From scattering amplitudes to gravitational waves observables

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An overview

Supported by U.S. DEPARTMENT OF ENERGY | Office of Science

Lots of theoretical and experimental experience with two-body problem in Newtonian mechanics

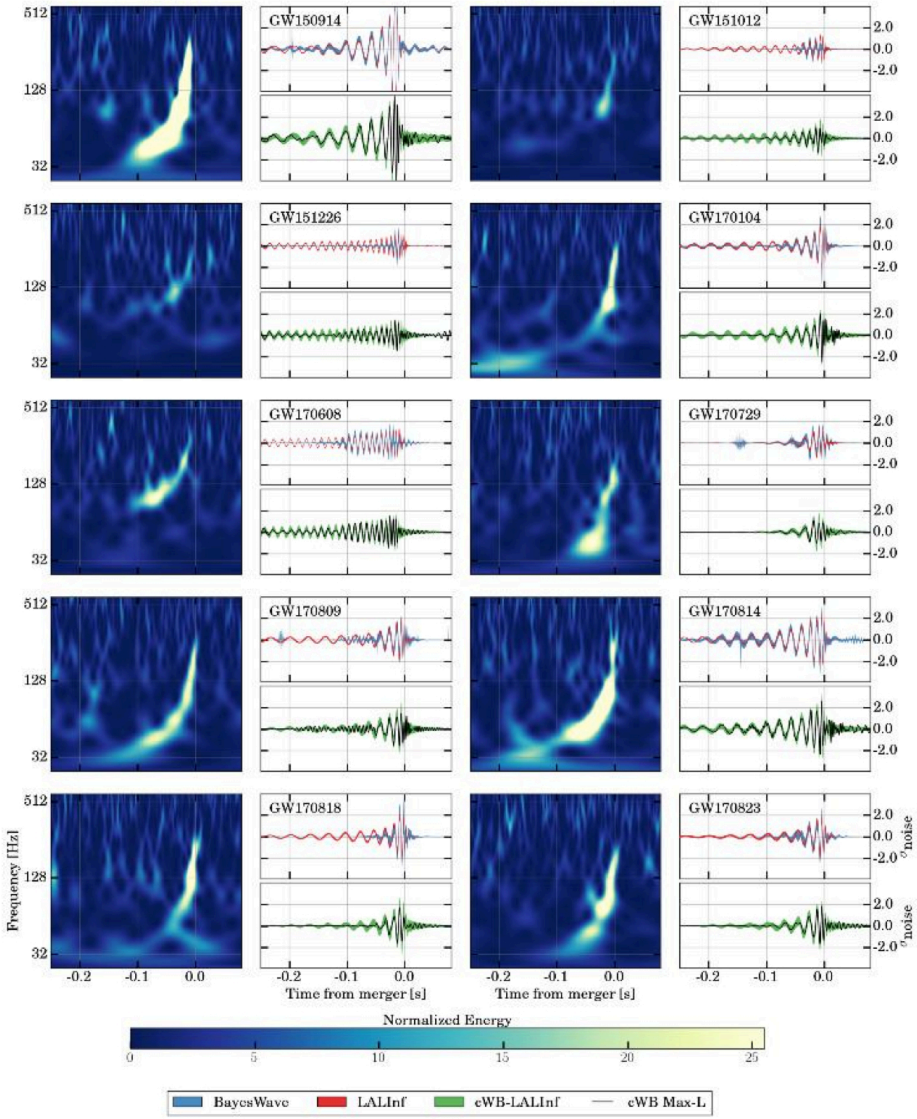
- Integrable
- Closed elliptical orbits; no perihelium precession
- Reducible to motion of effective particle around COM
- Laplace-Runge-Lenz vector

$$H = \frac{p^2}{2m} - \frac{k}{r} \qquad \vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} \qquad SO(3) \to SO(4)$$

- This conservation law is broken by relativistic effects, but a QFT generalization exists – dual conformal symmetry

Generally-relativistic two-body problem is much more difficult and interesting

- Departure form Kepler's result provided first observational test of general relativity (perihelion precession of Mercury)
- Lots of data showing important GR effects

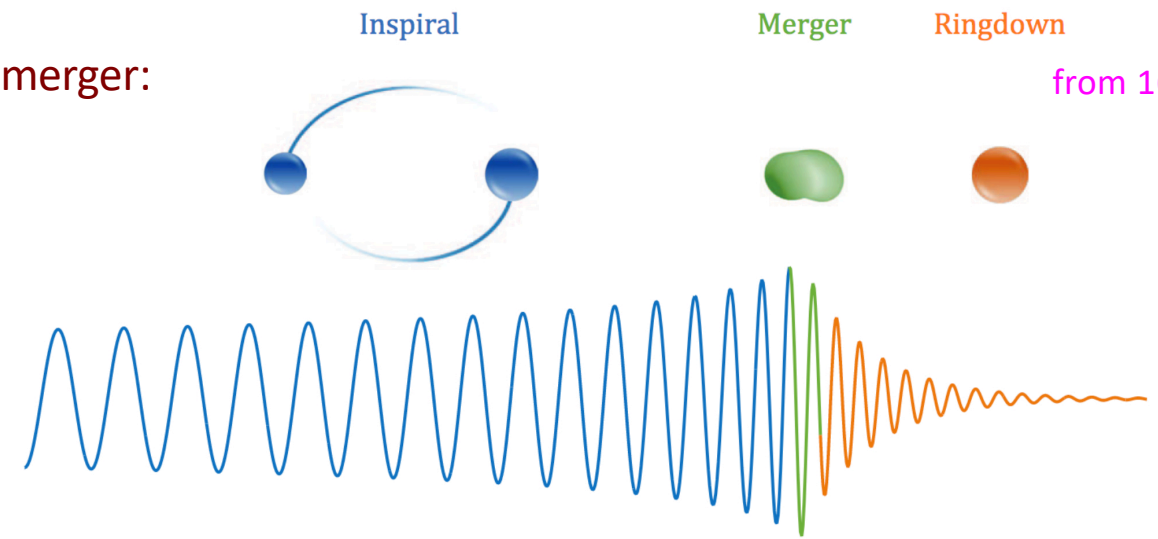


LIGO & VIRGO: GWTC-1: A NEW CATALOG OF GRAVITATIONAL-WAVE DETECTIONS

see next two talks for up-to-date data

Anatomy of a binary merger:

from 1610.03567 Antelis & Moreno



- Separation of scales: conservative dynamics vs. radiation emission

  ⟶ focus on conservative part

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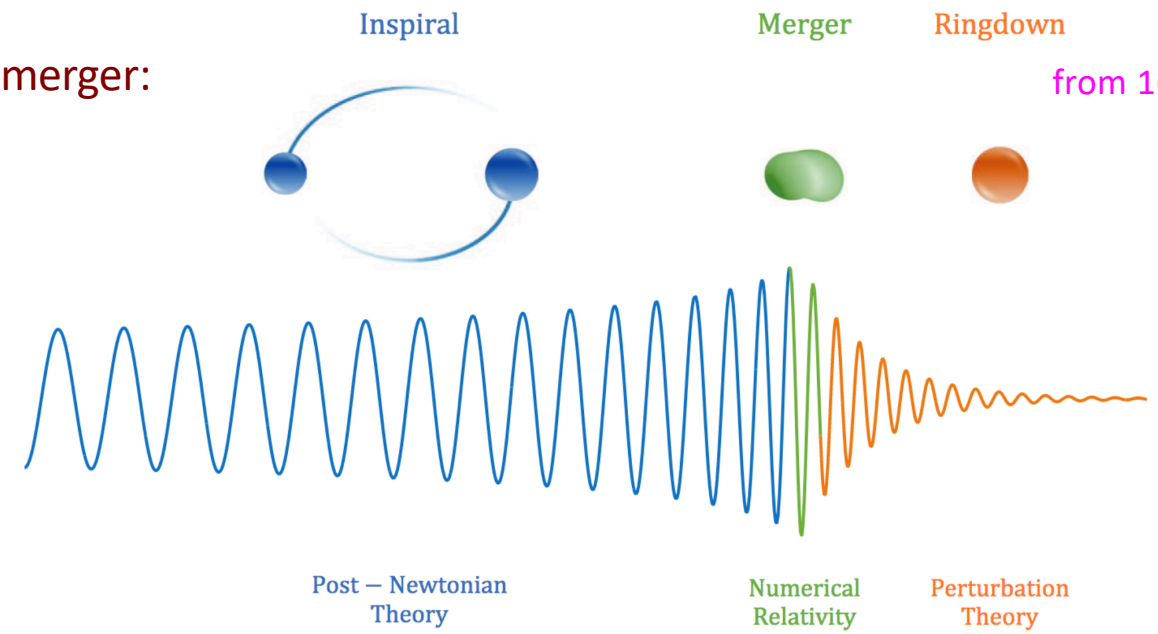

- Post-Newtonian expansion (weak field, nonrelativistic):
  - Closed orbits: expansion in two parameters: $\boldsymbol{v}^2 \sim \frac{GM}{|\boldsymbol{r}|} \ll 1$

Virial thm.

