Wess-Zumino-Witten models
and their correspondence to Chern-Simons theories

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DAMTP Part III essay - GitHub

Introduction

- Quantum Field Theories (QFTs) are mostly **perturbative** (Cons: confinement, mass gap in QCD, etc.)
- Non-perturbative QFTs: (mostly 2D) Conformal Field Theories (CFTs) & Topological QFTs.
- Rational 2D-CFTs [Belavin, Polyakov, Zamolodchikov '84]:
  - Use **infinite-dimensional symmetries** to reduce infinite problems to effective finite ones.
  - Hall-mark of chiral symmetry structures that realise sub-algebras of **Virasoro algebra**.
- An extended model of 2D-CFTs, realising **affine Kac-Moody algebra** (the simplest extension of semi-simple Lie algebras), namely **Wess - Zumino - (Novikov) - Witten (WZW) models**.

Contents

- Classical WZW actions = Non-linear sigma model + Wess - Zumino term
- Quantum theory - Current algebra (OPE)
- Affine Kac-Moody algebra
- Commutation relations for conserved currents
- Sugawara construction
- The algebra from language of “modes”
- Chern-Simons theories & correspondence to WZW models
- Further applications & Outlooks

Classical theory

Nonlinear sigma model

$g \equiv g(z, \bar{z}) : S^2 \to G$: element of semi-simple Lie group

$$S_0 = \frac{1}{4\lambda^2} \int_{S^2} \mathrm{d}^2 x \mathrm{tr}(g^{-1}\partial_\mu g g^{-1} \partial^\mu g).$$

Classical E.O.M.:

$$\partial_\mu (g^{-1} \partial^\mu g) = 0.$$

Conserved current $J^\mu = g^{-1}\partial^\mu g$ which equivalent to $J'^\mu = \partial^\mu g g^{-1}$,

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0.$$

Traceless tensor in CFT

$\Rightarrow$ These 2 terms varnish.

0

$$\begin{cases} \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} \cancel{T_{z\bar{z}}} = 0, \\ \partial_{\bar{z}} T_{zz} + \cancel{\partial_z T_{\bar{z}z}} = 0. \end{cases}$$

Problem: Not expected to vanish in general non-Abelian Lie algebra.

$\rightarrow$ Not separately conserved and the theory does not have (anti-)holomorphic currents.

Classical theory

Wess - Zumino (WZ) term

left-right invariant 3-form

$\tilde{g} : B^3 \to G$.

$$\Gamma[\tilde{g}] = i \int_{B^3} \tilde{g}^* \omega = -\frac{i}{12\pi} \int_{B^3} \mathrm{tr}(\tilde{g}^{-1} \mathrm{d}\tilde{g} \wedge \tilde{g}^{-1} \mathrm{d}\tilde{g} \wedge \tilde{g}^{-1} \mathrm{d}\tilde{g}),$$

With $\partial B^3 = S^2$, glueing with another WZ term along their boundaries $\mathscr{M} = (B^3 \sqcup B'^3)/S^2 = S^3$.

S^3 deform into $SU(2)$ subgoup of G.

Homotopy group $\pi_3(G) = \mathbb{Z}$

$$\Delta\Gamma = \Gamma[\tilde{g}]|_{B^3} - \Gamma[\tilde{g}']|_{B'^3} = \Gamma[\tilde{f}]|_{S^3} = 2\pi i.$$

$\tilde{f} : S^3 \to G$, taken to be identity.

Combined action

$$S[g] \equiv S_0[g] + k\Gamma[g], \quad k \in \mathbb{R}.$$

Path-integral

$$Z = \int \mathcal{D}g e^{-S[g]}.$$

$k \in \mathbb{Z}$: Level of model .

Next: WZW action & E.O.M.

