frac{2\pi b_1}{b_0}\mathbb{P}_N^{[0]}\right)\right]\mathbb{E}_N(t) + \mathcal{O}(\alpha^2) \tag{8}$$

- NS: eq. (8) be solved analytically.
- S/γ: a closed analytic form with α running does not exist $\rightarrow$ usually solved by means of numerical techniques.

Outline

Analytical solutions

Why?

- better understand of the details of the QED collinear dynamics
- because of the rapid growth of the electron PDF at $z \to 1$:
  - at LO, the electron PDF is equal to $\delta(1-z)$ and the LL evolution does smooth its behaviour, but the PDF remains very peaked towards $z = 1$.
  - with a NLO initial condition and a NLL evolution, the $z \to 1$ behavior in the $\overline{\text{MS}}$ scheme is actually worse (as we will show)

$\to$ crucial in the context of numerical computations

How?

- solving the evolution equations order by order in perturbation theory $\to$ recursive solution (calculated up to $\mathcal{O}(\alpha^3)$)
- using the large-N behaviour of the evolution operator in Mellin space $\to$ asymptotic large-z solution (all order in α)
- combining the two to obtain predictions in the whole z range.

Recursive solutions

Approach already known at LL (see e.g. [Skrzypek and Jadach (1991), Cacciari et al. (1992)]), we extended it at NLL with α running.
Starting point: rewriting the evolution equation eq. (2) in an integral form:

$$\frac{\partial \mathcal{F}(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}[\mathbb{P}\,\overline{\otimes}\,\mathcal{F}](z,\mu^2) \tag{9}$$

where:

$$\mathcal{F}(z,\mu^2) = \int_z^1 dy\,\Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,\mu^2) \tag{10}$$

and

$$g\,\overline{\otimes}_z h = \int_z^1 dx\, g(x)\, h\left(\frac{z}{x}\right) \tag{11}$$

$\mathcal{F}$ represented as a power series:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty}\left(\frac{t^k}{k!}\mathcal{J}_k^{\text{LL}}(z) + \frac{\alpha(t)}{2\pi}\frac{t^k}{k!}\mathcal{J}_k^{\text{NLL}}(z)\right) \tag{12}$$

By replacing eq. (12) in eq. (9), we find the following recurrence relations:

$$
\begin{aligned}
\mathcal{J}_k^{\text{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\text{LL}} \\
\mathcal{J}_k^{\text{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{J}_0^{\text{NLL}} \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \left(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{LL}} - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{LL}} \right)
\end{aligned}
$$

with the $\mathcal{J}_0^{\text{LL}}$ and $\mathcal{J}_0^{\text{NLL}}$ terms related to the integral of the initial conditions. The recursive solutions is then:

$$
\Gamma(z, \mu^2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(J_k^{\text{LL}}(z) + \frac{\alpha(t)}{2\pi} J_k^{\text{NLL}}(z) \right) \tag{13}
$$

with

$$
J_k^{\text{LL}}(z) = -\frac{d}{dz} \mathcal{J}_k^{\text{LL}}(z) \,, \qquad J_k^{\text{NLL}}(z) = -\frac{d}{dz} \mathcal{J}_k^{\text{NLL}}(z) \,. \tag{14}
$$

We calculated J_k^{LL} up to $k = 3$ and J_k^{NLL} up to $k = 2$ for three electron PDFs.

Non-singlet recursive solution up to $\mathcal{O}(\alpha^2)$: LL + NLL terms

$$\Gamma(z,t) = \frac{\alpha(t)}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-x) - 1\right)\right] + t\left(\frac{1+z^2}{1-z}\right)$$

$$+ t\frac{\alpha(t)}{2\pi}\Bigg[-12\frac{\log^2(1-z)}{1-z} + 12\log^2(1-z)$$

$$+ (8\pi b_0 + 8L_0 - 14)\frac{\log(1-z)}{1-z} + (-8\pi b_0 - 8L_0 + 10)\log(1-z)$$

$$+ \frac{1}{1-z}\left(-\frac{4\pi b_1}{b_0} + (6-4\pi b_0)L_0 + 4\pi b_0 - \frac{20N_F}{9} + \frac{4\pi^2}{3} + 1\right)$$

$$+ \frac{4\pi b_1}{b_0} + (4\pi b_0 - 2)L_0 - 4\pi b_0 + \frac{32N_F}{9} - \frac{4\pi^2}{3} - 2 + \hat{J}_1^{\mathrm{NLL}}(z)\Bigg]$$

$$+ \frac{t^2}{2}\left[\frac{1}{1-z}\left(4\left(z^2+1\right)\log(1-z) + (z+4)z - (1+3z^2)\log z + 1\right)\right] + \mathcal{O}(\alpha^3)$$

where $L_0 = \log\frac{\mu_0^2}{m^2}$, $\hat{J}_1^{\mathrm{NLL}}(z)$ are terms vanishing in the $z \to 1$ limit.