- Post-Minkowskian expansion (weak-field, relativistic): $\frac{GM}{|\boldsymbol{r}|} \ll \boldsymbol{v}^2 \sim 1$
  - Expansion in G
  - Resummation of the velocity expansion

- "Self-force" – expansion in a small mass ratio $\boldsymbol{v}^2 \sim GM/|\boldsymbol{r}| \sim 1$

PN vs PM expansion for non-spinning compact objects

$$\mathcal{L} = -mc^2 + \frac{1}{2}\mu v^2 + G\frac{m\mu}{r} + \mathcal{O}(c^{-2}) + \mathcal{O}(c^{-4}) + \dots$$

$$k\mathrm{PN} \sim (v^2)^n \left(\frac{Gm}{r}\right)^{k-n} \longrightarrow k\mathrm{PM} \sim c(p^2, p\cdot e_r)\left(\frac{Gm}{r}\right)^k$$

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN	
1PM	$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ v^{14}\ +\ \cdots)$	G
2PM		$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}\ +\ \cdots)$	G^2
3PM			$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}\ +\ \cdots)$	G^3
4PM				$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8\ +\ \cdots)$	G^4
5PM					$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6\ +\ \cdots)$	G^5
6PM						$(1$	$+\ v^2$	$+\ v^4\ +\ \cdots)$	G^6

1687 1938 1980 2000 2014

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6PM						$(1$	$+\ v^2$	$+\ v^4\ +\ \cdots)$	G^6

1980

2018

2019

2021*

PM results: Westfahl (79), Westfahl, Goller (80), Portilla (79-80), Bell et al (81), Ledvinka et al (10), Damour (16-17), Guevara (17), Vines (17), Bini, Damour (17-18)

recent PM results: Bern, Cheung, RR, Solon, Shen, Zeng (19), Cheung, Solon (20), Kalin, Porto (20); Parra-Martinez, Ruf, Zeng (20), Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng (21); Herrmann, Parra-Martinez, Ruf, Zeng (21)

How do scattering amplitudes fit here?

(1)

Scattering and bound state dynamics
are governed by the same (effective) Hamiltonian

Cheung, Rothstein, Solon;
Bern, Cheung, RR, Solon, Shen, Zeng

find Hamiltonian from scattering amplitude considerations → use to study bound state dynamics

"Integrate out the gravitons"

(1.5)

Generating function of scattering observables
from
scattering amplitudes

Bern, Parra-Martinez, RR,
Ruf, Solon, Shen, Zeng

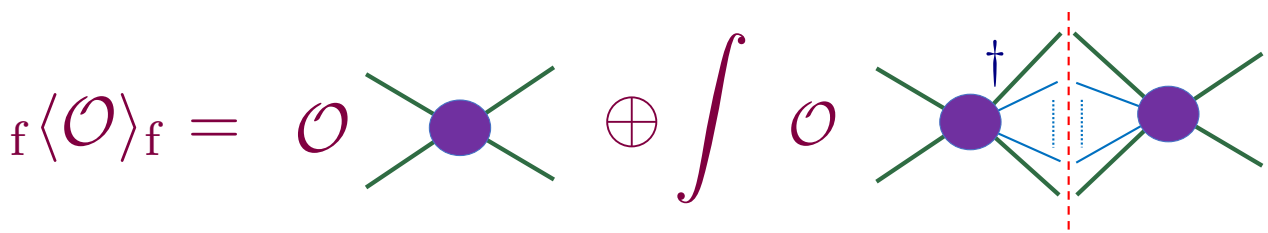
$\longrightarrow$ Analytic continuation* to bound observables

(2)

Final-state scattering observables

Kosower, Maybee, O'Connell

$$ {}_{\mathrm{f}}\langle \mathcal{O} \rangle_{\mathrm{f}} = {}_{\mathrm{f}}\langle \Psi | i[\mathcal{O}, T] | \Psi \rangle_{\mathrm{f}} + {}_{\mathrm{f}}\langle \Psi | T^{\dagger}[\mathcal{O}, T] | \Psi \rangle_{\mathrm{f}} $$



→ Analytic continuation* to bound observables

Why PM expansion --

- Increase precision/extend the reach of PN calculations

- Unmodeled LIGO searches for transient events

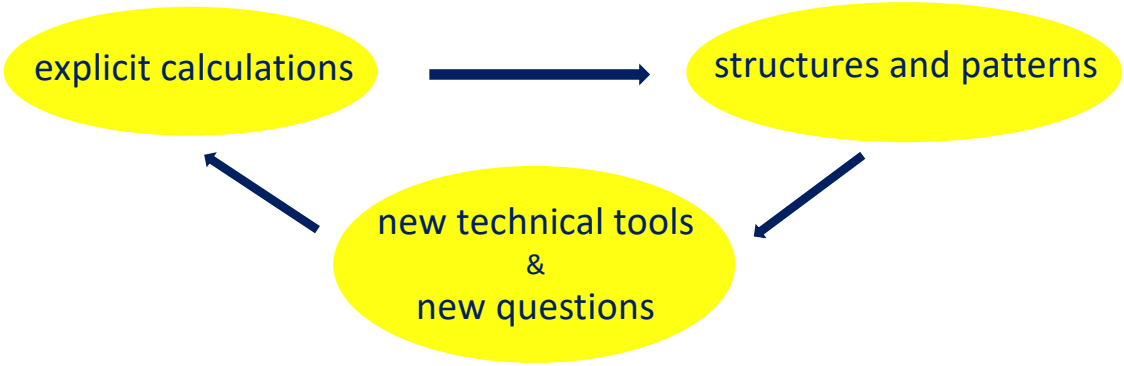
- Explore the structure of gravitational perturbation theory

  - Possible generalization of Laplace-Runge-Lenz symmetry to GR; such generalization is present at $\mathcal{O}(G)$

  - Unexpected functional structures (both at $\mathcal{O}(G^3)$ and $\mathcal{O}(G^4)$)

  - Unexpected structure for spin-dependent observables

Caron-Huot, Zahraee



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Caron-Huot, Zahraee

- Complementary approach; new perspective on gravitational interactions

- Information for semi-analytic/semi-numerical approaches

  - E.g. functional basis required for fitting numerical data

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High-energy gravitational scattering and the general relativistic two-body problem

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(Received 29 October 2017; published 26 February 2018)

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

Scattering amplitudes – bread and butter of quantum field theory calculations

(Older) Textbook approach – Feynman diagrams

-- excessively complex beyond tree diagrams

Novel methods make possible previously unimaginable calculations in gauge and gravity theories

- On-shell recursion relations — Britto, Cachazo, Feng, Witten
- Generalized unitarity — Bern, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng
- Color-kinematics duality — Bern, Carrasco, Johansson
- Double copy construction
- New integration methods

The key: manifest gauge invariance at all intermediate stages

Some milestones:

4-point amplitudes through 5 loops full color $\mathcal{N}$=4 super-Yang-Mills theory

4-point amplitudes through 5 loops $\mathcal{N}$=8 supergravity

4-point amplitudes through 4 loops $\mathcal{N}$=4 supergravity

4-point amplitudes through 2 loops full color $\mathcal{N}$=2 super-QCD

- Supersymmetry is not essential, but it does make things simpler
- Higher-point amplitudes have also been computed through these methods

Post-Minkowskian expansion plays on amplitudes' strengths

- All-orders in v/c plays on strengths of amplitude technology
- Great tools for perturbative gravity calculations: Unitarity, KLT relations, BCJ, integration techniques
  Bern, Dixon, Dunbar, Kosower; Kaway, Lewellen, Tye; Bern, Carrasco, Johansson
- Use EFT techniques to integrate out gravitons Goldberger, Rothstein; Neill, Rothstein; Cheung, Rothstein, Solon
- Model black holes as point-particles; suitable at large separation
- Amplitudes-based approach can easily incorporate other properties, such as spin and finite size Goldberger, Ridgway; Shen; etc; Vadya; Guevara et al; Donoghue et al. ; Guevara, Ochirov, Vines; O'Connell, Maybee, Vines; Bern, Luna, RR, Shen, Zeng; etc

Where are we at?

