UV and IR divergence cancellation in elastic scattering QED process

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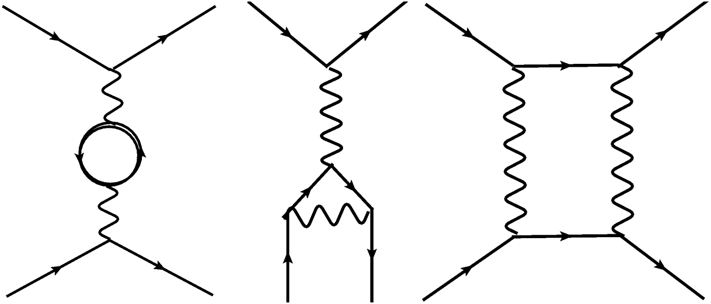

From left to right : Vacuum diagram, Vertex correction diagram, Box diagram. The LSZ reduction formula :

$$i\mathcal{M}_{total} = \prod_{i=1}^{4} \sqrt{\tilde{Z}_i} \left(i\mathcal{M}_{LO} + i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd} \right), \quad (1)$$

One-loop integrals

One out of the one-loop integrals we will encounter :

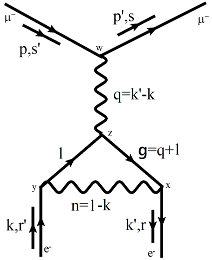

Figure 1: The Vertex correction diagrams

Vertex diagram's amplitude :

$$
\begin{aligned}
i\mathcal{M}_{1,vc} &= \bar{u}_e^r(k')(-e^3)\int \frac{d^4 l}{(2\pi)^4} g_{\rho\nu} \frac{\gamma^\nu(\not{g}+m_e)\gamma^\alpha(\not{l}+m_e)\gamma^\rho}{n^2(g^2-m_e)(l^2-m_e)} u_e^{r'}(k)\frac{-ig_{\alpha\beta}}{q^2}.\bar{u}_\mu^s(p')(-ie\gamma^\beta)u_\mu^{s'}(p) \\
&= e^4\int \frac{d^4 l}{(2\pi)^4}\frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2}\bar{u}_e^r(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u_e^{r'}(k).\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p).
\end{aligned} \tag{2}
$$

Dimensional regularization

At one-loop level, we will however meet some interesting problems of one-loop integrals with UV and IR-divergence. With UV-divergence, we are going to use dimensional regularization to parameterize UV divergent values [1] :

$$\frac{16\pi^2}{i}\int\frac{d^4q}{(2\pi)^4}... \to \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq..., \tag{3}$$

D-dimensional basic integral:

$$I_n(A) = \int d^Dk\frac{1}{(k^2-A+i\epsilon)^n} = i(-1)^n\pi^{D/2}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}(A-i\epsilon)^{D/2-n} \tag{4}$$

Scalar two-point function

$$B_0(p, m_0, m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{(q^2 - m_0^2 + i\epsilon)[(q+p)^2 - m^2 + i\epsilon]}, \quad (5)$$

using Feynman parametrization :

$$\to B_0(p, m_0, m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \int_0^1 dx \frac{1}{\left[(q+xp)^2 - x^2p^2 + x(p^2 - m^2 + m_0^2) - m_0^2 + i\epsilon\right]^2} \quad (6)$$

$$= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int_0^1 \int \underbrace{d^D q' \frac{1}{\left(q'^2 - A + i\epsilon\right)^2}}_{=I_2(A)} \qquad \text{set}: \ q' = q + xp \quad (7)$$

$$= (4\pi\mu^2)^{\frac{4-D}{2}} \Gamma\left(\frac{4-D}{2}\right) \int_0^1 dx \left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right]^{\frac{D-4}{2}} \quad (8)$$

Scalar two-point function

when $D \to 4$:

$$\left(4\pi\mu^2\right)^{\frac{4-D}{2}} = 1 + \frac{4-D}{2}\log\left(4\pi\mu^2\right) + O((D-4)^2), \tag{9}$$

$$\Gamma\left(\frac{4-D}{2}\right) = \frac{2}{4-D} - \gamma_E + O(D-4), \tag{10}$$

$$\left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right]^{\frac{D-4}{2}} = 1 + \frac{D-4}{2}\log\left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right] \tag{11}$$

$$\Rightarrow (4\pi\mu^2)^{\frac{4-D}{2}}\Gamma\left(\frac{4-D}{2}\right)\left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right]^{\frac{D-4}{2}} \tag{12}$$

$$= \frac{2}{4-D} - \gamma_E + \log\left(4\pi\mu^2\right) - \log\left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right] \qquad \gamma_E = -\Gamma'(1) \tag{13}$$

$$\Rightarrow B_0(p, m_0, m) = \triangle - \int_0^1 dx \log\left[\frac{x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon}{\mu^2}\right] + O(D-4) \tag{14}$$