Full non-singlet recursive solution up to $\mathcal{O}(\alpha^3)$

$$\hat{J}^{\rm LL}_{\rm NS,0} = 1 - z$$

$$\hat{J}^{\rm LL}_{\rm NS,1} = \frac{1}{z-1} \times \Big(3z^2\log(z) + (1-z)(z+3) - 4(z-1)^2\log(1-z) + \log(z)\Big)$$

$$\hat{J}^{\rm LL}_{\rm NS,2} = \frac{1}{4(z-1)} \times \Big(-24\big(z^2-1\big)\mathrm{Li}_2(z) + 24\big(-z^2+2\big(z^2+1\big)\log(z)+3\big)\log(1-z) - 2\log(z)\big(-9z^2+7z^2\log(z)+$$
$$\log(z)-3\big) + (z-1)\big(4\pi^2(3z-1)-3(5z+19)\big) - 48(z-1)^2\log^2(1-z) - 48z(\log(2-2z)-\log(2z))\Big)$$

$$\hat{J}^{\rm NLL}_{\rm NS,0} = (1-z)(L_0 - 2\log(1-z) - 1)$$

$$\hat{J}^{\rm NLL}_{\rm NS,1} = \frac{1}{18b_0(z-1)(z+1)} \times \Big(36\pi b_0^2L_0z^3 - 36\pi b_0^2L_0z^2 - 36\pi b_0^2L_0z + 36\pi b_0^2L_0 - 36\pi b_0^2z^3 - 72\pi b_0^2z^3\log(1-$$
$$z) + 36\pi b_0^2z^2 + 72\pi b_0^2z^2\log(1-z) + 36\pi b_0^2z + 72\pi b_0^2z\log(1-z) - 72\pi b_0^2\log(1-z) - 36\pi b_0^2 - 18b_0L_0z^3 -$$
$$72b_0L_0z^3\log(1-z) + 54b_0L_0z^3\log(z) - 54b_0L_0z^2 + 72b_0L_0z^2\log(1-z) + 54b_0L_0z^2\log(z) + 18b_0L_0z +$$
$$72b_0L_0z\log(1-z) + 18b_0L_0z\log(z) - 72b_0L_0\log(1-z) + 18b_0L_0\log(z) + 54b_0L_0 + 44b_0N_Fz^3 +$$
$$12b_0N_Fz^3\log(z) - 68b_0N_Fz^2 + 12b_0N_Fz^2\log(z) - 44b_0N_Fz + 12b_0N_Fz\log(z) + 12b_0N_F\log(z) +$$
$$68b_0N_F + 72b_0\big(z^3-z^2-2z+2\big)\mathrm{Li}_2(-z) + 216b_0z\mathrm{Li}_2\Big(\frac{1}{z+1}\Big) - 36b_0(z-1)(z+1)^2\mathrm{Li}_2(1-z) -$$
$$216b_0\mathrm{Li}_2\Big(\frac{1}{z+1}\Big) + 153b_0z^3 - 6\pi^2b_0z^3 + 108b_0z^3\log^2(1-z) - 27b_0z^3\log^2(z) + 54b_0z^3\log(1-z) -$$
$$90b_0z^3\log(z) - 72b_0z^3\log(1-z)\log(z) + 72b_0z^3\log(z)\log(z+1) - 135b_0z^2 + 6\pi^2b_0z^2 - 108b_0z^2\log^2(1-$$
$$z) + 9b_0z^2\log^2(z) + 18b_0z^2\log(1-z) - 90b_0z^2\log(z) - 72b_0z^2\log(1-z)\log(z) - 72b_0z^2\log(z)\log(z+$$
$$1) - 153b_0z - 18\pi^2b_0z - 108b_0z\log^2(1-z) - 9b_0z\log^2(z) + 108b_0z\log^2(z+1) + 108b_0\log^2(1-z) +$$
$$27b_0\log^2(z) - 108b_0\log^2(z+1) - 54b_0z\log(1-z) + 72b_0z\log(z) - 144b_0z\log(z)\log(z+1) - 18b_0\log(1-$$
$$z) + 72b_0\log(z) + 144b_0\log(z)\log(z+1) + 135b_0 + 18\pi^2b_0 + 36\pi b_1z^3 - 36\pi b_1z^2 - 36\pi b_1z + 36\pi b_1\Big)$$