Classical theory

WZW action

$$S[g] \equiv S_0[g] + k\Gamma[g], \qquad k \in \mathbb{Z}.$$

Varying action

$$\delta\Gamma = \frac{-i}{4\pi}\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{tr}\left(g^{-1}\delta g\left(-\bar{\partial}(g^{-1}\partial g) + \partial(g^{-1}\bar{\partial}g)\right)\right). \quad \delta S_0 = \frac{-i}{4\lambda^2}\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{tr}\left(g^{-1}\delta g(\partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\partial g))\right).$$

Classical E.O.M.

$$\partial(g^{-1}\bar{\partial}g)\left(1 + \frac{\lambda^2 k}{\pi}\right) + \bar{\partial}(g^{-1}\partial g)\left(1 - \frac{\lambda^2 k}{\pi}\right) = 0.$$

Choosing $\lambda^2 = \pi/k \ \ (k > 0)$ separately anti(-holomorphic) conserved current $\bar{J} = kg^{-1}\bar{\partial}g,\ J = -k\partial g g^{-1}$.

$$\partial(g^{-1}\bar{\partial}g) = 0 = g\partial(g^{-1}\bar{\partial}g)g^{-1} = \bar{\partial}(\partial g g^{-1}).$$

This exhibits the local $G(z) \times G(\bar{z})$ symmetry

$$g(z,\bar{z}) \mapsto g_L(z)g(z,\bar{z})g_R^{-1}(z).$$

Quantum theory - Current algebra

Quantization: Consider Operator Product Expansion (OPE) structure instead of canonical quantization

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p}.$$

constrain by current commutation and association

Conformal dimension $2-p$ is non-negative

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^a(w)}{z-w}.$$

OPE for (anti-)holomorphic current

"level"

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w},$$

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^{ab}{}_c \bar{J}^c(\bar{w})}{\bar{z}-\bar{w}}.$$

Affine Kac-Moody algebra

Under group multiplication, we derive **"Polyakov-Wiegmann identity"**:

$$S[gh] = S[g] + S[h] + W_k[g,h].$$

With convention of $W_k[g,h] \equiv kW_1[g,h] \equiv \frac{k}{8\pi}\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{tr}\left(g^{-1}\bar{\partial}g \wedge \partial h h^{-1}\right).$

W_k transform as a 2-cocycle:

$$S[(gh)l] = S[g(hl)]$$
$$\iff W_k[gh,l] + W_l[g,h] = W_k[g,hl] + W_k[h,l].$$

Virasoro algrebra is central extension of Witt algebra

Exact sequence of the algebra & group

$$0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0,$$
$$1 \to U(1) \to \hat{G} \to LG \to 1.$$

Virasoro algebra structure

$$[X \otimes t^n, Y \otimes t^m] = [X,Y] \otimes t^{n+m} + \kappa(X,Y) n \delta_{n+m,0} c.$$

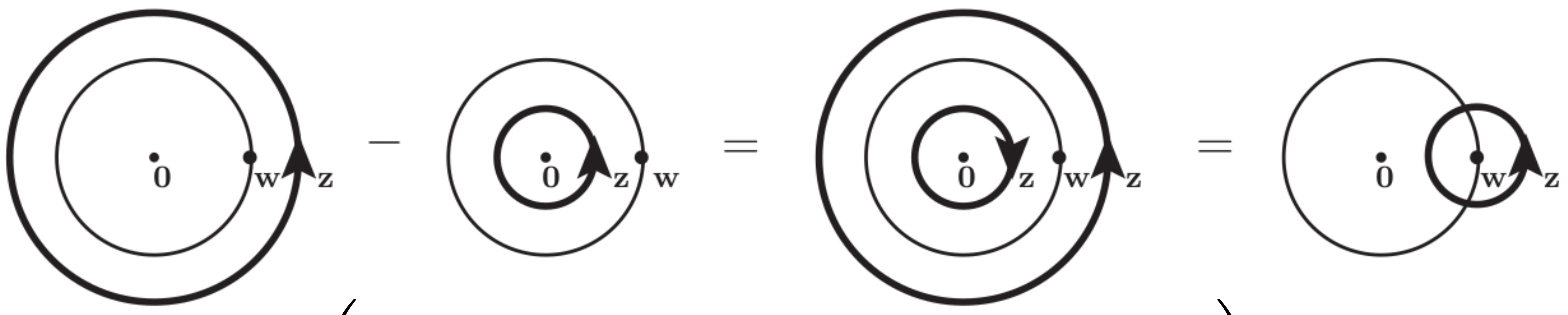
Commutation relations for conserved currents

