Well-posedness in modified theories of gravity

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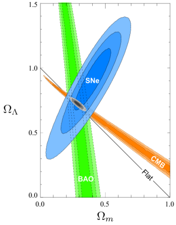
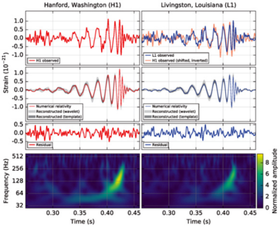
Modifying gravity: why?

- **Unexplained phenomena**: dark energy, dark matter, inflation related problems...
- **Theoretical shortcomings**: non-renormalizability of GR, unavoidable formation of singularities...
- **Brand-new observation channel with gravitational waves** $\rightarrow$ Model dependent tests





Well-posed or ill-posed?

System described by

- differential equation(s) determining its evolution
- initial data set

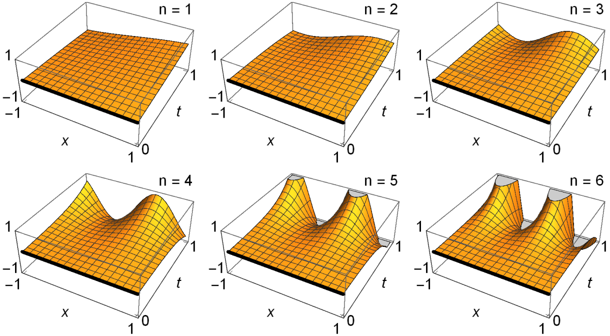
Well-posed Cauchy problem Hadamard 1902

The solution

1. exists (at least locally)
2. is unique
3. depends smoothly on the initial data

Laplace equation: $\partial_t^2\phi + \partial_x^2\phi = 0$

$$\phi_n(t,x) = \frac{e^{-\sqrt{n}}}{n}\sin(nx)\sinh(nt)$$



- $\phi_n(0,x) \underset{n\to\infty}{\to} 0,\ \partial_x\phi_n(0,x) \underset{n\to\infty}{\to} 0$
- **But** $\phi_n(t,x)$ blows up for $t \neq 0$ when $n \to \infty$

1. Hyperbolic PDEs
2. Horndeski theory: ill-posed?
3. Curing the ill-posedness

Second order PDEs

Linear PDE

$$\boldsymbol{P^{\mu\nu}}\partial_\mu\partial_\nu u + Q^\mu\partial_\mu u + Ru = 0$$

- μ, ν: spacetime indices (0-3 here)
- u: N-dimensional vector
- $P^{\mu\nu} = P^{(\mu\nu)}$, Q^μ and R: $N \times N$ matrices

- Spatial Fourier transform $\tilde{u}(t, \xi^i)$
- $\tilde{w} \equiv \left(\sqrt{1+\xi^2}\tilde{u}, -i\partial_t \tilde{u}\right)$

$$\rightarrow \tilde{w}(t, \xi^i) = e^{i\mathcal{M}(\xi^i)t}\tilde{w}(0, \xi^i)$$

Smooth dependence on initial data

$$||\tilde{w}||_{L^2}(t) < f(t)||\tilde{w}||_{L^2}(0) \Leftrightarrow ||e^{i\mathcal{M}(\xi^i)t}|| < f(t)$$

$$\Leftrightarrow ||e^{iM(\xi^i)t}|| < k$$

where M is the high-frequency part of $\mathcal{M}$

$$M(\xi^i) = \begin{bmatrix} 0 & \mathbb{1} \\ -A^{-1}C(\xi^i) & -A^{-1}B(\xi^i) \end{bmatrix}$$

with $A \equiv P^{00}$, $B(\xi^i) \equiv P^{0i}\xi^i$, $C(\xi^i) \equiv P^{ij}\xi^i\xi^j$

Weak hyperbolicity

- M must have only **real eigenvalues**
- Otherwise, $||e^{iM(\xi^i)t}||$ grows like $e^{\# t}$

$$\begin{bmatrix} \lambda_1 & 1 & & & & & & & \\ & \lambda_1 & 1 & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & 1 & & & & \\ & & & & \lambda_2 & & & & \\ & & & & & \lambda_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_n & 1 \\ & & & & & & & & \lambda_n \end{bmatrix}$$

Jordan decomposition

$M = S^{-1}JS$ where J is like on the right

Strong hyperbolicity

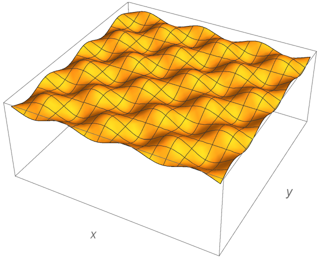
- M must have no non-trivial Jordan block
- $\Leftrightarrow$ M is diagonalizable
- Otherwise, $||e^{iM(\xi^i)t}||$ grows like t^p

Possible extensions

- Trivial extension to PDEs with non-constant coefficients $P^{\mu\nu}(t, x^i)$, ...
- Non-linear equations?