$$\hat{J}^{\rm NLL}_{\rm NS,2} = \frac{1}{36b_0z\big(z^2-1\big)} \times \Big((1-z)z\big(144(L_0-1)\pi^2\big(z^2-1\big)b_0^3 - 8\pi\big(18L_0(z+1)(z+3) - 2N_F(z+1)(11z-$$
$$17)+3\big(3\big(-8+\pi^2\big)z^2-15z+\pi^2+9\big)\big)b_0^2+\big(2\big(-2N_F\big(4\pi^2(z-1)-19z+63\big)(z+1)+9(57z-73)(z+$$
$$1)+6\pi^2\big(z(17z+3)+12b_1\big(z^2-1\big)-2\big)\big)-9L_0(z+1)\big(8\pi^2(z-1)-3(5z+19)\big)\big)b_0-144b_1\pi(z+1)(z+$$
$$3)\big) + 2\big(6b_0\log^3(z)z^4 - 18b_0\log^2(z)z^4 - 63b_0L_0\log^2(z)z^4 - 36b_0N_F\log^2(z)z^4 + 54b_0^2\pi\log^2(z)z^4 -$$
$$216b_0^2\pi(\log(1-z)-\log(z))z^4 - 342b_0\log(z)z^4 + 81b_0L_0\log(z)z^4 - 156b_0N_F\log(z)z^4 +$$
$$72b_0\log^2(2)\log(z)z^4 - 72b_0\log(2)\log(z)z^4 - 18b_0\pi^2\log(z)z^4 - 216b_1\pi\log(z)z^4 - 216b_0^2L_0\pi\log(z)z^4 -$$
$$24b_0^2N_F\pi\log(z)z^4 + 18b_0\log^3(z)z^3 + 144b_0\log^2(z)z^3 - 63b_0L_0\log^2(z)z^3 - 36b_0N_F\log^2(z)z^3 -$$
$$18b_0^2\pi\log^2(z)z^3 + 234b_0\log(z)z^3 + 297b_0L_0\log(z)z^3 - 12b_0N_F\log(z)z^3 - 72b_0\log^2(2)\log(z)z^3 +$$
$$144b_0\log(2)\log(z)z^3 + 42b_0\pi^2\log(z)z^3 + 288b_0^2\pi\log(z)z^3 - 216b_1\pi\log(z)z^3 - 216b_0^2L_0\pi\log(z)z^3 -$$
$$24b_0^2N_F\pi\log(z)z^3 - 6b_0\log^3(z)z^2 + 54b_0\log^2(z)z^2 - 9b_0L_0\log^2(z)z^2 - 12b_0N_F\log^2(z)z^2 +$$
$$18b_0^2\pi\log^2(z)z^2 + 306b_0\log(z)z^2 + 243b_0L_0\log(z)z^2 + 68b_0N_F\log(z)z^2 - 72b_0\log^2(2)\log(z)z^2 +$$
$$288b_0\log(2)\log(z)z^2 + 66b_0\pi^2\log(z)z^2 - 36b_0^2\pi\log(z)z^2 - 72b_1\pi\log(z)z^2 - 72b_0^2L_0\pi\log(z)z^2 -$$
$$24b_0^2N_F\pi\log(z)z^2+288b_0(z-1)^2(z+1)\log^3(1-z)z-18b_0\log^3(z)z-48b_0(z-4)(z-1)(z+4)\log^3(z+1)z-$$
$$144b_0\log^2(z)z - 9b_0L_0\log^2(z)z - 12b_0N_F\log^2(z)z - 54b_0^2\pi\log^2(z)z - 270b_0\log(z)z + 27b_0L_0\log(z)z -$$
$$76b_0N_F\log(z)z + 72b_0\log^2(2)\log(z)z - 144b_0\log(2)\log(z)z - 42b_0\pi^2\log(z)z - 108b_0^2\pi\log(z)z -$$
$$72b_1\pi\log(z)z-72b_0^2L_0\pi\log(z)z-24b_0^2N_F\pi\log(z)z-36b_0(z+1)\log^2(1-z)\big((6L_0(z-1)+12b_0\pi(z-1)-$$
$$5z-7)(z-1)+\big(11z^2+1\big)\log(z)\big)z+\log(1-z)\big(144\pi^2(z-1)^2(z+1)b_0^3+72\pi\big(4L_0(z+1)(z-1)^2-3z(z-1)+$$
$$3\big)b_0^2-(z-1)\big(108L_0(z+1)(z+3)+32N_F((3-8z)z+11)+3\big(-27z(7z+4)+4\pi^2\big(9z^2+z-6\big)+120\log^2(2)+$$
