Numerical study of a dressed black hole solution in asymptotically AdS spacetime

Black Holes and Neutron Stars in Modified Gravity Workshop

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Context and objectives

Short introduction

- PhD student at LUTH since October 1st, supervised by Philippe Grandclément.
  $\Rightarrow$ About the Evolution Problem in Numerical Relativity.

- **Today:** About my 3-month internship at the end of my Master (April-June 2019).

Context

- **Black hole no-hair theorem**: a stationary black hole is uniquely described by its mass, angular momentum and electric charge.

- Break asymptotic flatness hypothesis
  $\Rightarrow$ **Asymptotically Anti-de Sitter** (AAdS) spacetime
  - Curved background (negative cosmological constant)
  - Stabilizes radiative systems (radiation reaches infinity in finite time)

Static dressed black hole

- Solve Einstein's equations with $\Lambda = -3/l^2$ and matter content obeying the Klein-Gordon equation.

- The matter consists in a minimally coupled scalar field with self-interacting potential:

$$V(\phi) = -\frac{3}{4\pi G l^2} \sinh^2\left(\sqrt{\frac{4\pi G}{3}}\,\phi\right)$$

Static dressed black hole

- **Analytic solution** given by Martínez *et al.* (2004) [hep-th/0406111]:

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2}\left[-\left(\frac{r^2}{l^2} - \left(1+\frac{G\mu}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{l^2} - \left(1+\frac{G\mu}{r}\right)^2\right)^{-1} dr^2 + r^2 d\sigma^2\right]$$

$$\phi = \sqrt{\frac{3}{4\pi G}}\,\mathrm{artanh}\left(\frac{G\mu}{r+G\mu}\right).$$

- $d\sigma^2$ is the line element of a surface of constant negative curvature.
- For $r \to \infty$ we recover AdS.
- One integration constant μ involved in the horizon radius and the intensity of the scalar field $\Rightarrow$ mass parameter.

Static dressed black hole

- If $\mu > -l/4G$, event horizon located at
$$r_+ = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right)$$
- The horizon has negative curvature.
- There is a 1-to-1 correspondence between r_+ and $\phi(r_+)$.
- Coordinate singularity on the horizon (Schwarzschild-like): not well suited for numerical purposes.

Goals of the internship

1. Obtain a numerical solution for the static black hole:
    - Parametrization of $d\sigma^2$,
    - Gauge and boundary conditions for regular coordinates.

2. Use it as starting configuration to simulate a rotating black hole.

Building the system of equations

Bulk equations

- 3+1 formalism: well-suited for a numerical approach.
  - 11 unknowns: N (lapse), B^i (shift), γ_{ij} (3D metric), ϕ (scalar field).
  - 11 equations: $H = 0$, $M^i = 0$, $E_{ij} = 0$, Klein-Gordon.

  $\Rightarrow$ 3+1 equations with Λ, projections of $T_{\mu\nu}$ and $\partial_t = 0$.

- Parametrization of the base manifold:

$$d\sigma^2 = \frac{1}{\cos^2\theta}(d\theta^2 + \sin^2\theta\ d\varphi^2), \qquad \theta \in [0, \pi/2), \varphi \in [0, 2\pi).$$

  $\Rightarrow$ This is the *whole* hyperbolic plane $\mathbb{H}^2$.
  - **Advantage:** Easily implemented within the Kadath library.
  - **Drawback:** Not a compact base manifold.

- 3-vector basis: $\left(\partial_r, \frac{\cos\theta}{r}\partial_\theta, \frac{\cos\theta}{r\sin\theta}\partial_\varphi\right)$ orthonormal for our reference metric $\overline{ds}^2 = dr^2 + r^2 d\sigma^2$.

Gauge fixing conditions

- Diffeomorphism invariance: we need to fix the gauge for the numerical resolution.

- **Maximal slicing condition**: $K = 0$
(extrinsic curvature $K_{ij} = -\mathcal{L}_{\boldsymbol{m}}\gamma_{ij}/2N$, $K = \gamma^{ij}K_{ij}$)

- **Spatial harmonic gauge**: $V^k = 0$ in a modified system where we replace $\mathcal{R}_{ij} \leftarrow \mathcal{R}_{ij} - D_{(i}V_{j)}$ and we define $V^k = \gamma^{ij}(\Gamma^k_{ij} - {}^{(AdS)}\Gamma^k_{ij})$
$\Rightarrow$ **Well-posed system** and we recover the solution of Einstein's equations in this gauge.
$\Rightarrow$ The ${}^{(AdS)}\Gamma^k_{ij}$ are fixed coordinate-dependent coefficients which act as gauge sources.