Scalar two-point function

Specific case $m_0 = 0$, we get:

$$B_0(p,0,m) = \triangle - \int_0^1 dx \log\left[\frac{x^2p^2 - x(p^2-m^2) - i\epsilon}{\mu^2}\right] \quad (15)$$

$$= \triangle + \log\left(\mu^2\right) + 1 - \int_0^1 dx \log\left[xp^2 - p^2 + m^2 - i\epsilon\right] \quad (16)$$

$$= \triangle + \log\left(\mu^2\right) + 1 - \frac{1}{p^2}\int_{m^2-p^2-i\epsilon}^{m^2-i\epsilon} dx \log[x] \quad (17)$$

$$= \triangle + \log\left(\mu^2\right) + 1 - \frac{1}{p^2}\left[x\log(x)\Big|_{m^2-p^2-i\epsilon}^{m^2-i\epsilon} - x\Big|_{m^2-p^2-i\epsilon}^{m^2-i\epsilon}\right] \quad (18)$$

$$= \triangle + \log\left(\frac{\mu^2}{m^2}\right) + 2 + \frac{m^2-p^2}{p^2}\log\left(\frac{m^2-p^2-i\epsilon}{m^2}\right) \quad (19)$$

UV divergent parts

UV divergent parts of N-point functions : (Represent UV divergent term by $\triangle$)

- $A_0(m) = m^2\triangle$,
- $B_0(p^2, m_0, m_1) = \triangle$,
- $B_\mu(p, m_0, m_1) = \frac{-1}{2} p_\mu \triangle,$
- B_1 $(B_\mu = p_\mu . B_1) = \frac{-1}{2}\triangle$,
- $B_{\mu\nu}(p, m_0, m_1) = \frac{-g_{\mu\nu}}{12}\left[p^2 - 3(m_0^2 + m_1^2)\right]\triangle + \frac{p_\mu p_\nu}{3}\triangle,$
- $C_{\mu\nu}(p, p', m_0, m_1, m_2) = g_{\mu\nu} C_{00}(p, p', m_0, m_1, m_2) = \frac{g_{\mu\nu}}{4}\triangle,$

where $\triangle = \frac{2}{4-D} - \gamma_E + 1$, D is the dimensions of the loop integrals and γ_E is the Euler constant.

The UV divergence of total amplitude

- The tree-level

$$i\mathcal{M}_0 \sim 1(\alpha).$$

- The Vacuum polariztion

$$i\mathcal{M}_{vp} \sim \frac{-2}{3}\triangle(\alpha^2).$$

- The Vertex correction

$$i\mathcal{M}_{vc} \sim \frac{1}{2}\triangle(\alpha^2).$$

- The Box diagrams

$$i\mathcal{M}_{bd} \sim \text{UV-convergent}(\alpha^2).$$

- LSZ factor

$$\tilde{Z}_i \sim 1 - \frac{1}{4}\triangle(\alpha).$$

The UV divergence of total amplitude

Total one-loop amplitude :

$$
\begin{aligned}
i\mathcal{M}_{total} &= \prod_{i=1}^{4} \sqrt{\tilde{Z}_i}(i\mathcal{M}_0 + i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd}) \\
&\sim [1 - \frac{1}{2}\triangle(\alpha)]. \left[1(\alpha) - \frac{2}{3}\triangle(\alpha^2) + \frac{1}{2}\triangle(\alpha^2)\right] \\
&\sim \frac{-2}{3}\triangle(\alpha^2) + \frac{1}{2}\triangle(\alpha^2) - \frac{1}{2}\triangle(\alpha^2) \sim \frac{-2}{3}\triangle(\alpha^2).
\end{aligned} \tag{20}
$$

Renormalization

To resolve the UV-singularities, we have to use an extra procedure, the Renormalization method, first of all, we must renormalize the QED Lagrangian.