$$81\big)\big)b_0+12\big(3(z-1)(z(2z+5)+1)\log^2(z)+(z+1)\big((30\pi b_0+27L_0+4N_F-30)z^2-24z+9L_0+4N_F+$$
$$6b_0\pi+27\big)\log(z)+24(z-1)(2\log(2)-\log(z))\log(z+1)\big)b_0+288b_1\pi(z-1)^2(z+1)\big)z-108b_0\log^2(z)-$$
$$18b_0\big(-16\log(2)z^4+8(\log(z)+2\log(2))z^3+4(15z-13)\log(z)z+8(z-1)(\log(1-z)+\log(z)+2\log(2))z+$$
$$(z-1)(z(12\pi b_0+z(14z+29)-23)-12)\big)\log^2(z+1)+12b_0\log(2)\big(-4\pi^2z\big(z^3+2z-1\big)+(z-1)(z+$$
$$1)(z(9z+7)+3)\log(8)+2z\big(z\big(4z^2+z+16\big)-11\big)\log^2(2)\big)-216b_0\log(2)\log(z)-12b_0\big(6z(z+1)(z(2z-$$
$$3)+2)\log^2(z)+6(z-1)z\big(-4z+2b_0\pi\big(z^2-2\big)-3\big)\log(z)+(z-1)\big(12z\log^2(2)\big(z^2+4\big)+\pi^2\big(z^3+z\big)+$$
$$6\big(z\big((-1+2\log(2))z^2+z+5\big)-2z(\log(1-z)+\log(2))+3\big)(\log(z)+\log(2))\big)\big)\log(z+1)\big)+18b_0\big(-$$
$$16z(z+1)\mathrm{Li}_3\Big(\frac{1-z}{2}\Big)(z-1)^2+16z(z+1)\mathrm{Li}_3\Big(\frac{z-1}{2z}\Big)(z-1)^2-56z(z+1)\mathrm{Li}_3\Big(\frac{z}{z+1}\Big)(z-1)^2+16z(z+$$
$$1)\mathrm{Li}_3\Big(\frac{2z}{z+1}\Big)(z-1)^2-16z\big((\log(z+1)-\log(1-z))z^2+\log(1-z)-\log(z+1)-5\log(2)\big)\mathrm{Li}_2\Big(\frac{1-z}{2}\Big)(z-$$
$$1)+8z(z+1)(7z+4)\mathrm{Li}_3(1-z)(z-1)+4z(z+1)(7z+5)\mathrm{Li}_3(z)(z-1)-16z\big(z^2+4\big)\mathrm{Li}_3\Big(\frac{z+1}{2}\Big)(z-1)+$$
$$4z(z+1)\big((4\pi b_0+3L_0-8)z^2+2(z(-8\log(1-z)+\log(z)+2\log(z+1))-4(\log(z+1)-\log(1-z)))z-3L_0+$$
$$\log\big((z-1)^8\big)-2\log(z)+4\log(z+1)-4b_0\pi+11\big)\mathrm{Li}_2(1-z)+8\big(z^2-1\big)(z(2\log(2)(z-1)-z+2)-2(z-$$
$$1)z\log(1-z)+3)\mathrm{Li}_2\Big(\frac{z-1}{2z}\Big)-4\big((z-1)\big(z\big(z(15z-1)+4b_0\pi\big(z^2-2\big)-20\big)-6\big)+4z\big(\log(z)z^3+\log(1-$$
$$z)z-\log(1-z)+\log(z)+(z(z+6)-5)\log(z+1)\big)\big)\mathrm{Li}_2(-z)+8z\big((z-1)\big(4z^2-2z+6\log(1-z)-6b_0\pi-$$
$$3\big)+2\big(z^3+5z-4\big)\log(z+1)\big)\mathrm{Li}_2\Big(\frac{1}{z+1}\Big)-4\big(z^2-1\big)(z(5z-8)+10(z-1)z\log(z+1)-6)\mathrm{Li}_2\Big(\frac{z}{z+1}\Big)+$$
$$8\big(z^2-1\big)(z(2\log(2)(z-1)-z+2)-2(z-1)z\log(z+1)+3)\mathrm{Li}_2\Big(\frac{2}{z+1}-1\Big)-8z^2\big(z^2-1\big)\mathrm{Li}_3\Big(\frac{z-1}{z}\Big)-$$
$$8z(z+1)(z(7z-16)+7)\mathrm{Li}_3(-z)+16z\big(-3z^3+z^2+z-3\big)\mathrm{Li}_3\Big(\frac{1}{z+1}\Big)+z(z(z(61z-69)+59)+29)\zeta(3)\big)\Big)$$