Laurent expansion around origin

$$J^a(z) \equiv \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}.$$

generalised modes of expansion

$$J^a_n = \oint dz J^a(z) z^n.$$



$$[J^a_n, J^b_m] = \frac{1}{(2\pi i)^2}\left(\oint dz \oint_{|z|>|w|} dw - \oint dz \oint_{|w|>|z|} dw\right) z^n w^m J^a(z) J^b(w)$$

$$= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz z^n w^m \left[\frac{i f^{ab}{}_c J^c(z)}{z-w} + \frac{k\delta^{ab}}{(z-w)^2}\right]$$

$$= i f^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}.$$

Virasoro algebra structure

$$[J^a \otimes s_n, J^b \otimes s_m] = [J^a, J^b] \otimes s_{n+m} + kn\delta^{ab}\delta_{n+m,0}.$$

when n = m = 0, back to finite-dim Lie algebra

$$[J^a_0, J^b_0] = i f^{ab}{}_c J^c_0.$$

Sugawara construction

Energy-momentum tensor

Classical version	Quantum version
$T(z) = \gamma' J^a(z) J^a(z).$	$T(z) = \gamma (J^a J^a)(z) \equiv \frac{\gamma}{2\pi i} \oint_z \frac{\mathrm{d}x}{x-z} J^a(x) J^a(z).$

Coexter number (quad. Casimir of ad. rep.)

$$f^{ab}{}_c f^{bc}{}_d = 2h^\vee \delta^{ad}.$$

TJ OPE:

$$\overbracket{T(z)J^a}(w) = \overbracket{J^a(w)T}(z) \sim 2\gamma(k+h^\vee)\left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\right).$$

$$\gamma \equiv \frac{1}{2(k+h^\vee)}.$$

TT OPE:

$$\overbracket{T(z)T}(w) \sim \frac{k \dim \mathfrak{g}}{2(k+h^\vee)(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}.$$

Central charge

$$c \equiv \frac{k \dim \mathfrak{g}}{k+h^\vee}.$$

The algebra from language of “modes”

Recall: Commutation relation for conserved currents

$$[J^a_n, J^b_m] = i f^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}.$$

Normal ordering term

$$L_n \equiv \gamma \left(\sum_{m\leq -1} J^a_m J^a_{n-m} + \sum_{m\geq 0} J^a_{-m} J^a_m \right) = \gamma \sum_{m\in\mathbb{Z}} : J^a_m J^a_{n-m} : .$$

A good exercise: The commutation relation for conserved currents

$$[L_n, J^a_m] = -2\gamma(k+h^\vee) m J^a_{n+m} = -m J^a_{n+m}.$$

Virasoro algebra (Vir)

$$[L_n, L_m] = \gamma \left[L_n, \sum_{l\leq -1} J^a_l J^a_{n-l} + \sum_{l\geq 0} J^a_{n-l} J^a_l \right]$$

$$= (n-m) L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}.$$

$$c = \frac{k \dim \mathfrak{g}}{k+h^\vee} = 2k\gamma \dim(\mathfrak{g}).$$

→ Affine Kac-Moody algebra is the Lie ideal of the universal enveloping algebra: Vir $\ltimes$ $\hat{\mathfrak{g}}_k$.