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e_0\bar{\psi}_0 A\!\!\!/\psi_0 \\ \rightarrow \mathcal{L}_R &= Z_\psi\bar{\psi}(i\partial\!\!\!/ - Z_m.m)\psi - \frac{1}{4}Z_A F^{\mu\nu}F_{\mu\nu} - Z_e Z_\psi\sqrt{Z_A}e\bar{\psi}A\!\!\!/\psi, \end{aligned} \tag{21}$$

with :

$$\begin{cases} \psi_0 = \sqrt{Z_\psi}\psi = \sqrt{1+\delta_\psi}\psi \\ A_0^\mu = \sqrt{Z_A}A^\mu = \sqrt{1+\delta_A}A^\mu \\ m_0 = Z_m.m = m + \delta_m \\ e_0 = Z_e.e = e + \delta_e \end{cases}, \tag{22}$$

where we have expanded perturbatively at one-loop order $Z_i = 1 + \delta_i(\alpha)$.

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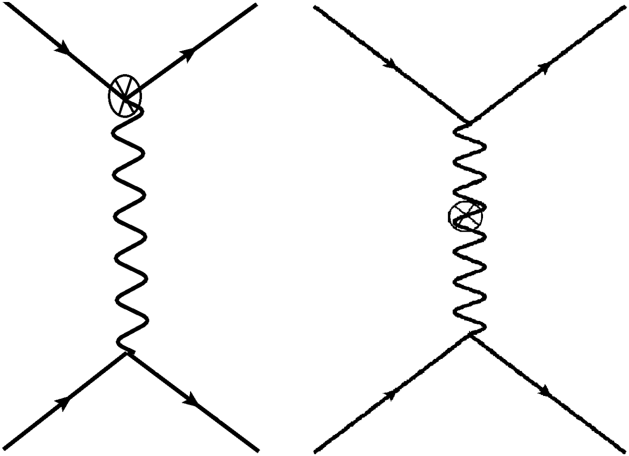
where we have expanded perturbatively at one-loop order $Z_i = 1 + \delta_i(\alpha)$.

Renormalized Lagrangian

The renormalized Lagrangian reads :

$$\begin{aligned} \mathcal{L}_R &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}A\!\!\!/\psi \\ &- \bar{\psi}\delta_m\psi + \delta_\psi\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}A\!\!\!/\psi(\delta_e + \delta_\psi + \frac{1}{2}\delta_A) \\ &= \mathcal{L}_R^0 + \mathcal{L}_{counterterm}. \end{aligned} \tag{23}$$

Counterterm diagram



UV cancellation

- Counterterm amplitude :

$$i\mathcal{M}_{ct} \sim \frac{1}{6}\triangle(\alpha^2). \tag{24}$$

- LSZ factors after Renormalization

$$\begin{aligned}\Rightarrow \tilde{Z}_p &= 1 - \frac{d\hat{\Sigma}^{ff}(\not{p})}{d\not{p}}|_{\not{p}=m} = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}|_{\not{p}=m} - \delta_\psi = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}|_{\not{p}=m} + 2m\frac{\partial\Sigma^{ff}(p)}{\partial p^2}|_{\not{p}=m} \\ &= 1 - \frac{d\Sigma^{ff}(d\not{p})}{d\not{p}}|_{\not{p}=m} + \frac{\partial\Sigma^{ff}(p)}{\partial p^2}\frac{2\not{p}\partial\not{p}}{\partial\not{p}}|_{\not{p}=m} = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}|_{\not{p}=m} + \frac{\partial\Sigma^{ff}(p)}{\partial p^2}\frac{\partial p^2}{\partial\not{p}}|_{\not{p}=m} = 1\end{aligned}. \tag{25}$$

The UV divergence of total amplitude after Renormalization :
Combining the results of renormalized LSZ factors and additional Feynman counterterm amplitude into our previous amplitude Eq.(20) to get UV convergent amplitude after renormalization :

$$\begin{aligned} i\mathcal{M}_{total} &= \prod_{i=1}^{4}\sqrt{\tilde{Z}_i}(i\mathcal{M}_0 + i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd} + i\mathcal{M}_{ct}) \\ &\sim 1.\left[1(\alpha) - \frac{2}{3}\triangle(\alpha^2) + \frac{1}{2}\triangle(\alpha^2) + \frac{1}{6}\triangle(\alpha^2)\right] \sim 0\triangle \rightarrow \textbf{UV convergent}. \end{aligned} \tag{26}$$