Similar results for the singlet and the photon PDFs

Asymptotic solutions

Key fact: the large-z region corresponds to the large-N region in the Mellin space. Then:

- calculation of E_N in the large-N region;
- analytical Mellin inverse transform: $\Gamma(z,\mu^2) = M^{-1}[E_N\,\Gamma_{0,N}]$.

LL solution for non-singlet [Gribov and Lipatov (1972)]

$$P_N^{[0]} \overset{N\to\infty}{\longrightarrow} -2\log\bar{N} + 2\lambda_0\,, \quad \bar{N} = N\,e^{\gamma_{\mathrm{E}}}, \quad \lambda_0 = \frac{3}{4} \tag{15}$$

$$\Gamma^{\mathrm{LL}}(z,\mu^2) = \frac{e^{-\gamma_{\mathrm{E}}\eta_0}\,e^{\lambda_0\eta_0}}{\Gamma(1+\eta_0)}\,\eta_0(1-z)^{-1+\eta_0}, \quad \eta_0 = \frac{\alpha}{\pi}\log\frac{\mu^2}{\mu_0^2} \tag{16}$$

We are resumming the $\log(1-z)/(1-z)$ divergent terms to all order in α
N.B. α is supposed as fixed here (since at LL we are entitled to neglect it)

NLL solution for non-singlet

Convenient to perform the convolution with initial condition in the z-space:

$$\Gamma^{\rm NLL}(z,\mu^2) = \left(\delta(1-x) + \frac{\alpha(\mu_0^2)}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-x) - 1\right)\right]_+\right) \otimes_z \; M^{-1}\big[\exp\big(\log E_N\big)\big] \tag{17}$$

By exploiting:

$$P_N^{[1]} \overset{N\to\infty}{\longrightarrow} \frac{20}{9} N_F \log\bar{N} + \lambda_1\,, \quad \lambda_1 = \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{N_F}{18}(3+4\pi^2) \tag{18}$$

one obtains:

$$M^{-1}\big[\exp\big(\log E_N\big)\big] = \frac{e^{-\gamma_{\rm E}\xi_1}e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\,\xi_1(1-z)^{-1+\xi_1} \tag{19}$$

Same structure of the LL result, with $\xi_1 = 2t + \mathcal{O}(\alpha^2)$ and $\hat{\xi}_1 = \frac{3}{2}t + \mathcal{O}(\alpha^2)$.

After having performed the convolution we obtain:

$$\Gamma^{\rm NLL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \Bigg\{ 1 + \frac{\alpha(\mu_0)}{\pi} \Bigg[\left(\log\frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \Bigg] \Bigg\} \tag{20}$$

with:

$$A(\xi_1) = \frac{1}{\xi_1} + \mathcal{O}(\xi_1)\,, \quad B(\xi_1) = -\frac{\pi^2}{6} + 2\zeta_3\xi_1 + \mathcal{O}(\xi_1^2) \tag{21}$$

- NLL still very peaked towards $z = 1$, with behavior worse than LL
- if $\mu_0 \simeq m_e$ and $\mu \simeq 100$ GeV, then $\xi_1 \simeq 0.05$
  $\rightarrow$ the $\log(1-z)$ term is much larger than the $\log^2(1-z)$ one, even for z values *extremely* close to one.

Singlet and photon cases

Dominant term of the splitting matrices in the large-N region are:

$$\mathbb{P}_{\mathrm{S},N} \xrightarrow{N\to\infty} \begin{pmatrix} -2\log\bar{N} + 2\lambda_0 & 0 \\ 0 & -\frac{2}{3}N_F \end{pmatrix} + \frac{\alpha}{2\pi}\begin{pmatrix} \frac{20}{9}N_F\log\bar{N} + \lambda_1 & 0 \\ 0 & -N_F \end{pmatrix} + \mathcal{O}(\alpha^2) \tag{22}$$

This is a diagonal matrix → independent evolution

- Singlet solution = non-singlet solution (i.e. the mixing with the photon does not affect the electron large-z behaviour)
- Photon solution:

$$\Gamma_\gamma(z,\mu^2) = \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\frac{\alpha(\mu_0)}{2\pi}\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right)\right] \tag{23}$$

Unfortunately, eq. (23) does not work . . . here mixing is important!

Improvement of photon large-z PDF

Solving the evolution equations by including off-diagonal elements implies a significant increase in complexity. Main idea: solve the matrix differential equation by treating the off-diagonal subdominant terms ($\mathcal{O}(1/N)$) as a "perturbation" of the "LO" diagonal result ($\log \bar{N}$ and constants):

$$\mathbb{E}_N(t) = \mathbb{E}_N^{(0)}(t)\, \mathbb{E}_N^{(1)}(t).$$

Then convolve with initial conditions and perform the Mellin anti-trasform. The final result is rather involved, but dominant terms in the $z \to 1$ limit are:

$$\Gamma_\gamma(z) \overset{z\to 1}{\longrightarrow} \frac{\alpha(\mu_0)}{\alpha(\mu)} \left[\left(\frac{\alpha(\mu_0)}{2\pi} \right) \frac{3}{\xi_{1,0}} \log(1-z) - \left(\frac{\alpha(\mu_0)}{2\pi} \right)^2 \frac{1}{2\,\xi_{1,0}} \log^3(1-z) \right]$$

where $\xi_{1,0} = 2 + \mathcal{O}(\alpha)$. Formally dominant term suppressed w.r.t the subdominant one by a factor proportional to α.