- Check a posteriori $K = 0$, $V = 0$ to assess numerical convergence.

Apparent horizon (Inner boundary conditions)

We follow the work done by Hugo Roussille at LUTH in 2018 for the simulation of a Kerr black hole.

- Apparent horizon *vs.* Event horizon (coincide for stationary systems)

- Non-expanding apparent horizon, shearless, at fixed $r = cst$:

$$\Theta = D_i \tilde{s}^i + K_{ij} \tilde{s}^i \tilde{s}^j = 0, \qquad B^i = N \tilde{s}^i + \omega (\partial_\varphi)^i,$$

with $\tilde{s}^i$ the unit normal to the horizon. In particular $\boxed{B^r = \sqrt{\gamma^{rr}} N}$.
$\Rightarrow$ **Everything about rotation is here:** ω parameter.

- 3 remaining degrees of freedom from the differential gauge:

$$N = 1/2, \qquad \gamma_{r\theta} = \gamma_{r\varphi} = 0.$$

Apparent horizon (Inner boundary conditions)

- The system of equations is degenerate on the boundary.
- 1D example: $E(x) \equiv a(x)y''(x) + b(x)y'(x) + c(x)y(x) + d(x) = 0$ for $x \in [0,1]$. If $a(0) = 0$, then $E(0)$ is 1st order and is its own boundary condition at $x = 0$.
- In our case, start from the 3+1 equations and write the matrix of the principal part normal to the horizon $\boldsymbol{A}\partial^2_{rr}\boldsymbol{u} + \cdots = 0$. $\boldsymbol{A}$ has three vanishing eigenvalues $\propto (B^r)^2 - \gamma^{rr}N^2$ with eigenvectors close to $\gamma_{\theta\theta}$, $\gamma_{\theta\varphi}$ and $\gamma_{\varphi\varphi}$, so we use:

$$E_{\theta\theta} = E_{\theta\varphi} = E_{\varphi\varphi} = 0$$

- **Klein-Gordon also degenerate:** we don't need to impose $\phi = \phi_0$.

$$\frac{1}{N} D_i \left(N \left(\gamma^{ij} - \frac{B^i B^j}{N^2} \right) D_j \phi \right) - \frac{dV}{d\phi} = 0$$

AdS border (Outer boundary conditions)

- Vanishing scalar field $\phi = 0$.
- AdS border: by definition, there exists $\Omega \to 0$ such that N^2, $\gamma_{ij} \propto \Omega^{-2}$.
  $\Rightarrow$ We need to **regularize** to deal with finite quantities.
- Define regularized quantities such as $\tilde{N} \equiv \Omega N$ and use the regularized equations in a domain reaching to the AdS border.
- Impose continuity of unknown fields and their normal derivatives.
- This process involves a change of variables to write the AdS metric as $ds^2_{AdS} = -N^2 dt^2 + f(r')(dr'^2 + r'^2 d\sigma^2)$ where $f \propto \Omega^{-2}$.

  $\Rightarrow$ This sets the AdS border at **finite radius**.

Numerical setting and resolution

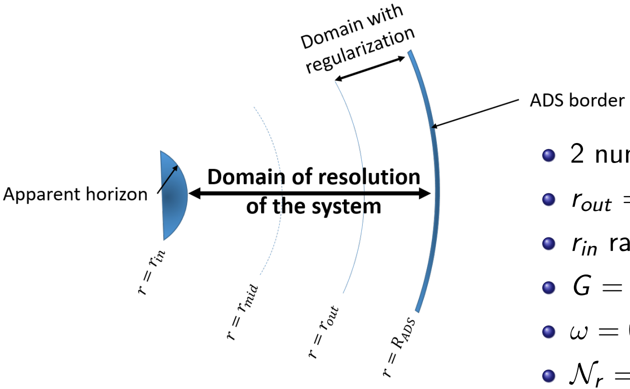
Spectral methods and the Kadath library

- **Spectral methods**: discrete approximation of a function.
  $\Rightarrow$ interpolation on a predefined basis of functions (e.g. Chebyshev polynomials, trigonometric functions).

- **Spectral convergence**: Very fast convergence to the true function (for smooth functions, faster than any polynomial).
  $\Rightarrow$ Good precision with a limited number of points.

- **Kadath library**: numerical code developed at LUTH which implements spectral methods and a Newton-Raphson scheme to solve non-linear PDEs.
  $\Rightarrow$ Very flexible in terms of geometry, equations to solve, designed with NR in mind, but no evolving systems yet (hence PhD).