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IR divergence of NLO differential cross section

- The Vacuum polariztion

$$i\mathcal{M}_{vp} \sim \text{IR convergent.}$$

- The Vertex correction

$$i\mathcal{M}_{vc} = \frac{e^2}{4\pi^2}\left[-2k'.k\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{\lambda}{m_e}\right) - 2p'.p\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{\lambda}{m_\mu}\right)\right].i\mathcal{M}_{LO}.$$

- The Box diagrams

$$i\mathcal{M}_{bd} = \frac{e^2}{4\pi^2}\left[-2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{\lambda^2}{-q^2-i\epsilon}\right)\right.$$
$$\left.-2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{\lambda^2}{-q^2-i\epsilon}\right)\right].i\mathcal{M}_{LO}.$$

- The counterterm diagrams

$$i\mathcal{M}_{ct} = \frac{e^2}{4\pi^2}\left[-\log\frac{\lambda}{m_e} - \log\frac{\lambda}{m_\mu}\right].i\mathcal{M}_{LO}.$$

IR divergence of NLO differential cross section

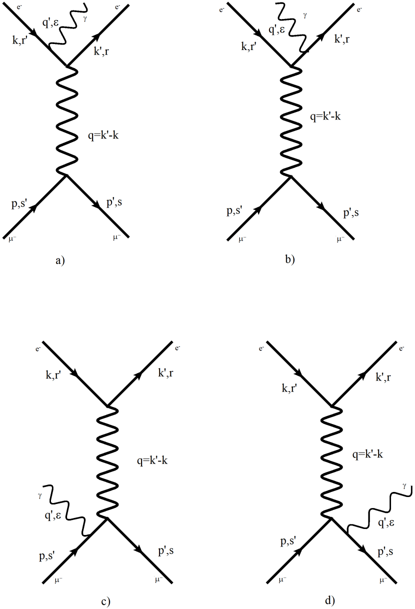
The total virtual differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{Virt}}_{\text{NLO}} = \frac{e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{\text{LO}}$$

$$\times\, 2Re\left[-2k'.k\,\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{\lambda}{m_e}\right) - 2p'.p\,\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{\lambda}{m_\mu}\right) - \log\frac{\lambda}{m_e} - \log\frac{\lambda}{m_\mu}\right.$$

$$\left.-2k'p'\,\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{\lambda^2}{-q^2-i\epsilon}\right) - 2k'p\,\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{\lambda^2}{-q^2-i\epsilon}\right)\right](\alpha^3). \tag{27}$$

Photon radiation



Photon radiation

For the calculation at NLO, we have to include the emission of one additional photon. We split this real emission process into two parts as follows :

$$d\sigma_{real}(\alpha^3) = d\sigma_{Soft}(\alpha^3) + d\sigma_{hard}(\alpha^3), \tag{28}$$

where the soft-photon region is defined by $E_\gamma \leq \triangle E$ with $\triangle E$ being a cutoff parameter. The value of $\triangle E$ must be very small compared to the colliding energy.

Soft-photon corrections

Because of the electrons and muons are charged particles, they always emit photons (electromagnetic radiation). The photon emission is therefore an essential part of QED scattering processes. The $e^-\mu^- \to e^-\mu^-$ scattering without photon emission is actually unphysical and we can't observe this process separately.

In soft-photon emission, we neglect the momentum q' of the radiative photon everywhere except in the denominator of the fermion propagator. We then get the following result for the differential cross section :

$$\left(\frac{d\sigma}{d\Omega}\right)_{Soft} = -\left(\frac{d\sigma}{d\Omega}\right)_{\text{LO}} \cdot \frac{e^2}{(2\pi)^3} \int_{|\mathbf{q'}| \leq \triangle E} \frac{d^3 q'}{2\omega_{q'}} \left[\frac{k^2}{(q'k)^2} + \frac{k'^2}{(q'k')^2} - \frac{2kk'}{q'k.q'k'} + \frac{p^2}{(q'p)^2} + \frac{p'^2}{(q'p')^2} - \frac{2pp'}{q'k.q'k'} \right.$$
$$\left. + 2\text{Re}\left(\frac{p'k'}{p'q'.k'q'} - \frac{p'k}{p'q'.kq'} - \frac{pk'}{pq'.k'q'} + \frac{pk}{pq'.kq'} \right) \right] (\alpha^3), \quad (29)$$

with $\omega_{q'} = \sqrt{|\vec{q}'|^2 + \lambda^2}$, where λ is the photon mass.