Non-linear PDEs

Small arbitrary deformation of the inital data set:



- Non-linearly well posed problem $\Rightarrow$ All linearizations around small deviations well-posed
- **Converse result** holds in general: if all arbitrary small deviations yield well-posed linearized problems, then well-posed non linear problem Kreiss & Lorenz '89

Gauge & constraints (1/2)

Example: electromagnetism in vacuum

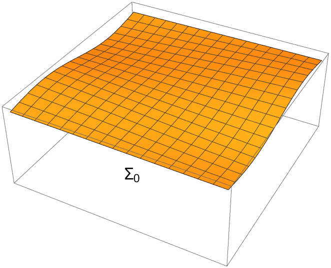
- $\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$: 4 equations and 4 unknowns
- **But** t component: $\nabla^2 A_0 - \vec{\nabla} \cdot (\partial_t \vec{A}) = 0$ ($\Leftrightarrow \vec{\nabla} \cdot \vec{E} = 0$)
  $\rightarrow$ Underdetermined evolution for A_μ
- Fine, because 1 function absorbed by gauge

Gauge & constraints (2/2)

Good gauge?

Start with arbitrary initial data $(u, \partial_t u)$ on Σ_0 and gauge condition $G = 0$.
Then check:

1. Possible to impose $G = 0$ on Σ_0?
2. Constraints $\Rightarrow \partial_t G = 0$ on Σ_0?
3. $G = 0$ propagated by evolution equations?



$+$ Of course, are evolution equations well-posed in this gauge?

Well-posedness of GR (1/3)

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$

(Linearized) harmonic gauge

$$H_\alpha \equiv \mathfrak{g}_{\alpha\beta}{}^{\mu\nu} \nabla^\beta h_{\mu\nu} = 0$$

$$\mathfrak{g}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{2}(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\nu_\alpha \delta^\mu_\beta - g_{\alpha\beta} g^{\mu\nu})$$

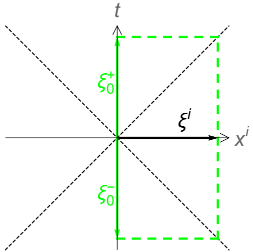
Good gauge in the above sense

Well-posedness of GR (2/3)

Einstein equations in harmonic gauge

$$P_{\alpha\beta}{}^{\rho\sigma\,\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} + \ldots = 0$$

- Principal symbol $P_{\alpha\beta}{}^{\rho\sigma\,\mu\nu} = \mathfrak{g}_{\alpha\beta}{}^{\rho\sigma} g^{\mu\nu}$
- $N = \dfrac{4(4+1)}{2} = 10$
- M has $2N$ eigenvectors: $v = (t^{(\alpha\beta)}, \xi_0^{\pm} t^{(\alpha\beta)})$; $Mv = \xi_0^{\pm} v$



$\rightarrow$ **GR well-posed in harmonic gauge**

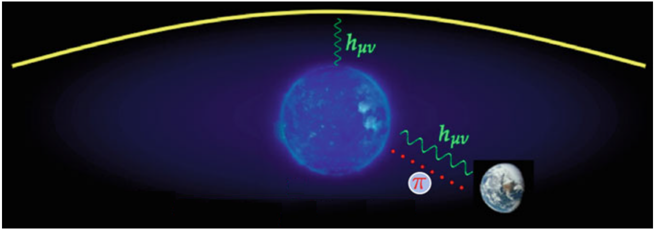
Well-posedness of GR (3/3)

NB

- Well-posedness **depends on gauge**; GR not well-posed in ADM formalism when lapse and shift gauge-fixed
- Stronger statements about well-posedness in GR

Modifying gravity: why with a scalar field?

- Simplest additional degree of freedom
- Many theories related in specific regimes



Modifying gravity with a scalar field: how?