Matching

Combine the recursive and the asymptotic solution by means of an additive formula:

$$\Gamma_{\rm mtc}(z) = \Gamma_{\rm rec}(z) + \Big(\Gamma_{\rm asy}(z) - \Gamma_{\rm subt}(z)\Big) G(z)\,, \quad \lim_{z\to 1} G(z) = 1 \tag{24}$$

Choice of subtraction term $\Gamma_{\rm subt}$ and matching function G dictated by:

$$\Gamma_{\rm mtc} \sim \Gamma_{\rm asy} \quad z \simeq 1 \tag{25}$$

$$\Gamma_{\rm mtc} \sim \Gamma_{\rm rec} \quad \text{small- and intermediate-}z \tag{26}$$

After technical studies:

- $\Gamma_{\rm subt}$ chosen as $\mathcal{O}(\alpha^3)$ expansion of $\Gamma_{\rm asy}$
- different strategy for G:
  - NS/S: $G(z) \equiv 1$ ($\Gamma_{\rm asy}(z) - \Gamma_{\rm subt}(z)$ cancel very well in the small-z region)
  - γ: non trivial G needed ($\Gamma_{\rm asy}$ problematic in the small-z region)! $G(\hat{z}_0, \hat{z}_1, p)$ (transition between $\Gamma_{\rm rec}$ and $\Gamma_{\rm asy}$ in the region $\hat{z}_0 < -\log_{10}(1-z) < \hat{z}_1$ with p used to adjust the abruptness of the transition)

Outline

ePDF

Public numerical code, written in `C++`, available here:

`https://github.com/gstagnit/ePDF`

Well known techniques for solving DGLAP equation:

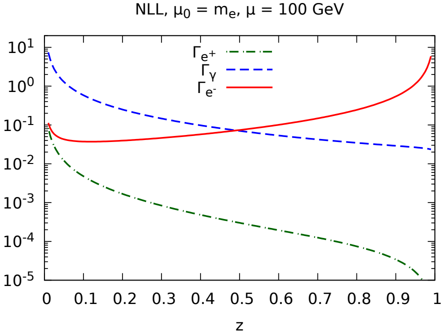
- Evolution equation solved in Mellin space by means of a discretised path-ordered product (see e.g. [Bonvini (2012)]) or adopting the U-matrix formalism (see e.g. [Vogt (2005)])
- Numerical inverse Mellin transform with an algorithm based on an optimized path in the complex plane (Talbot path)

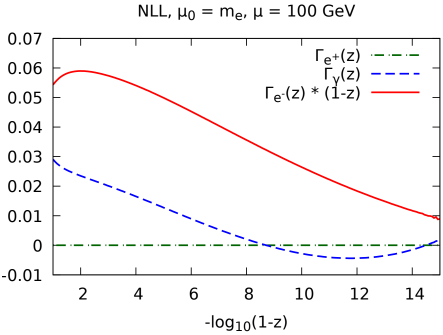
In the code you can also find:

- a routine for the evolution of α at NLL
- all the analytical solutions (recursive, asymptotic and matched)

Outline

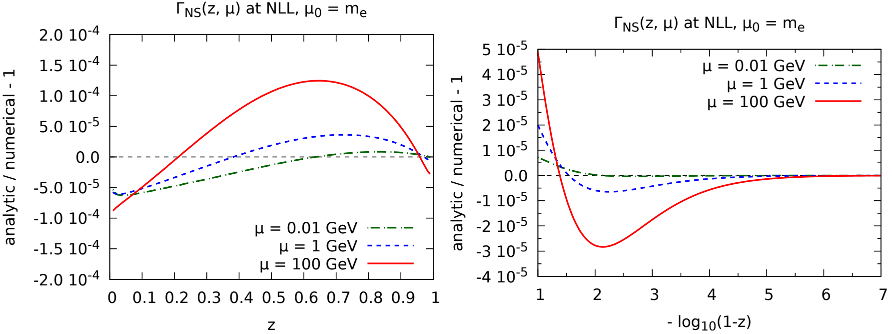
NLL electron PDFs





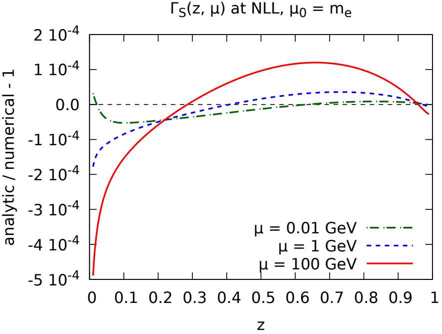
- Electron still dominates at large z, while photon at small z (however remember the constraint $z_+ z_- \geq M^2/s$)
- At large z, Γ_γ is smaller than Γ_{e^-} by $[-\log_{10}(1-z)]$ orders of magnitude.

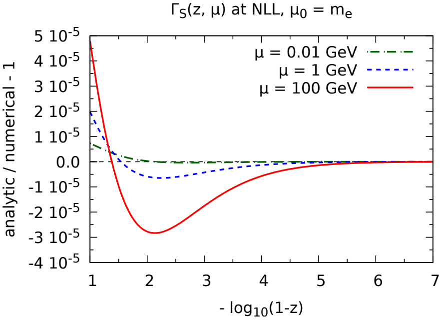
Analytical vs. numerical



- Worst-case scenario ($\mu = 100$ GeV): agreement at the $10^{-4} - 10^{-5}$ level
- On the linear scale, largest discrepancy at small z's for the singlet
- Photon problematic on the log scale, but small in absolute value