IR divergent part of the *Soft-photon radiation*

The total IR divergent part of the *Soft-photon radiation* differential cross section reads :

$$\left(\frac{d\sigma}{d\Omega}\right)_{Soft} = \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{LO}.Re\left\{4\log\left(\frac{2\triangle E}{\lambda}\right) + 4kk'\frac{x_{te}}{m_e^2(1-x_t^2)}\log(x_{te})\log\left(\frac{2\triangle E}{\lambda}\right)\right.$$
$$+ 4pp'\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{2\triangle E}{\lambda}\right) + 2\left[2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right. \quad (30)$$
$$\left.\left. +2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right]\right\}(\alpha^3).$$

We can see that the cross section of soft photon radiation process as IR divergent as the virtual corrections, but with a sign difference.

IR convergent cross section

The IR-divergent part of the NLO cross section reads :

$$\left(\frac{d\sigma}{d\Omega}\right)^{IR}_{\text{NLO}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Virt}} + \left(\frac{d\sigma}{d\Omega}\right)_{Soft} = \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{\text{LO}} Re\left[\log\left(\frac{2\triangle E}{m_e}\right)^2 + \log\left(\frac{2\triangle E}{m_\mu}\right)^2 \right.$$
$$+4kk'\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{2\triangle E}{m_e}\right) + 4pp'\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{2\triangle E}{m_\mu}\right) \quad (31)$$
$$\left. +\ 4k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{4\triangle E^2}{-q^2-i\epsilon}\right) + 4k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{4\triangle E^2}{-q^2-i\epsilon}\right)\right](\alpha^3),$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{NLO}} \text{ is IR convergent .}$$

Next-to-leading order cross section

Finally, we define here the NLO cross section :

$$
\begin{aligned}
d\sigma_{NLO} &= d\sigma_{LO}(\alpha^2) + d\sigma_{virt}(\alpha^3) + d\sigma_{soft}(\alpha^3). \\
&= f(s, t, u, m_e, m_\mu, \triangle E, A_0, B_0, C_0, D_0, ...)
\end{aligned}
\tag{32}
$$

Conclusion

We have successful cancelled out all divergences occurring at next-to-leading order, UV divergence is cancelled by renormalization and IR divergence by adding soft-photon corrections. We note that the photon radiation is an indispensable part of the scattering process of charged particles.

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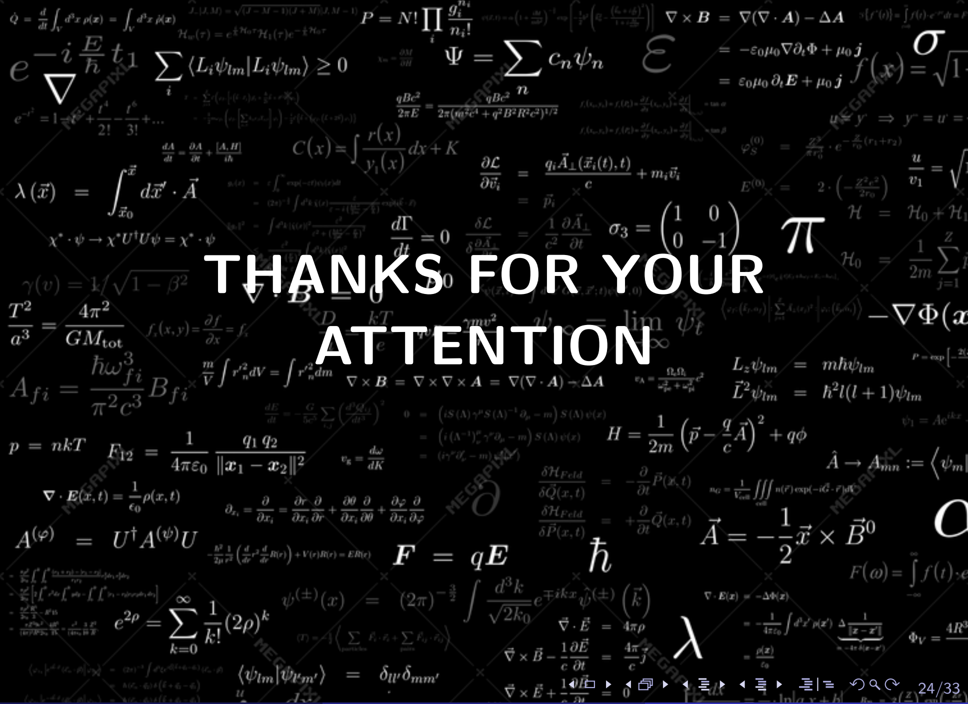