Horndeski action

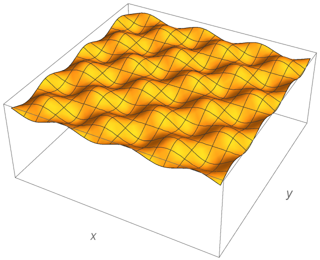
$$S = \int \sqrt{-g}\, d^4x\, (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)$$

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X) \\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
&\quad - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]
\end{aligned}$$

$$X = -\frac{1}{2}(\nabla\phi)^2$$

$\rightarrow$ **2^{nd} order field equations**

Horndeski theory: ill-posed?



Claims Papallo & Reall '17

- Horndeski models with $G_4(\phi, X)$ or $G_5(\phi, X)$ are **ill-posed** around a generic background in any **generalized harmonic gauge**
- Horndeski models with $G_{4X} = 0$ and $G_5 = 0$ are **well-posed** in the same context

Assumptions

Weak field background

- Technical conditions on the $G_{iX,\phi}$ (small non-Einstein terms)
- NB1: Weak field background **doesn't** imply well-posedness
- NB2: Highly symmetric backgrounds may still be well-posed

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$
$$\phi \to \phi + \psi$$

Generalized harmonic gauge

$$H_\alpha \equiv \mathfrak{g}_{\alpha\beta}{}^{\mu\nu} \nabla^\beta h_{\mu\nu} + \mathfrak{f}_{\alpha\beta} \nabla^\beta \psi = 0$$

Only other obvious "good gauge"

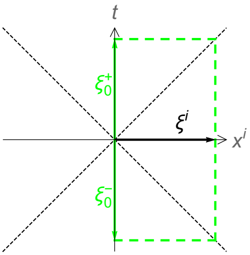
Results

Weak hyperbolicity

- Symmetries of the principal symbol $\Rightarrow$ Weak hyperbolicity for **all Horndeski models**
- Independent of the gauge

Strong hyperbolicity

- **Suitable $\mathfrak{f}_{\alpha\beta}$ for G_2 and G_3**
- **Impossible for G_4 and G_5** (2×2 Jordan block for ξ_0^+ and for ξ_0^-)



End of the story?

1 Hyperbolic PDEs

2 Horndeski theory: ill-posed?

3 Curing the ill-posedness

Disformal transformation

Einstein-scalar action

$$S = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\left[M_{\mathrm{Pl}}^2\tilde{R} - (\tilde{\nabla}\phi)^2\right]$$

$$\downarrow \quad g_{\mu\nu} = \tilde{g}_{\mu\nu} - D\partial_\mu\phi\partial_\nu\phi$$

Horndeski action

$$G_2(X) = \frac{X}{\sqrt{1-2DX}}, \quad G_4(X) = \sqrt{1-2DX}$$

$$\phi \to \phi + \psi$$
$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \quad \Leftrightarrow \quad \tilde{g}_{\mu\nu} \to \tilde{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$$

Original harmonic gauge

$$\tilde{\mathfrak{g}}_{\alpha\beta}{}^{\mu\nu} \nabla^{\beta} \tilde{h}_{\mu\nu} = 0$$

$\downarrow$ Disformal transformation

Non-generalized harmonic gauge!

$$\tilde{\mathfrak{g}}_{\alpha\beta}{}^{\mu\nu} \nabla^{\beta} h_{\mu\nu} + \mathfrak{f}_{\alpha\beta} \nabla^{\beta} \psi + \ldots = 0 \quad : (G)$$

$\tilde{\mathfrak{g}}_{\alpha\beta}{}^{\mu\nu}$ instead of $\mathfrak{g}_{\alpha\beta}{}^{\mu\nu}$

Good gauge?

1. Possible to impose $G = 0$ on Σ_0?
2. Constraints $\Rightarrow \partial_t G = 0$ on Σ_0?
3. $G = 0$ propagated by evolution equations?

Good gauge?

1. Possible to impose $G = 0$ on Σ_0? ✓
2. Constraints $\Rightarrow \partial_t G = 0$ on Σ_0?
3. $G = 0$ propagated by evolution equations? ✓

Conclusions

- Claims about generic ill-posedness of G_4 and G_5 Horndeski models
- Not generic enough gauge
- Ongoing proof of well-posedness for some G_4 model

Prospects

- How to generalize to more quartic and quintic models?
- Is well-posedness so essential?

Thank you!