- Computation of effective interaction potentials between massive bodies ($\mathcal{O}(G^3)$ and $\mathcal{O}(G^4)^*$)
- Conservative radiation contributions to observables ($\mathcal{O}(G^3)$)
- Dissipative observables (energy/momentum loss, $\mathcal{O}(G^3)$)
- Spin-dependent observables; conjectured generating fct of open-orbit observables $\mathcal{O}(S^2G^2)$
- Leading order finite-size effects; structure

More structure: an amplitude-action relation

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

S-matrix unitarity

$$SS^\dagger = 1 \qquad S = 1 + iT \quad \Longrightarrow \quad 2\mathrm{Im}T = TT^\dagger$$

Purely-elastic/conservative processes: only 2-particle states

Theory of a single scalar $\longrightarrow$ better solution: $S = e^{i\hat{\Phi}}$ phase shift operator

Classical limit: $\langle \boldsymbol{p}+\boldsymbol{q}, E|S|\boldsymbol{p}, E\rangle = e^{i\tilde{I}_r(\boldsymbol{q})}\big(1+\mathcal{O}(\hbar)\big) = \sum_k \mathcal{M}_k(\boldsymbol{q})$

What is $\tilde{I}_r(\boldsymbol{q})$:

- With fixed time and coordinate:

$$\langle t_f, x_f|S|t_i, x_i\rangle = e^{iS_{cl}(\boldsymbol{x}(t),\boldsymbol{p}(t))}\big(1+\mathcal{O}(\hbar)\big) \qquad S_{cl}(\boldsymbol{x}(t),\boldsymbol{p}(t)) = \int_{\text{trajectory}} (\boldsymbol{p}d\boldsymbol{x} - H(\boldsymbol{p},\boldsymbol{x}))$$

- Legendre-transform to fixed energy: (use assumed conservative nature of scattering)

It is the radial action: $I_r = S_{cl} + Et\Big|_{t_i}^{t_f} = \int_{\text{trajectory}} \boldsymbol{p}d\boldsymbol{x} \qquad dI_r = -\chi dJ + \Delta t dE + \ldots$

- Direct relation between elastic scattering amplitudes in classical limit and radial action

The two-body matter Hamiltonian through 4PM:

Hamiltonian – general form:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+c_1(\boldsymbol{p}^2)\frac{G}{\boldsymbol{r}}+c_2(\boldsymbol{p}^2)\frac{G^2}{\boldsymbol{r}^2}+c_3(\boldsymbol{p}^2)\frac{G^3}{\boldsymbol{r}^3}+c_4(\boldsymbol{p}^2)\frac{G^4(\boldsymbol{r}^2\mu^2e^{2\gamma_E})^{4\epsilon}}{\boldsymbol{r}^4}+\mathcal{O}(G^5)$$

Coefficient functions:

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right)-\frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi}-\frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right]$$

Westphal; Damour

$$c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right)-\frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}-\frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)}\right.$$
$$\left.-\frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi}-\frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2}+\frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}+\frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right]$$

Bern, Cheung, RR, Solon, Shen, Zeng

$$c_4 = \frac{m^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\mathrm{p}}+\nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon}+\mathcal{M}_4^{\mathrm{f}}-10\mathcal{M}_4^{\mathrm{t}}\right)\right]+\mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right]+\mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2}+\frac{E\xi(3\xi-1)}{2}\right)c_1^4-2E^2\xi^2c_1^2c_2\right]$$
$$+\left(\mathcal{D}+\frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3+c_2^2)+\left(\frac{4\xi-1}{4E}+\frac{2E^3\xi^3}{\boldsymbol{p}^4}+\frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4+\left((1-3\xi)-\frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2c_2\right]$$

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

Some definitions:

$$m = m_1+m_2\,, \quad \nu = \frac{m_1m_2}{m^2}\,, \quad E = E_1+E_2\,, \quad \xi = \frac{E_1E_2}{E^2}\,, \quad \gamma = \frac{E}{m}\,, \quad \sigma = \frac{p_1\cdot p_2}{m_1m_2}\,, \quad \mathcal{D} = \frac{d}{d\boldsymbol{p}^2}$$

The classical limit of the potential-scattering part of the 4-scalar amplitude at 3 loops:

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)} \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\begin{aligned}
\mathcal{M}_4^{\mathrm{f}} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)
\end{aligned}$$

$$\begin{aligned}
h_1 &= \frac{1151-3336\sigma+3148\sigma^2-912\sigma^3+339\sigma^4-552\sigma^5+210\sigma^6}{12(\sigma^2-1)} \\
h_2 &= \frac{1}{2}\left(5-76\sigma+150\sigma^2-60\sigma^3-35\sigma^4\right) \\
h_3 &= \sigma\frac{(-3+2\sigma^2)}{4(\sigma^2-1)}\left(11-30\sigma^2+35\sigma^4\right) \\
h_4 &= \frac{1}{144(\sigma^2-1)^2\sigma^7}(-45+207\sigma^2-1471\sigma^4+13349\sigma^6 \\
&\quad -37566\sigma^7+104753\sigma^8-12312\sigma^9-102759\sigma^{10}-105498\sigma^{11} \\
&\quad +134745\sigma^{12}+83844\sigma^{13}-101979\sigma^{14}+13644\sigma^{15}+10800\sigma^{16}) \\
h_5 &= \frac{1}{4(\sigma^2-1)}(1759-4768\sigma+3407\sigma^2-1316\sigma^3+957\sigma^4 \\
&\quad -672\sigma^5+341\sigma^6+100\sigma^7) \\
h_6 &= \frac{1}{24(\sigma^2-1)^2}(1237+7959\sigma-25183\sigma^2+12915\sigma^3+18102\sigma^4 \\
&\quad -12105\sigma^5-9572\sigma^6+2973\sigma^7+5816\sigma^8-2046\sigma^9) \\
h_7 &= 2\sigma\frac{(-852-283\sigma^2-140\sigma^4+75\sigma^6)}{3(\sigma^2-1)} \\
h_8 &= \frac{\sigma}{8(\sigma^2-1)^2}(-304-99\sigma+672\sigma^2+402\sigma^3-192\sigma^4-719\sigma^5 \\
&\quad -416\sigma^6+540\sigma^7+240\sigma^8-140\sigma^9) \\
h_9 &= \frac{1}{2}\left(52-532\sigma+351\sigma^2-420\sigma^3+30\sigma^4-25\sigma^6\right) \\
h_{10} &= 2\left(27+90\sigma^2+35\sigma^4\right) \\
h_{11} &= 20+111\sigma^2+30\sigma^4-25\sigma^6 \\
h_{12} &= \frac{834+2095\sigma+1200\sigma^2}{2(\sigma^2-1)} \\
h_{13} &= -\frac{1183+2929\sigma+2660\sigma^2+1200\sigma^3}{2(\sigma^2-1)} \\
h_{14} &= \frac{7\left(169+380\sigma^2\right)}{4(\sigma-1)}
\end{aligned}$$

NB: 4PM *potential* does not contain the complete conservative physics.

- signal: IR divergence; scheme dependence
- resolution: additional terms needed, related gravitational radiation emitted and reabsorbed by the two-body system (tail effect)

known in a PN expansion, but not in a PM expansion

Energetics of two-body Hamiltonians in post-Minkowskian gravity

Andrea Antonelli,[1] Alessandra Buonanno,[1,2] Jan Steinhoff,[1] Maarten van de Meent,[1] and Justin Vines[1]
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orbits to merger

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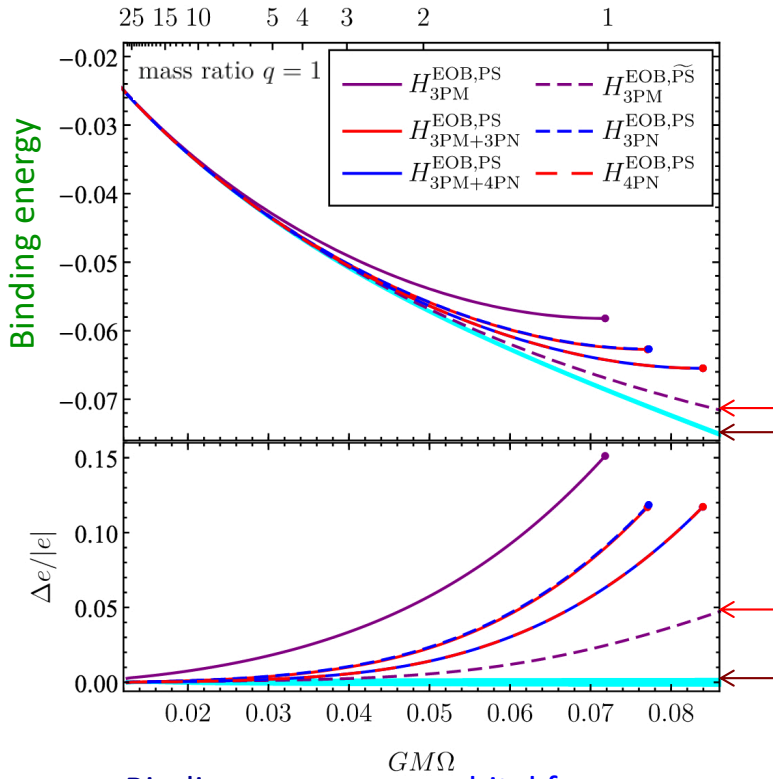