Analytical vs. numerical





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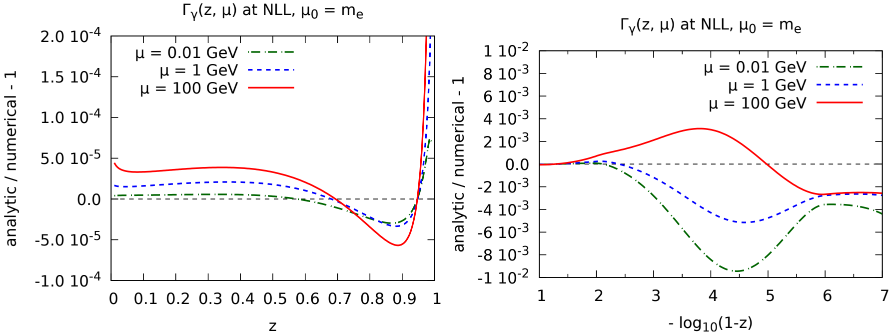

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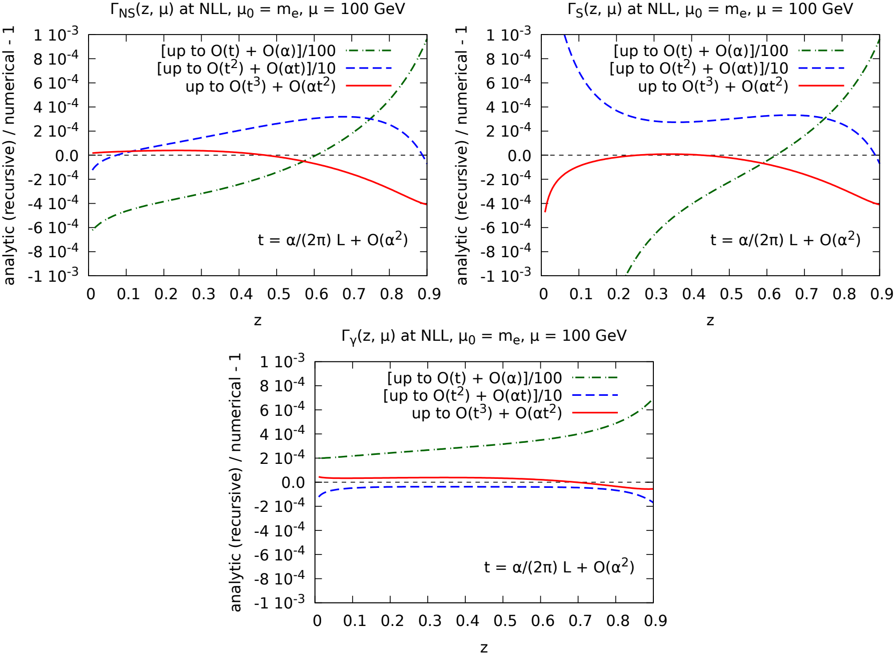
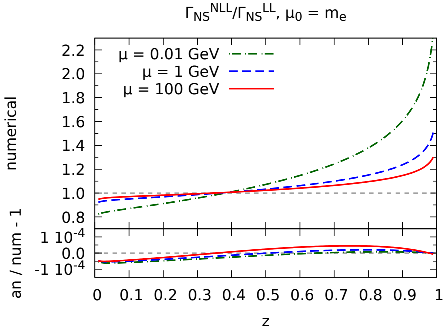
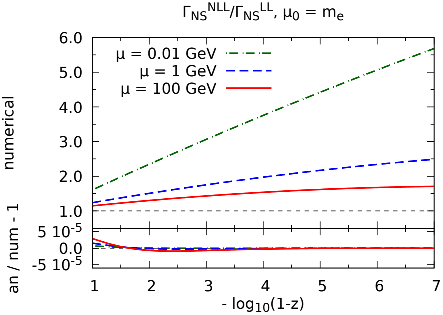

Γ_NS(z, μ) at NLL, μ_0 = m_e, μ = 100 GeV
Γ_S(z, μ) at NLL, μ_0 = m_e, μ = 100 GeV
Γ_γ(z, μ) at NLL, μ_0 = m_e, μ = 100 GeV
analytic (recursive) / numerical - 1
[up to O(t) + O(α)]/100
[up to O(t^2) + O(αt)]/10
up to O(t^3) + O(αt^2)
t = α/(2π) L + O(α^2)
z
1 10^-3
8 10^-4
6 10^-4
4 10^-4
2 10^-4
0.0
-2 10^-4
-4 10^-4
-6 10^-4
-8 10^-4
-1 10^-3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