Numerical setting

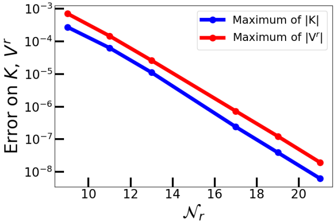


- 2 numerical domains,
- $r_{out} = 30,\ R_{AdS} = 40,$
- r_{in} ranging from 7 to 11,
- $G = 1,\ l = 20,$
- $\omega = 0$ (no rotation),
- $\mathcal{N}_r = 17,\ \mathcal{N}_\theta = 5,\ \mathcal{N}_\varphi = 4.$

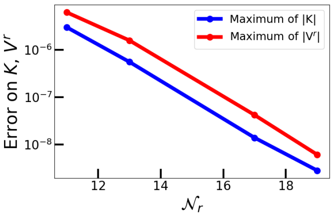
$\Rightarrow$ We vary slowly the horizon radius (0.2 step) and use the current solution as initial guess for the following one.

Tests and results

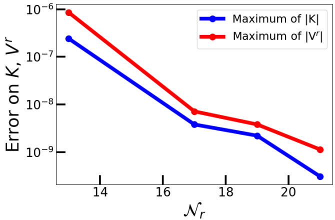
Convergence tests



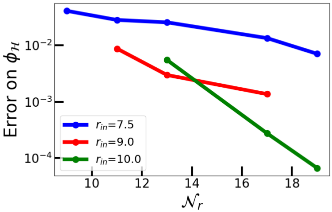
$r_{in} = 7.5$



$r_{in} = 9.0$

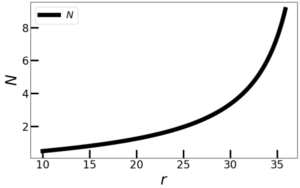


$r_{in} = 10.0$

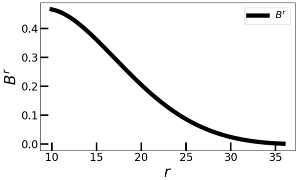


ϕ on horizon

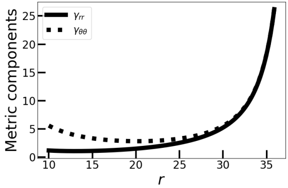
Static Black Hole



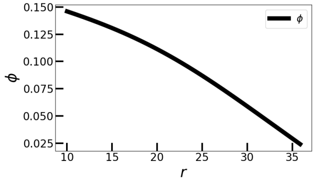
N



B^r

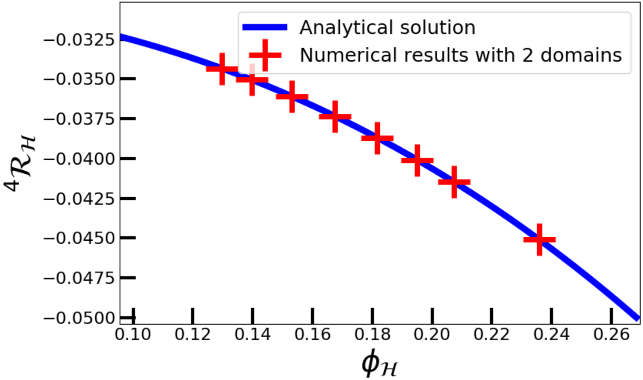


γ_{rr} and $\gamma_{\theta\theta}$



ϕ

Comparison with analytic solution



- Smaller radii correspond to more relativistic configurations.
- Analytic curve computed with SageManifolds.

Comparison with analytic solution

- Probably not a fully comprehensive test but still performs well.
- Global quantities?
  $\Rightarrow$ Can be tricky in asymptotically AdS
  $\Rightarrow$ Because the surface of $\mathbb{H}^2$ is infinite, the mass is infinite!

Rotating case and challenges

- Recall $B^\varphi = \omega \dfrac{r \sin\theta}{\cos\theta}$ on horizon: additional regularization required.
- More importantly, our base manifold is not compact!
  $\Rightarrow$ Need a parametrization of a compact surface without boundary and with constant negative scalar curvature.
- We haven't found such a parametrization, and should we get it, it would need to be implemented into Kadath.
- In this case, the boundary conditions on the horizon may be changed.