Binding energy vs. orbital frequency

Comparison of various EOB-type models *including 3PM data* to numerical GR

based on 3PM

NGR

Radiation not included
⟶ not completely conclusive

based on 3PM

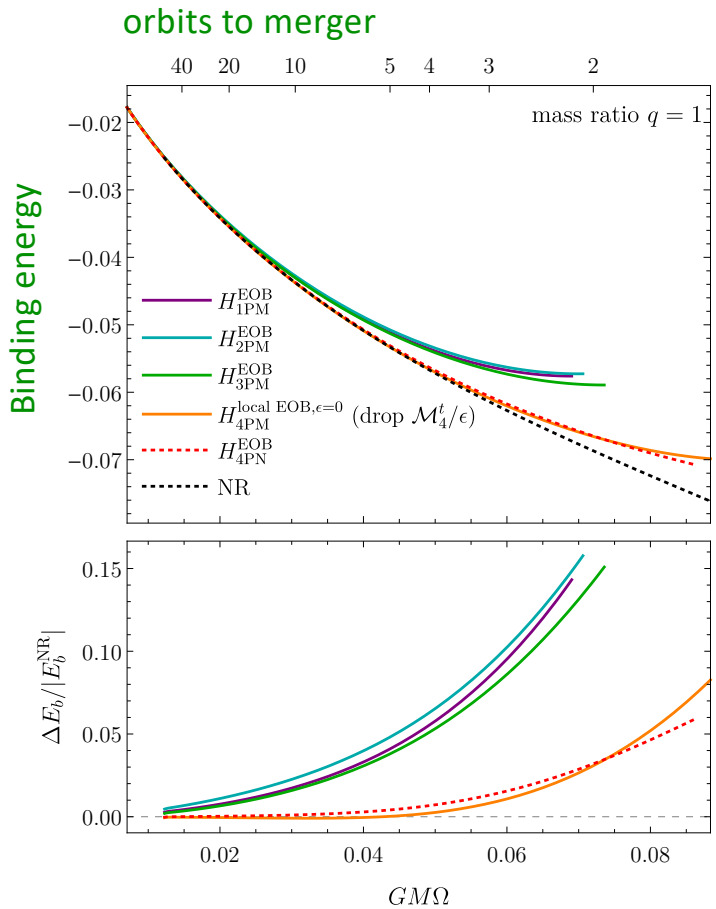
NGR

"This rather encouraging result motivates a more comprehensive study of EOB resummations of PM results."

Energetics with 4PM potential

Khalil, Buonanno, Steinhoff, Vines (in preparation)

Even with incomplete Hamiltonian, it is worth probing energetics to see if we are on track



Binding energy vs. orbital frequency

$H_{\rm 4PM}$ w/o tail

$H^{\rm EOB}_{\rm 4PN}$

NGR

Various EOB-type models *including partial 4PM data*

vs.

numerical GR

- Only the finite part of potential is used
  - $\frac{1}{\epsilon}\mathcal{M}^{\rm t}_4$ is dropped
  - Scale dependence also gone

- Not yet conclusive but quite encouraging

- Motivation to complete the tail calculation

Motion of spinning bodies

An all-order conjecture for open-orbit observables: Bern, Luna, RR, Shen, Zeng

$$\Delta\mathcal{O} = e^{-i\chi\mathcal{D}}[\mathcal{O}, e^{i\chi\mathcal{D}}]$$

$$\chi\mathcal{D}g \equiv \chi g + \mathcal{D}_{SL}(\chi, g) \qquad \mathcal{D}_{SL}(\chi, g) \equiv -\sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial\chi}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

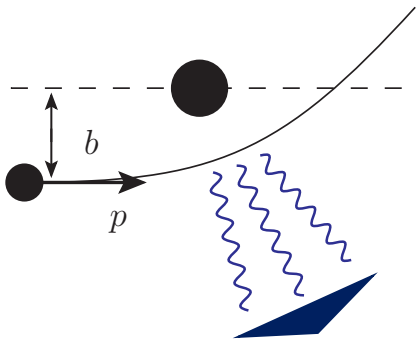
$$\mathcal{M}_{S_1 S_2 \to S_1 S_1} = C e^{i\chi} - 1$$

$\chi \equiv$ conservative eikonal phase of the $S_1 S_2 \to S_1 S_2$ amplitude

- General proof at $\mathcal{O}(G_N)$; explicit check at $\mathcal{O}(G_N^2)$ through orders ($S_1 S_2$, S_1^2, S_2^2)
  recent checks at S_1^1 and S_2^2 Luna, Kosmopoulos vs. Liu, Porto, Yang
- Points to remarkable simplicity and structure in the solution to spinning Hamilton's eqs.
- One might expect that a direct relation between χ and observables exists to all orders
- The eikonal phase or its improved version, the radial action $I_r = \int_\gamma p \cdot dx$, are generating functions of open-orbit observables

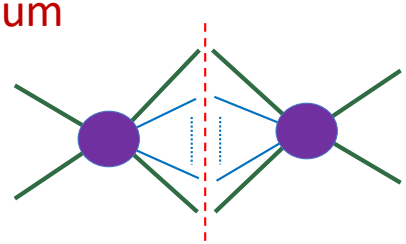
Other amplitudes-based approaches: Maybee, O'Connell, Vines ; Guevara, Ochirov, Vines

Radiative observables in (classical) scattering processes



Example: radiated 4-momentum

$$ {}_{\mathrm{f}}\langle \Delta p^{\mu}_{\mathrm{rad}} \rangle_{\mathrm{f}} = \int \sum_{\text{gravitons}} l^{\mu}_{i} $$



Kosower, Maybee, O'Connell

LO & NLO:

$$ \Delta p^{\mu,(1)}_{1,\perp} = \frac{G^2 m^3 \nu}{|b|^2} \frac{3\pi}{4} \frac{(5\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \frac{b^{\mu}}{|b|} $$

$$ \Delta p^{\mu,(1)}_{1,u} = \frac{G^2 m^4 \nu^2}{|b|^2} \frac{2\,(1 - 2\sigma^2)^2}{(\sigma^2 - 1)} \left[\frac{1}{m_1} \check{u}^{\mu}_{1} - \frac{1}{m_2} \check{u}^{\mu}_{2} \right] $$

$$ \Delta p^{\mu,(2)}_{1,\mathrm{rad}} = \frac{G^3 m^4 \nu^2}{|b|^3} \left\{ \frac{4}{\sqrt{\sigma^2 - 1}} \frac{b^{\mu}}{|b|} \left[f^{\mathrm{LS}}_{1}(\sigma) + f^{\mathrm{LS}}_{3}(\sigma) \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma - 1}{2}}}{\sqrt{\sigma^2 - 1}} \right] \right. $$

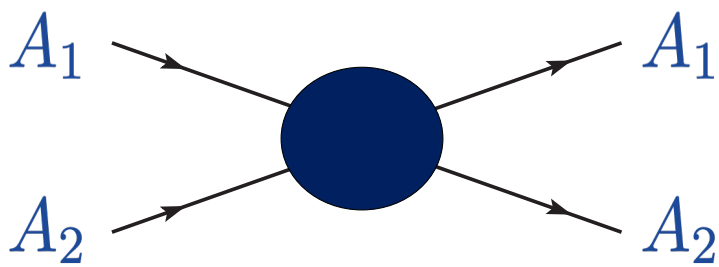
$$ \left. + \pi\, \check{u}^{\mu}_{2} \left[f_1(\sigma) + f_2(\sigma) \log\left(\frac{\sigma + 1}{2} \right) + f_3(\sigma) \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma - 1}{2}}}{\sqrt{\sigma^2 - 1}} \right] \right\} $$

Herrmann, Parra-Martinez, Ruf, Zeng

Agreement with other approaches at this order (divergence of $\mathcal{O}(G^4)$ potential)