NLL vs. LL

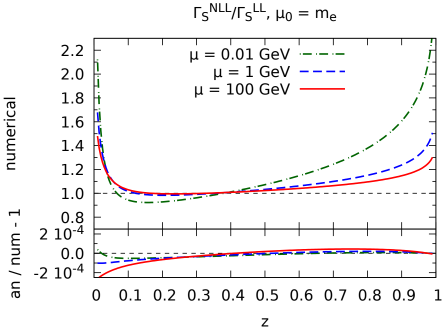


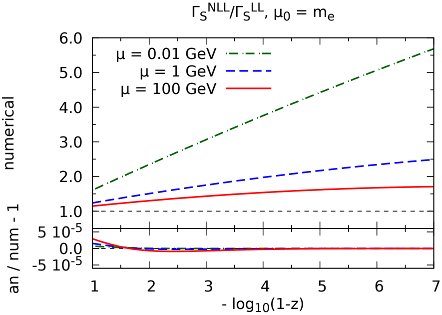


$$\Gamma^{\rm NLL}_{\rm NS/S}(z,\mu^2) \sim LL \left(1 + \frac{\alpha(\mu_0)}{\pi}\left[a + \frac{b}{\alpha(\mu)\log(\mu^2/\mu_0^2)}\log(1-z) - \log^2(1-z)\right]\right)$$

$$\text{Insets:} \quad \left(\frac{\rm PDF_{NLL}}{\rm PDF_{LL}}\right)_{\rm an} \Bigg/ \left(\frac{\rm PDF_{NLL}}{\rm PDF_{LL}}\right)_{\rm num} - 1$$

NLL vs. LL





$$\Gamma^{\rm NLL}_{\rm NS/S}(z,\mu^2) \sim LL\left(1+\frac{\alpha(\mu_0)}{\pi}\left[a+\frac{b}{\alpha(\mu)\log(\mu^2/\mu_0^2)}\log(1-z)-\log^2(1-z)\right]\right)$$

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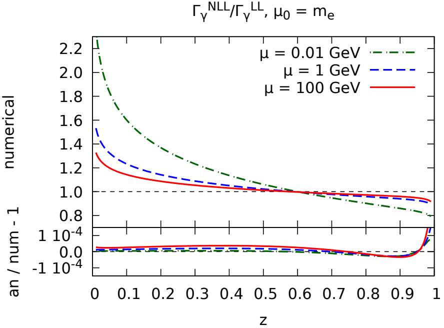
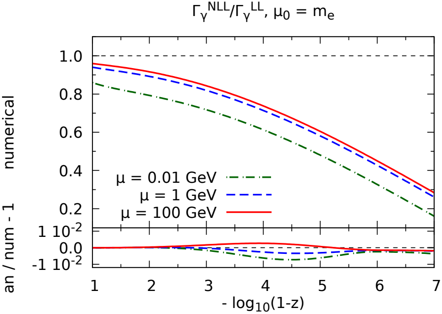
NLL vs. LL

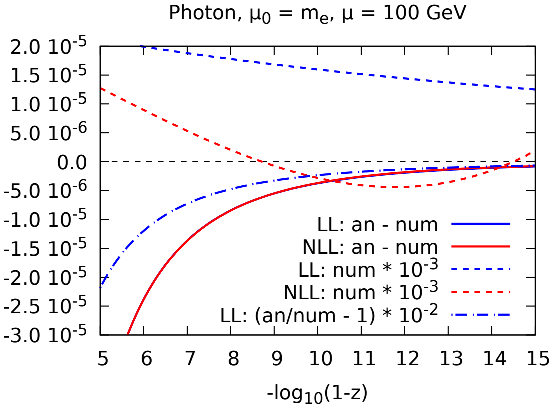




$$\text{Insets:} \quad \left(\frac{\text{PDF}_{\text{NLL}}}{\text{PDF}_{\text{LL}}}\right)_{\text{an}} \Bigg/ \left(\frac{\text{PDF}_{\text{NLL}}}{\text{PDF}_{\text{LL}}}\right)_{\text{num}} - 1$$

Asymptotic photon behaviour

$$\Gamma_\gamma(z) \xrightarrow{z\to1} \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\left(\frac{\alpha(\mu_0)}{2\pi}\right)\frac{3}{\xi_{1,0}}\log(1-z) - \left(\frac{\alpha(\mu_0)}{2\pi}\right)^2\frac{1}{2\,\xi_{1,0}}\log^3(1-z)\right]$$



Onset of the true asymptotic regime occurs at much larger z values!

Outline

Conclusions

- We computed the electron, positron and photon PDFs of the unpolarised e^- at NLL (and by charge conjugation also the ones of the incoming e^+).
- They are obtained by means of both numerical and analytical methods, which agree extremely well in the region relevant for phenomenology.
- Analytical results stem from an additive matching between a recursive solution and an asymptotic $z \to 1$ one.
- At NLL the large-z peak is even more pronunced than at LL $\to$ this is in part an artefact of the $\overline{\text{MS}}$ scheme and in future works will explore the adoption of alternative subtraction schemes
- QED PDFs are the first ingredients towards a full NLO framework for the computation of observables relevant to e^+e^- collider: phenomenological implications of NLL PDFs only after a convolution with the subtracted cross sections.