  $\Rightarrow$ **Suggestions are welcome!**

Conclusion

- We have obtained a numerical solution for the static black hole.
- To the extent of our tests, it is consistent with the analytic solution.
- Rotating case: need for a new parametrization of the base manifold, then probably redo the static solution and use further regularization.
- Seems quite involved and not the main purpose of the PhD: stand-by status.
- Suggestions and comments appreciated!

Analytic metric

$$S = \int d^4x\sqrt{-g}\left(\frac{\mathcal{R}+6l^{-2}}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \tag{1}$$

$$V(\phi) = -\frac{3}{4\pi G l^2}\sinh^2\sqrt{\frac{4\pi G}{3}}\phi. \tag{2}$$

$$G_{\mu\nu} - \frac{3}{l^2}g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3a}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{dV}{d\phi} = 0, \tag{3b}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V(\phi). \tag{4}$$

Analytic metric

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2}\left[-\left(\frac{r^2}{l^2}-\left(1+\frac{G\mu}{r}\right)^2\right)dt^2+\left(\frac{r^2}{l^2}-\left(1+\frac{G\mu}{r}\right)^2\right)^{-1}dr^2+r^2d\sigma^2\right] \tag{5a}$$

$$\phi = \sqrt{\frac{3}{4\pi G}}\,\mathrm{artanh}\left(\frac{G\mu}{r+G\mu}\right). \tag{5b}$$

$$d\sigma^2 = \frac{1}{\cos^2\theta}(d\theta^2+\sin^2\theta\ d\varphi^2). \tag{6}$$

$$ds^2_{AdS} = -\left(\frac{r^2}{l^2}-1\right)dt^2+\left(\frac{r^2}{l^2}-1\right)^{-1}dr^2+r^2d\sigma^2. \tag{7}$$

3+1 equations

$$m^\alpha = Nn^\alpha = (\partial_t)^\alpha - B^\alpha \tag{8}$$

$$\mathcal{L}_{\boldsymbol{m}} = \partial_t - \mathcal{L}_{\boldsymbol{B}} \tag{9}$$

$$\mathcal{R} + K^2 - K_{ij}K^{ij} - 2\Lambda = 16\pi G\rho, \tag{10}$$

$$D_k K_i^k - D_i K = 8\pi G p_i, \tag{11}$$

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = -D_i D_j N + N\left[\mathcal{R}_{ij} + KK_{ij} - 2K_{ik}K_j^k - \Lambda\gamma_{ij} - 8\pi G\left(S_{ij} - \frac{S-\rho}{2}\gamma_{ij}\right)\right]. \tag{12}$$

$$\frac{1}{N} D_i\left(N\left(\gamma^{ij} - \frac{B^i B^j}{N^2}\right) D_j\phi\right) - \frac{dV}{d\phi} = 0. \tag{13}$$

Reference metric

$$\overline{ds}^2 = dr^2 + r^2 d\sigma^2 = dr^2 + \frac{r^2}{\cos^2\theta}(d\theta^2 + \sin^2\theta d\varphi^2) \tag{14}$$

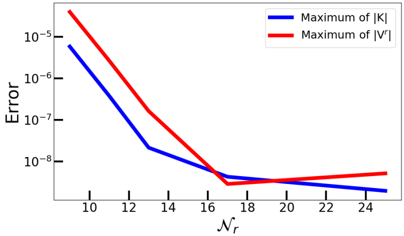
$$ds^2_{AdS} = -N^2 dt^2 + f(r')(dr'^2 + r'^2 d\sigma^2). \tag{15}$$

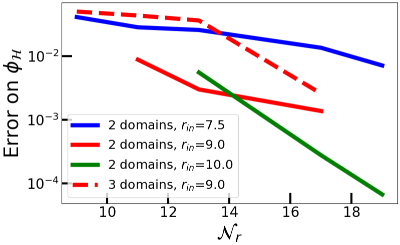
$$\begin{cases} r' &= R_{AdS} \exp\left[-\arctan\left(\frac{1}{\sqrt{\frac{r^2}{l^2} - 1}}\right)\right] \\ r &= \frac{l}{\sin \ln \frac{R_{AdS}}{r}} \end{cases} \tag{16}$$

$$f(r') = \frac{l^2}{r'^2 \sin^2 \ln \frac{R_{AdS}}{r'}}, \qquad N^2 = \frac{\cos^2 \ln R_{AdS}/r'}{\sin^2 \ln R_{AdS}/r'}. \tag{17}$$

$$\Omega \equiv \sin \ln \frac{R_{AdS}}{r'}. \tag{18}$$

Convergence tests with 3 domains





Static Black Hole with 3 domains