Damour

From Amplitudes to effective two-body Hamiltonian



Construct an effective theory of positive-energy matter particles whose scattering amplitudes in the classical limit are the same as those of General Relativity coupled to scalar fields in the classical limit

$$L_{\mathrm{EFT}} = \bar{A}_1 \partial_t A_1 + \bar{A}_2 \partial_t A_2 - H$$

$$H(\boldsymbol{p}, \boldsymbol{r}, G) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p}, \boldsymbol{r}, G)$$

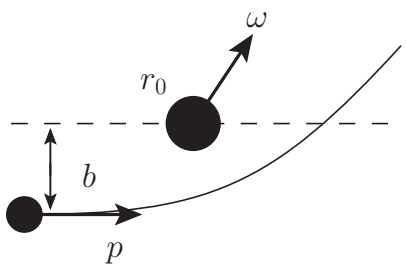
Goal: interaction of classical heavy particles with spin while integrating out short distance gravitons

GR + (spinning) matter $\xrightarrow{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{off shell}} + h_{\mu\nu}^{\text{radiation}}}$ $S_{eff} = S_{eff}(\text{matter}, h_{\mu\nu}^{\text{radiation}})$

Subtlety: radiation can also be off shell

(Semi) classical limit: "all conserved charges are large"

In $\hbar = 1$ units: "classical": de Broglie wavelength λ of particles is much smaller than their



- separation: $\lambda \sim \dfrac{1}{|\boldsymbol{p}|} \ll |\boldsymbol{b}| \sim \dfrac{1}{|\boldsymbol{q}|} \quad \Longrightarrow \quad |\boldsymbol{L}| = |\boldsymbol{b} \times \boldsymbol{p}| \gg 1 \; ; \; |\boldsymbol{p}| \gg |\boldsymbol{q}|$

- size: $\lambda \sim \dfrac{1}{|m r_0 \boldsymbol{\omega}|} \ll r_0 \quad \Longrightarrow \quad |\boldsymbol{S}| = m r_0^2 |\boldsymbol{\omega}| \gg 1$

Classical limit/expansion: $\mathcal{O}(1/|\boldsymbol{L}|) \sim \mathcal{O}(1/|\boldsymbol{S}|) \sim \mathcal{O}(|\boldsymbol{q}|/m) \sim \mathcal{O}(q/m)$

Newton's potential is classical $V_{\text{Newton}} \sim \dfrac{mG}{b}$

Structure of two-body classical potential: $\mathcal{F}_{\boldsymbol{r}}\Big[V(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{S}), \boldsymbol{q}\Big] \sim \dfrac{c_{ijk}(\boldsymbol{p})}{|\boldsymbol{q}|^3} \left(Gm|\boldsymbol{q}| \right)^i \left(\dfrac{\boldsymbol{q} \cdot \boldsymbol{S}}{m} \right)^j \left(R\,|\boldsymbol{q}| \right)^k$

► Loops contain classical physics

From Amplitudes to effective Hamiltonian: the flowchart

Full theory (GR + scalar fields)		Effective theory (positive-energy matter)
Amplitudes methods: double copy, KLT ↓		↓ ansatz
Graviton-matter tree amplitudes		$V(\boldsymbol{p},\boldsymbol{q})\bar{A}_1 A_1 \bar{A}_2 A_2$
Generalized unitarity, double copy ↓		↓ Feynman graphs, other
Integral representation of full theory 4-pt loop amplitude		Integral representation of EFT 4-pt loop amplitude
Multi-loop integration ↓		↓ Multi-loop integration
Full-theory 4-pt matter amplitude	$=$ same physics	EFT 4pt amplitude

From Amplitudes to Classical Hamiltonian: the flowchart

Full theory
(GR + scalar fields)

Amplitudes methods:
double copy, KLT

Graviton-matter
tree amplitudes

Generalized unitarity
double copy

+ classical limit

Integral representation of
full theory 4-pt loop amplitude
in the classical limit

Multi-loop integration

+ classical limit

Full-theory
4-pt matter amplitude
in the classical limit

same physics

Effective theory
(positive-energy matter)

ansatz

$$V(\boldsymbol{p},\boldsymbol{q})\bar{A}_1 A_1 \bar{A}_2 A_2$$

Feynman graphs,
other

Integral representation of
EFT 4-pt loop amplitude
in classical limit

classical limit

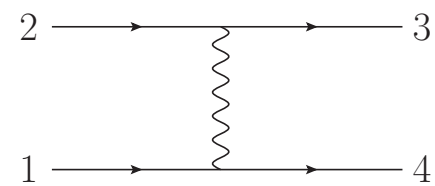
+ Multi-loop integration

EFT 4pt amplitude
in the classical limit

Example -- scalar-coupled GR at tree level: contributions to the classical Hamiltonian

$$\mathcal{M}^{\rm NR}_{\rm tree} = \frac{16\pi G_N}{4E_1E_2}\frac{1}{q^2}(2(p_1\cdot p_2)^2 - m_1^2m_2^2) = \frac{16\pi G_N m_1^2 m_2^2}{4E_1E_2}\frac{1}{q^2}(2\sigma^2-1)$$

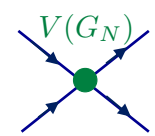
$$\sigma = \frac{p_1\cdot p_2}{m_1 m_2}$$



Center of mass

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + V(\boldsymbol{p},\boldsymbol{r})\ , \quad V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G_N}{|\boldsymbol{r}|}\right)^i$$

$$c_1 = \frac{\nu^2}{\gamma^2\xi}(m_1+m_2)^2(1-2\sigma^2)$$



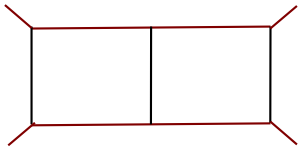
$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \qquad \xi = \frac{E_1E_2}{(E_1+E_2)^2} \qquad \gamma = \frac{E_1+E_2}{m_1+m_2}$$

Gravitational scattering amplitudes

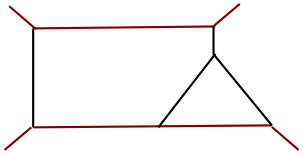
Feynman graph gravitational perturbation theory is complicated

Estimate for the 2-loop 4-scalar amplitude:

- 3 terms per s-s-graviton vertex
- about 100 terms per 3-graviton vertex
- 2-3 terms per graviton propagator
- some (large) number of 2-loop graphs



$\sim 10^4$ terms



$\sim 10^5$ terms

Most are gauge artifacts and cancel out in final expression

Avoid handling them at all $\longleftrightarrow$ loop amplitudes from tree amplitudes

Generalized unitarity

Reorganization of Feynman graph perturbation theory

Bern, Dixon, Dunbar, Kosower

1-loop improvements: Britto, Cachazo, Feng

Amplitudes' integrands = rational functions with prescribed poles and residues

Poles = graph structure with given number of external lines and loops

Residues = generalized cuts = products of tree amplitudes = "generalized cuts"

► Very convenient for our purpose: can weed out pieces that are not classical

Amplitudes from generalized unitarity and double copy

- Gravity trees from gauge theory trees through KLT relations: Kawai, Lewellen, Tye

$$M_4^{\rm tr}(1,2,3,4) = -is_{12}A_4^{\rm tr}(1,2,3,4)A_4^{\rm tr}(1,2,4,3)$$
$$M_5^{\rm tr}(1,2,3,4,5) = \quad is_{12}s_{34}A_5^{\rm tr}(1,2,3,4,5)A_5^{\rm tr}(2,1,4,3,5) + (2 \leftrightarrow 3)$$
$$M_6^{\rm tr} = 12 \text{ terms of the type } s^3 A_6 A_6$$

- Hold state-by-state for external lines, following addition of helicities

e.g. $\quad \text{scalar} \longleftrightarrow \text{scalar} \times \widetilde{\text{scalar}} \qquad h^{++} \longleftrightarrow A^+ \times \tilde{A}^+ \qquad \varphi^{+-} \longleftrightarrow A^+ \times \tilde{A}^-$

- Hold in any dimension; implements all simplifications required by gauge invariance