THANKS FOR YOUR ATTENTION

Renormalization conditions

These conditions require that those renormalized fuctions have a tree-level form in the on-shell limit ($p^2 = m^2$). This is the on-shell renormalization sheme :

- Condition 1 - Dirac equation :

$$\tilde{Re}\hat{\Gamma}^{ff}(p)u(p)|_{p^2=m^2} = 0$$

$$\Rightarrow \delta_m = \tilde{Re}\Sigma^{ff}(m) = \frac{e^2}{8\pi^2}\left[mB_1(m^2, 0, m) - mB_0(m^2, 0, m) + \frac{m}{2}\right].$$

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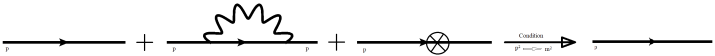
- Condition 2 :



$$\lim_{p^2 \to m^2} \frac{\not{p} + m}{p^2 - m^2} \tilde{Re}\hat{\Gamma}^{ff}(p)u(p) = iu(p)$$

$$\Rightarrow \delta_{\psi_j} = -2m_j \tilde{Re} \frac{\partial \Sigma^{ff}(p)}{\partial p^2}|_{\not{p}=m_j} = -2m_j \frac{\partial \Sigma^{ff}(p)}{\partial p^2}|_{\not{p}=m_j}$$

$$= \frac{m_j e^2}{4\pi^2} \left[-\frac{B_1(m_j^2, 0, m_j)}{2m_j} - \frac{B_0(m_j^2, 0, m_j)}{2m_j} - m_j \frac{\partial B_1(p^2, 0, m_j)}{\partial p^2}\bigg|_{p^2=m_j^2} \right.$$

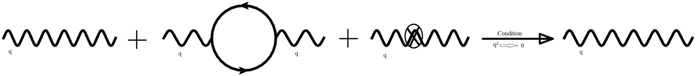
$$\left. + \; m_j \frac{\partial B_0(p^2, 0, m_j)}{\partial p^2}\bigg|_{p^2=m_j^2} + \frac{1}{4m_j} \right].$$

slide

Renormalization conditions

These conditions require that those renormalized fuctions have a tree-level form in the on-shell limit $(p^2 = m^2)$. This is the on-shell renormalization sheme :

- Condition 3:



$$\lim_{q^2 \to 0} \frac{1}{q^2} Re \hat{\Gamma}^{AA}_{\mu\nu}(q)\epsilon^{\nu}(q) = -i\epsilon_{\mu}(q)$$

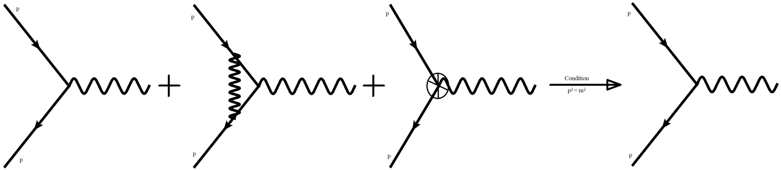
$$\Rightarrow \delta_A = \frac{-e^2}{4\pi^2} \sum_{j=e,\mu} Re \left[-\frac{1}{9} + \frac{2}{3} m_j^2 B_0'(q^2, m_j, m_j)|_{q^2=0} + \frac{B_1(0, m_j, m_j)}{3} + \frac{B_0(0, m_j, m_j)}{2} \right].$$

slide

Renormalization conditions

These conditions require that those renormalized fuctions have a tree-level form in the on-shell limit ($p^2 = m^2$). This is the on-shell renormalization sheme :