Free field construction

At quantum level, n-free fermions is just the same theory to $\mathfrak{so}(n)_1$ of the WZW model

Action of n massless free fermion with $SO(N)$ symmetry:

$$S = \int_{S^2} \mathrm{d}^2 z \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i, \quad (i = 1, ..., n).$$

Generator of fun. rep. of $\mathfrak{so}(n)$

(Anti-)holomorphic conserved current

$$J^a(z) = \frac{1}{2} T^a_{ij} (\psi^i \psi^j)(z), \qquad \bar{J}^a(\bar{z}) = \frac{1}{2} T^a_{ij} (\bar{\psi}^i \bar{\psi}^j)(\bar{z}).$$

OPE

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w}. \qquad J^a(z)\psi^j(w) = -\frac{T^a_{lj}\psi^j(w)}{z-w}.$$

In $\mathfrak{so}(n)_1$, it is $\dfrac{(n-1)n}{n(n-1)} = 1$

$$J^a(z)J^b(w) \sim \boxed{\frac{\mathcal{C}(\lambda)\dim(\lambda)}{2\dim(\mathfrak{g})}} \frac{\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w},$$

→ “Non-abelian bosonization” transform local fermionic to bosonic fields & retained all symmetries.

Chern-Simons theory

2+1 dim Chern-Simons (CS) theory:

A: Connection of the principal G-bundle on $\mathscr{M}$

since, $\pi(G) = \mathbb{Z}$,
gauge invariant $\Rightarrow k \in \mathbb{Z}$

$$S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right).$$

Canonical Quantization:

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{tr}(\epsilon^{ij} A_i \partial_t A_j).$$

$$\left[A_i^a(x), A_j^b(y)\right] = \frac{\pi i}{k}\delta^{ab}\epsilon_{ij}\delta^2(x-y).$$

$\mathcal{M} = \mathbb{R} \times \Sigma$ (time x 2D surface) form of $S = \int p_i \partial_t q^i$

$$\left[A_z^a(z_1, \bar{z}_1), A_{\bar{z}}^b(z_2, \bar{z}_2)\right] = \frac{\pi}{k}\delta^{ab}\delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2).$$

Hilbert space $\mathscr{H}$ of all holomorphic functional $\Psi(A_{\bar{z}})$ & its functional derivatives of A_z

$$A_z^a \Psi(A_{\bar{z}}) = \frac{\pi}{k}\frac{\delta}{\delta A_{\bar{z}}^a}\Psi(A_{\bar{z}}).$$

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int \mathrm{d}t \int \mathrm{tr}(A_z \partial_t A_{\bar{z}} + A_t F_{z\bar{z}}).$$

Identity

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right)\Psi(A_z) = \frac{k}{\pi}\partial_z A_{\bar{z}}\Psi(A_z).$$

Correspondence to WZW model

WZW partition function

$$Z[A] = \left\langle \exp\left(\frac{1}{\pi} \int A^b_{\bar{w}} J^b_w \right) \right\rangle_{\text{WZW}}.$$

From the known $J^a(z)J^b(w)$, we derive $\partial_{\bar{z}} J^a(z)J^b(w)$ and use relation

$$\frac{\delta}{\delta A^a_{\bar{z}}} Z[A] = \frac{1}{\pi} J^a(z) Z[A].$$

We derive

Identity

$$\boxed{\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right] \right) \Psi(A_z) = \frac{k}{\pi} \partial_z A_{\bar{z}} \Psi(A_z).}$$

→ Hilbert states of CS theory connect to generating function (for a given source) of WZW models.

→ Exploit it to calculate CS observables (e.g. n-Wilson loops on S^3) using WZW models.

Further applications & Outlooks

- Other aspects (refer to my essay - GitHub):
  - Representation (Primary fields, Verma modules, why called “Rational” CFT?)
  - Knizhnik–Zamolodchikov (KZ) equations (4-point correlation function)
  - Mathematical construction of Affine Kac-Moody algebra - in the Appendix
- Further developments:
  - $SL(2,\mathbb{R})/U(1)$ gauged WZW model as Witten's 2-dim Euclidean black holes [Witten '92]
  - WZW models of universal cover group of $SL(2,R)$ describes bosonic strings on AdS_3 [Maldacena & Oorugi '01], [Maldacena et. al '01], [Maldacena & Oorugi '02]
  - Plateau transition in the integer quantum Hall effect [Zirnbauer '19]