- Color/kinematics, double-copy and generalized double-copy Bern, Carrasco, Johansson; Bern, Carrasco, Chen, Johansson, RR

e.g. $$i\mathcal{A}_4^{\rm tree}(1,2,3,4) = g^2\left(\frac{c_s n_s(p,\epsilon)}{s} + \frac{c_t n_t(p,\epsilon)}{t} + \frac{c_u n_u(p,\epsilon)}{u}\right)$$

$$c_s + c_t + c_u = 0 \qquad\qquad n_s + n_t + n_u = 0$$

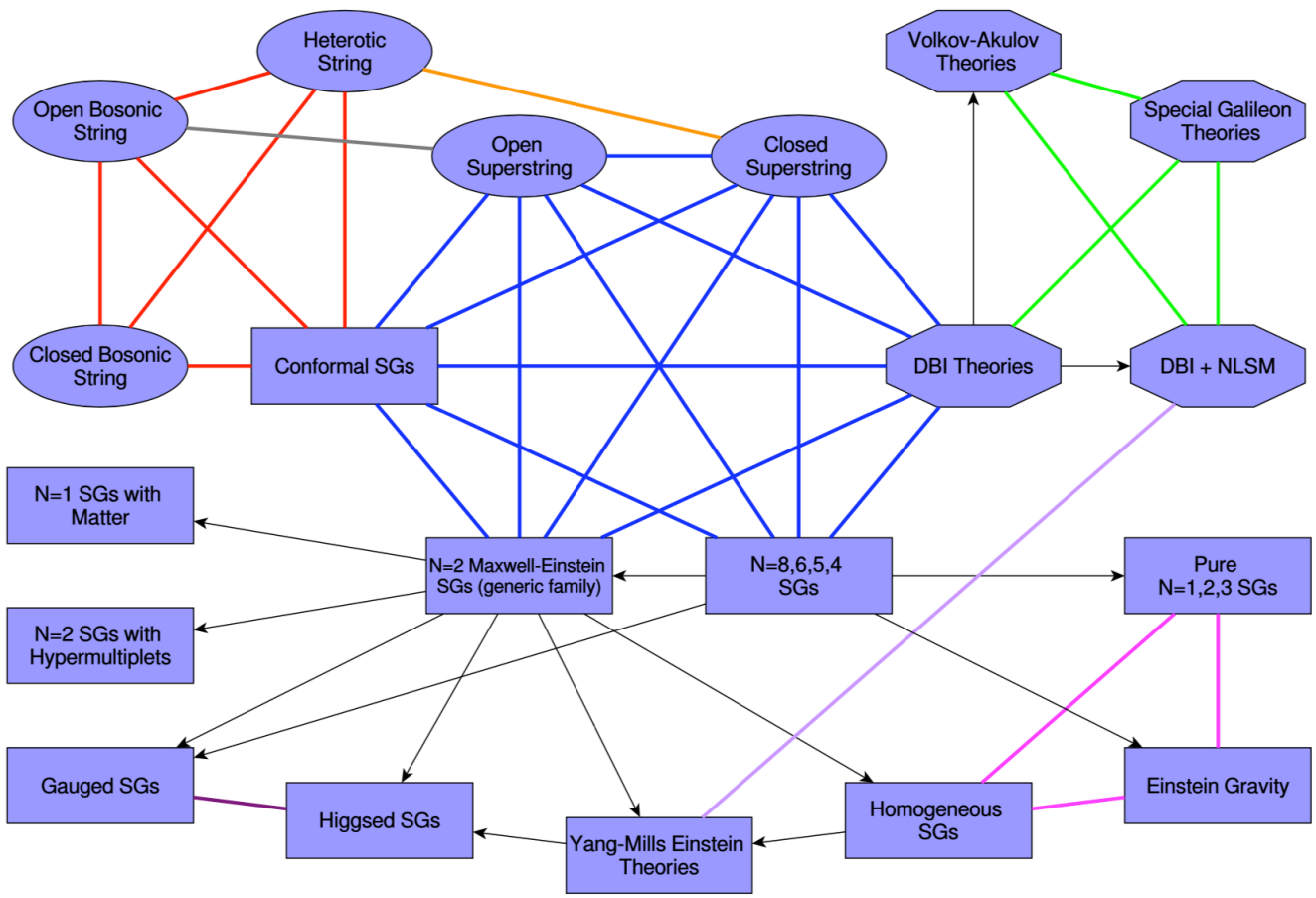
$$i\mathcal{M}_4^{\rm tree}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^2\left(\frac{n_s(p,\epsilon)\tilde{n}_s(p,\epsilon)}{s} + \frac{n_t(p,\epsilon)\tilde{n}_t(p,\epsilon)}{t} + \frac{n_u(p,\epsilon)\tilde{n}_u(p,\epsilon)}{u}\right)$$

- Full power in relating loop level amplitudes; implications beyond amplitudes and gravitational theories

- Here used cut-by-cut; important for obtaining a graph-based organization

5 A WEB OF DOUBLE-COPY-CONSTRUCTIBLE THEORIES

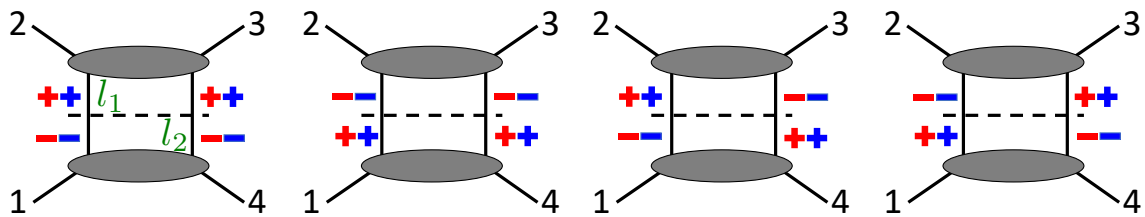
from Bern, Carrasco, Chiodaroli, Johansson, RR



Example -- scalar-coupled GR at 1 loop: contributions to the classical limit

- Long-range force: scalar lines cannot touch
- Classical limit: every loop has at least one matter line

Neill, Rothstein;
Bjerrum-Bohr, Damgaard,
Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon



Gauge theory building blocks: $A_4(1^s,2^+,3^+,4^s) = i\,\dfrac{m_1^2[23]}{\langle 23\rangle(2p_1\cdot p_2)}$ $\qquad A_4(1^s,2^+,3^-,4^s) = i\,\dfrac{\langle 3|1|2]^2}{(2p_2\cdot p_3)\,(2p_1\cdot p_2)}$

Gauge theory cut: $C_{\rm YM}=\sum_{h_5,h_6} A_4(4^s,1^s,5^{h_5},6^{h_6})A_4(2^s,3^s,-6^{-h_6},-5^{-h_5})$

$$C_{\rm YM} = 2\left(\frac{\mathcal{E}^2+\mathcal{O}^2}{s_{23}^2}+m_1^2m_2^2\right)\frac{1}{t_{1\ell_1}\,t_{2\ell_1}}$$

$$\mathcal{E}^2 = \frac{1}{4}\Big[-t_{12}s_{23}+s_{23}t_{1\ell_1}-s_{23}t_{2\ell_1}+2t_{1\ell_1}t_{2\ell_1}\Big]^2$$
$$\mathcal{O}^2 = \mathcal{E}^2-(s_{23}m_1^2+s_{23}t_{1\ell_1}+t_{1\ell_1}^2)(s_{23}m_2^2-s_{23}t_{2\ell_1}+t_{2\ell_1}^2)$$

Gravity cut: $C_{\rm GR}=-s_{23}^2\sum_{h_5,h_6} A_4(4^s,1^s,5^{h_5},6^{h_6})A_4(1^s,4^s,5^{h_5},6^{h_6})A_4(2^s,3^s,-6^{-h_6},-5^{-h_5})A_4(3^s,2^s,-6^{-h_6},-5^{-h_5})$

$$C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4+\mathcal{O}^4+6\mathcal{E}^2\mathcal{O}^2\right)+m_1^4m_2^4\right]\left[\frac{1}{t_{1\ell_1}}+\frac{1}{t_{4\ell_1}}\right]\left[\frac{1}{t_{2\ell_1}}+\frac{1}{t_{3\ell_1}}\right]$$

Example -- scalar-coupled GR at 1 loop; contributions to the classical limit

- Long-range force: scalar lines cannot touch
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Neill, Rothstein;
Bjerrum-Bohr, Damgaard,
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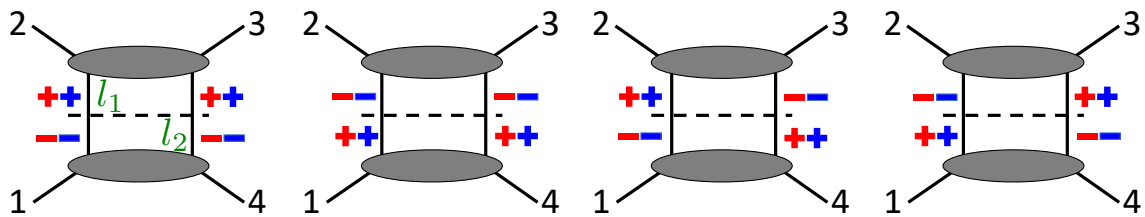
Gravity cut:

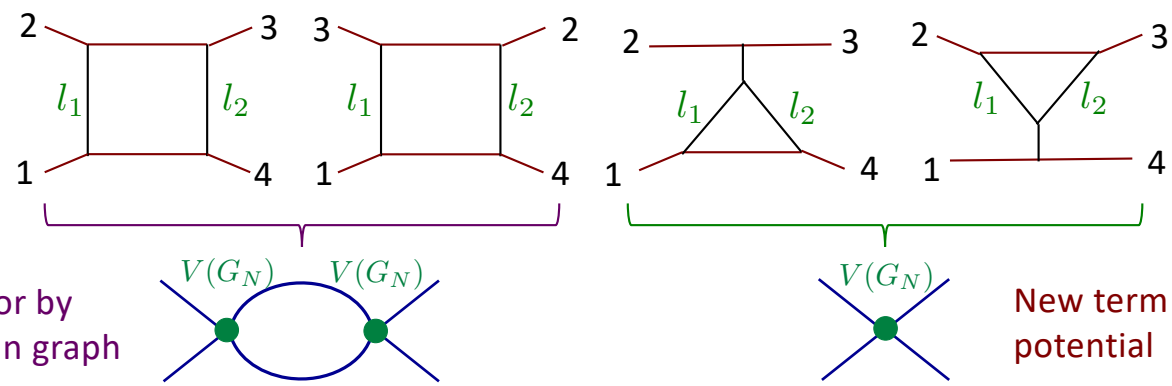


$$C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2\mathcal{O}^2\right) + m_1^4 m_2^4\right]\left[\frac{1}{t_{1\ell_1}} + \frac{1}{t_{4\ell_1}}\right]\left[\frac{1}{t_{2\ell_1}} + \frac{1}{t_{3\ell_1}}\right]$$

$$\mathcal{E}^2 = \frac{1}{4}\left[-t_{12}s_{23} + s_{23}t_{1\ell_1} - s_{23}t_{2\ell_1} + 2t_{1\ell_1}t_{2\ell_1}\right]^2$$

$$\mathcal{O}^2 = \mathcal{E}^2 - (s_{23}m_1^2 + s_{23}t_{1\ell_1} + t_{1\ell_1}^2)(s_{23}m_2^2 - s_{23}t_{2\ell_1} + t_{2\ell_1}^2)$$

Graph organization:



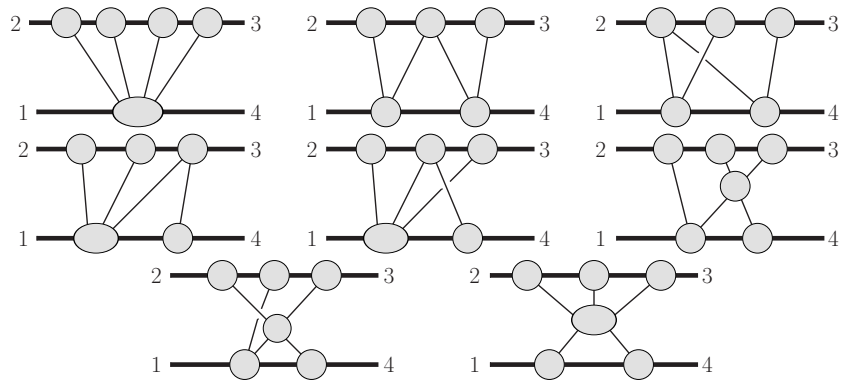
Fully accounted for by
1-loop EFT Feynman graph

New term in the
potential

Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

The classical limit of the potential 4-scalar at 3 loops; 3-loop radial action:

Bern, Parra-Martinez, RR, Ruf, Solon, Shen, Zeng



$$\tilde{I}_{r,4}(\boldsymbol{q}) = G^4 m^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}}\right)\right]$$

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)} \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\begin{aligned}
\mathcal{M}_4^{\mathrm{f}} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)
\end{aligned}$$

$$m = m_1 + m_2\ , \quad \nu = \frac{m_1 m_2}{m^2}\ , \quad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$h_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2-1)}$$

$$h_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right)$$

$$h_3 = \sigma\frac{\left(-3+2\sigma^2\right)}{4(\sigma^2-1)}\left(11 - 30\sigma^2 + 35\sigma^4\right)$$

$$\begin{aligned}
h_4 = {} & \frac{1}{144(\sigma^2-1)^2\sigma^7}\big(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 \\
& - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} \\
& + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}\big)
\end{aligned}$$

$$\begin{aligned}
h_5 = {} & \frac{1}{4(\sigma^2-1)}\big(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4 \\
& - 672\sigma^5 + 341\sigma^6 + 100\sigma^7\big)
\end{aligned}$$

$$\begin{aligned}
h_6 = {} & \frac{1}{24(\sigma^2-1)^2}\big(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4 \\
& - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9\big)
\end{aligned}$$

$$h_7 = 2\sigma\frac{\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)}$$

$$\begin{aligned}
h_8 = {} & \frac{\sigma}{8(\sigma^2-1)^2}\big(-304 - 99\sigma + 672\sigma^2 + 402\sigma^3 - 192\sigma^4 - 719\sigma^5 \\
& - 416\sigma^6 + 540\sigma^7 + 240\sigma^8 - 140\sigma^9\big)
\end{aligned}$$

$$h_9 = \frac{1}{2}\left(52 - 532\sigma + 351\sigma^2 - 420\sigma^3 + 30\sigma^4 - 25\sigma^6\right)$$

$$h_{10} = 2\left(27 + 90\sigma^2 + 35\sigma^4\right)$$

$$h_{11} = 20 + 111\sigma^2 + 30\sigma^4 - 25\sigma^6$$

$$h_{12} = \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2-1)}$$

$$h_{13} = -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)}$$

$$h_{14} = \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma-1)}$$

Simple dependence on symmetric mass ratio consistent with arguments of Damour & Bini, Damour, Geralico

Explore gravitational interactions of spinning bodies

New conceptual and technical challenges

- description of higher-spin fields
- their minimal and nonminimal interactions with gravity
- EFT description of spin

Vast PN literature provides guidance:

Barker & R. O'Connell; Porto, Rothstein;+Ross; Damour; +Bini; +Nagar; Steinhoff, Schäfer, Hergt, Hartung, Levi, Steinhoff; Holstein, Ross, Vaidya; Levi, McLeod, von Hippel; Siemonsen, Vines; Blanchet, Buonanno, Faye; Khalil, Buonanno, Steinhoff, Vines; …

Various amplitudes-based PM and PN approaches:

Vaidya; Chung, Huang, Kim, Lee, Guevara, Ochirov, Vines; Maybee, O'Connel, Vines; Vines, Steinhoff, Buonanno;…

We use Lagrangian that covariantizes the most general parity-even stress tensor Bern, Luna, RR, Shen, Zeng

$$T^{\mu\nu}(p_1,q) = \frac{p_1^\mu p_1^\nu}{m}\sum_{n=0}^{\infty}\frac{C_{ES^{2n}}}{(2n)!}\left(\frac{q\cdot S(p_1)}{m}\right)^{2n} - \frac{i}{m}q_\rho p_1^{(\mu}S(p_1)^{\nu)\rho}\sum_{n=1}^{\infty}\frac{C_{BS^{2n+1}}}{(2n+1)!}\left(\frac{q\cdot S(p_1)}{m}\right)^{2n}$$

- Many useful properties, including double-copy structure
- Kerr black hole: all unit coefficients from comparison w/ Kerr stress tensor of Vines

Same general strategy in presence of spin:

Bern, Luna, RR, Shen, Zeng

- Tree-level and 1-loop amplitudes to $\mathcal{O}(S_1 S_2)$:

$$\frac{\mathcal{M}_4^{\text{tree}}}{4E_1E_2} = \frac{4\pi G}{\boldsymbol{q}^2}\left[a_1^{(0)} + i\sum_{j=1}^{2} a_1^{(1,j)}(\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{S}_j + a_1^{(2,1)}\boldsymbol{q}\cdot\boldsymbol{S}_1\,\boldsymbol{q}\cdot\boldsymbol{S}_2\right]$$

coefficients in paper

rest-frame spin vectors

$$\frac{\mathcal{M}_4^{1\text{ loop}}}{4E_1E_2} = \frac{2\pi^2G^2}{|\boldsymbol{q}|}\left[a_2^{(0)} + i\sum_{j=1}^{2} a_2^{(1,j)}(\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{S}_j + a_2^{(2,1)}\boldsymbol{q}\cdot\boldsymbol{S}_1\,\boldsymbol{q}\cdot\boldsymbol{S}_2 + a_2^{(2,2)}\boldsymbol{q}^2\,\boldsymbol{S}_1\cdot\boldsymbol{S}_2 + a_2^{(2,3)}\boldsymbol{q}^2\,\boldsymbol{p}\cdot\boldsymbol{S}_1\,\boldsymbol{p}\cdot\boldsymbol{S}_2\right]$$
$$- ia_{\text{B}}I_{\text{B}} - ia_{\bar{\text{B}}}I_{\bar{\text{B}}}$$