- Condition 4 :



$$\bar{u}(p)\hat{\Gamma}^{Aff}_{\mu}(p,p)u(p)\Big|_{p^2=m^2} = -ie\bar{u}(p)\gamma_{\mu}u(p)$$

$$\Rightarrow \delta_e = -\frac{1}{2}\delta_A.$$

slide

Notations

$$A_0(m) = \langle | \left(q^2 - m^2 + i\epsilon\right)^{-1} | \rangle_q \tag{33}$$

$$B_0(p, m_0, m_1) = \langle | \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{34}$$

$$B_\mu(p, m_0, m_1) = \langle | \, q_\mu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{35}$$

$$B_{\mu\nu}(p, m_0, m_1) = \langle | \, q_\mu q_\nu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{36}$$

$$C_0(p, p', m_0, m_1, m_2) = \langle | \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p')^2 - m_2^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{37}$$

$$C_\mu(p, p', m_0, m_1, m_2) = \langle | \, q_\mu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p')^2 - m_2^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{38}$$

$$C_{\mu\nu}(p, p', m_0, m_1, m_2) = \langle | \, q_\mu q_\nu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p')^2 - m_2^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{39}$$

$$D_0(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle | \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p_1)^2 - m_2^2 + i\epsilon\right)\left((q+p_2)^2 - m_3^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{40}$$

$$D_\mu(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle | \, q_\mu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p_1)^2 - m_2^2 + i\epsilon\right)\left((q+p_2)^2 - m_3^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{41}$$

$$D_{\mu\nu}(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle | \, q_\mu q_\nu \left[(q^2 - m_0^2 + i\epsilon)\left((q+p)^2 - m_1^2 + i\epsilon\right)\left((q+p_1)^2 - m_2^2 + i\epsilon\right)\left((q+p_2)^2 - m_3^2 + i\epsilon\right)\right]^{-1} | \rangle_q \tag{42}$$

Vacuum polarization

$$i\mathcal{M}_{1,vp} = \frac{-ie^4}{q^4 4\pi^2}\bar{u}_e^r(k')\gamma^\mu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu u_\mu^{s'}(p)\Big\{2B_{\mu\nu}(q, m_e, m_e) + q_\mu B_\nu(q, m_e, m_e) + q_\nu B_\mu(q, m_e, m_e) + \left[\frac{q^2}{2}B_0(q^2, m_e, m_e) - A_0(m_e)\right] g_{\mu\nu}\Big\}, \tag{43}$$

and :

$$i\mathcal{M}_{2,vp} = \frac{-ie^4}{q^4 4\pi^2}\bar{u}_e^r(k')\gamma^\mu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu u_\mu^{s'}(p)\Big\{2B_{\mu\nu}(q, m_\mu, m_\mu) + q_\mu B_\nu(q, m_\mu, m_\mu) + q_\nu B_\mu(q, m_\mu, m_\mu) + \left[\frac{q^2}{2}B_0(q^2, m_\mu, m_\mu) - A_0(m_\mu)\right] g_{\mu\nu}\Big\}. \tag{44}$$

Vertex correction

$$i\mathcal{M}_{1,vc} = \frac{ie^4}{q^2 16\pi^2}\bar{u}_e^r(k')\Big\{-2\gamma^\alpha - 2\gamma^\mu\gamma^\alpha\gamma^\nu\Big[C_{\mu\nu}(k,k',0,m_e,m_e) + k_\mu C_\nu(k,k',0,m_e,m_e) + k_\nu C_\mu(k,k',0,m_e,m_e)$$
$$+k_\mu k_\nu C_0(k,k',0,m_e,m_e) + q_\nu C_\mu(k,k',0,m_e,m_e) + q_\nu k_\mu C_0(k,k',0,m_e,m_e)\Big] + 8m_e\Big[C^\alpha(k,k',0,m_e,m_e)$$
$$+k^\alpha C_0(k,k',0,m_e,m_e)\Big] + 4m_e q^\alpha C_0(k,k',0,m_e,m_e) - 2m_e^2\gamma^\alpha C_0(k,k',0,m_e,m_e)\Big\} u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\alpha u_\mu^{s'}(p), \tag{45}$$

and :

$$i\mathcal{M}_{2,vc} = \frac{ie^4}{q^2 16\pi^2}\bar{u}_e^r(k')\gamma_\alpha u_\mu^{r'}(k)\bar{u}_\mu^s(p')\Big\{-2\gamma^\alpha - 2\gamma^\mu\gamma^\alpha\gamma^\nu\Big[C_{\mu\nu}(p,p',0,m_\mu,m_\mu) + p_\mu C_\nu(p,p',0,m_\mu,m_\mu)$$
$$+p_\nu C_\mu(p,p',0,m_\mu,m_\mu) + p_\mu p_\nu C_0(p,p',0,m_\mu,m_\mu) - q_\nu C_\mu(p,p',0,m_\mu,m_\mu) - q_\nu p_\mu C_0(p,p',0,m_\mu,m_\mu)\Big]$$
$$-4m_\mu q^\alpha C_0(p,p',0,m_\mu,m_\mu) + 8m_\mu\Big[C^\alpha(p,p',0,m_\mu,m_\mu) + p^\alpha C_0(p,p',0,m_\mu,m_\mu)\Big]$$
$$-2m_\mu^2\gamma^\alpha C_0(p,p',0,m_\mu,m_\mu)\Big\} u_\mu^{s'}(p). \tag{46}$$