- Tree-level and 1-loop eikonal phase:

$$\chi_1 = \frac{1}{4m_1m_2\sqrt{\sigma^2-1}}\int\frac{d^{2-2\epsilon}\boldsymbol{q}_\perp}{(2\pi)^{2-2\epsilon}}e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp}\mathcal{M}^{\text{tree}}(\boldsymbol{q}_\perp,\boldsymbol{p})$$

$$\chi_2 = \frac{1}{4m_1m_2\sqrt{\sigma^2-1}}\int\frac{d^{2-2\epsilon}\boldsymbol{q}_\perp}{(2\pi)^{2-2\epsilon}}e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp}\mathcal{M}^{\triangle+\triangledown}(\boldsymbol{q}_\perp,\boldsymbol{p})$$

$$\mathcal{D}_{SL}(f,g) \equiv -\sum_{a=1,2}\epsilon^{ijk}S_a^k\frac{\partial f}{\partial S_a^i}\frac{\partial g}{\partial L^j}$$

- Observables: $\mathcal{O}\in\{\nabla_b, S_1, S_2\}$

$$\Delta_2\mathcal{O} = [\mathcal{O}, i\chi_2] + \frac{1}{2}[-i\chi_1,[\mathcal{O},i\chi_1]] + i\mathcal{D}_{SL}(-i\chi_1,[\mathcal{O},i\chi_1]) + [\mathcal{O},\frac{i}{2}\mathcal{D}_{SL}(i\chi_1,i\chi_1)]$$

aligned spin – agreement w/Guevarra, Ochirov, Vines; PN – agreement w/ Steinhoff, Schäffer, +Hergt, +Hartung

Motion of spinning bodies

Bern, Luna, RR, Shen, Zeng

An all-order conjecture for open-orbit observables:

$$\Delta\mathcal{O} = e^{-i\chi\mathcal{D}}[\mathcal{O}, e^{i\chi\mathcal{D}}]$$

$$\chi\mathcal{D}g \equiv \chi g + \mathcal{D}_{SL}(\chi, g) \qquad \mathcal{D}_{SL}(\chi, g) \equiv -\sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial\chi}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

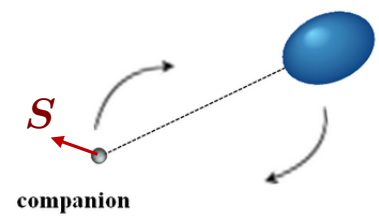
$$\mathcal{M}_{S_1 S_2 \to S_1 S_1} = C e^{i\chi} - 1$$

$\chi \equiv$ conservative eikonal phase of the $S_1 S_2 \to S_1 S_2$ amplitude

- General proof at $\mathcal{O}(G_N)$; explicit check at $\mathcal{O}(G_N^2)$ through orders ($S_1 S_2$, S_1^2, S_2^2)
  recent checks at S_1^1 and S_2^2 Luna, Kosmopoulos vs. Liu, Porto, Yang
- Points to remarkable simplicity and structure in the solution to spinning Hamilton's eqs.
- One might expect that a direct relation between χ and observables exists to all orders
- The eikonal phase or its improved version, the radial action $I_r = \int_\gamma p \cdot dx$, are generating functions of open-orbit observables

Spinning and nonspinning particles interacting with tidal deformations

Large literature on tidal deformations, mainly from world line perspective



Flanagan, Hinderer; Damour, Nagar; Carney, Wade, Irwin; Bini, Damour; + Geralico; Steinhoff, Hinderer, Buonanno, Taracchini; Henry, Faye, Blanchet; Goldberger, Rothstein;...

- We use a QFT description of tidal effects

Gravitational generalization of linear and nonlinear susceptibilities in electrodynamics

- Lagrangian terms for linear and nonlinear tidal deformations:

Bern, Parra-Martinez, RR, Sawyer, Solon

$$S_{\text{tidal}}^{\text{QFT}} = m \int d^4x \sqrt{-g} \sum_{k\geq 2} \left(\rho_e^{(k)}\, \phi \hat{E}_{\mu_1}{}^{\mu_2} \hat{E}_{\mu_2}{}^{\mu_3} \cdots \hat{E}_{\mu_k}{}^{\mu_1} \phi + \rho_m^{(k)}\, \phi \hat{B}_{\mu_1}{}^{\mu_2} \hat{B}_{\mu_2}{}^{\mu_3} \cdots \hat{B}_{\mu_k}{}^{\mu_1} \phi\right) + \cdots$$

Not all multilinears are independent; use relations to verify calculations and look for structure

see Aoude, Haddad, Helset for relations between multilinears

Example of structure in finite-size effects

Bern, Parra-Martinez, RR, Sawyer, Shen

Consider nonlinear tidal operators: $(E^n) \equiv \phi E_{\mu_1}{}^{\mu_2} E_{\mu_2}{}^{\mu_3} \dots E_{\mu_n}{}^{\mu_1} \phi \quad (B^n) \equiv \phi B_{\mu_1}{}^{\mu_2} B_{\mu_2}{}^{\mu_3} \dots B_{\mu_n}{}^{\mu_1} \phi$

- *E* and *B* have rank 3, reduced from 4 – not all are independent!

$$(E^n) = n \sum_{2p+3q=n} \frac{1}{2^p 3^q} \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q+1)} (E^2)^p \, (E^3)^q$$

- E^2, B^2 and E^3 have the same structure; for scattering off a spinless particle:

$$(\mathcal{O}) \in \{(E^2), (B^2), (E^3)\} : (\mathcal{O}) = \frac{1}{r^h} \left(a_{(\mathcal{O})} + b_{(\mathcal{O})} \frac{(\boldsymbol{r} \cdot \boldsymbol{u}_1)^2}{\boldsymbol{r}^2} + c_{(\mathcal{O})} \frac{(\boldsymbol{r} \cdot \boldsymbol{u}_1)^4}{\boldsymbol{r}^4} \right)$$

Potential/Hamiltonian and eikonal phase at leading order:

$$V_{(\mathcal{O})^n}(\boldsymbol{p}, \boldsymbol{r}) = -\frac{\mathcal{N}_{(\mathcal{O})^n}}{4E_1 E_2 \, |\boldsymbol{r}|^{nh}} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} a_{\mathcal{O}}^{n-k} \, b_{\mathcal{O}}^{l} \, c_{\mathcal{O}}^{k-l} \, (\sigma^2 - 1)^{2k-l} \frac{\Gamma(\frac{1}{2} + 2k - l)\Gamma(\frac{1}{2}hn)}{\sqrt{\pi}\Gamma(2k - l + \frac{1}{2}hn)} \, ,$$

$$\chi(\mathcal{O})^n(\boldsymbol{p}, \boldsymbol{b}) = \frac{\mathcal{N}_{(\mathcal{O})^n}}{4m_1 m_2 \, |\boldsymbol{b}|^{nh-1}} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} a_{\mathcal{O}}^{n-k} \, b_{\mathcal{O}}^{l} \, c_{\mathcal{O}}^{k-l} \, (\sigma^2 - 1)^{2k-l-1/2} \frac{\Gamma(\frac{1}{2} + 2k - l)\Gamma(\frac{1}{2}(hn-1))}{\Gamma(2k - l + \frac{1}{2}hn)}$$

agreement with world-line type approach of Cheung, Solon

Summary and outlook

Amplitudes – new powerful ways to look at gravitational problems with and without spin

- pushed state of the art for spinless interaction potential calculations to 3PM and 4PM
- pushed state of the art for spinning 2PM Hamiltonians
- a method to computing radiative observables
- a systematic approach to finite-size effects

- provide inspiration to find hidden structure and hidden simplicity
- close relation between amplitude and radial action
- analytic dependence of observables on velocity
- spinning eikonal conjecture; explicit tests through quadratic order in spin
- interactions of spinning and spinless particles and tidal deformations

Summary and outlook

- Applications of amplitudes/particle physics methods to GW physics are only at the beginning

- Many immediate and longer term questions, both conceptual and technical/computational

  E.g. closer connection between open-orbit and closed-orbit observables w/ radiation

- Methods and techniques are not exhausted and can be further improved

  E.g. at least one more order is accessible

- Methods (may) have applications to other areas of gravitational physics

Future looks bright; Expect renewed progress in the future