Box diagrams

$$\begin{aligned} i\mathcal{M}_{1,bd} = \frac{ie^4}{16\pi^2} \Big[& 4\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)k'p'D_0(-q,-k',p',0,0,m_e,m_\mu) \\ & + \bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')2\not{k}'\gamma^\alpha\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',p',0,0,m_e,m_\mu) \\ & - \bar{u}_e^r(k')2\not{p}'\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',p',0,0,m_e,m_\mu) \\ & - \bar{u}_e^r(k')\gamma^\mu\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\mu\gamma^\beta\gamma_\nu u_\mu^{s'}(p)D_{\alpha\beta}(-q,-k',p',0,0,m_e,m_\mu)\Big], \end{aligned} \tag{47}$$

and :

$$\begin{aligned} i\mathcal{M}_{2,bd} = \frac{ie^4}{16\pi^2} \Big[& 4\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)k'pD_0(-q,-k',-p,0,0,m_e,m_\mu) \\ & - \bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu\gamma_\alpha 2\not{k}' u_\mu^{s'}(p)D_\alpha(-q,-k',-p,0,0,m_e,m_\mu) \\ & - \bar{u}_e^r(k')2\not{p}\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',-p,0,0,m_e,m_\mu) \\ & + \bar{u}_e^r(k')\gamma^\mu\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu\gamma^\beta\gamma_\mu u_\mu^{s'}(p)D_{\alpha\beta}(-q,-k',-p,0,0,m_e,m_\mu)\Big]. \end{aligned} \tag{48}$$

LSZ factors

- $$\tilde{Z}_p = 1 + \frac{e^2}{16\pi^2}\{[-2B_0(m_\mu, 0, m_\mu) - 2B_1(m_\mu, 0, m_\mu) + 1] + \not{p}\frac{d\Sigma_v(p)}{d\not{p}} + 2m_\mu \frac{d\Sigma_s(p)}{d\not{p}}\}|_{\not{p}=m_\mu}. \tag{49}$$

- $$\tilde{Z}_{p'} = 1 + \frac{e^2}{16\pi^2}\{[-2B_0(m_\mu, 0, m_\mu) - 2B_1(m_\mu, 0, m_\mu) + 1] + \not{p}'\frac{d\Sigma_v(p')}{d\not{p}'} + 2m_\mu \frac{d\Sigma_s(p')}{d\not{p}'}\}|_{\not{p}'=m_\mu}. \tag{50}$$

- $$\tilde{Z}_{k'} = 1 + \frac{e^2}{16\pi^2}\{[-2B_0(m_e, 0, m_e) - 2B_1(m_e, 0, m_e) + 1] + \not{k}'\frac{d\Sigma_v(k')}{d\not{k}'} + 2m_e \frac{d\Sigma_s(k')}{d\not{k}'}\}|_{\not{k}'=m_e}. \tag{51}$$

- $$\tilde{Z}_k = 1 + \frac{e^2}{16\pi^2}\{[-2B_0(m_e, 0, m_e) - 2B_1(m_e, 0, m_e) + 1] + \not{k}\frac{d\Sigma_v(k)}{d\not{k}} + 2m_e \frac{d\Sigma_s(k)}{d\not{k}}\}|_{\not{k}=m_e}. \tag{52}$$

UV divergent parts

UV divergent parts of N-point functions : (Represent UV divergent term by $\triangle$)

- $A_0(m) = m^2\triangle$,
- $B_0(p^2, m_0, m_1) = \triangle$,
- $B_\mu(p, m_0, m_1) = \frac{-1}{2} p_\mu \triangle$,
- B_1 $(B_\mu = p_\mu . B_1) = \frac{-1}{2}\triangle$,
- $B_{\mu\nu}(p, m_0, m_1) = \frac{-g_{\mu\nu}}{12}\left[p^2 - 3(m_0^2 + m_1^2)\right]\triangle + \frac{p_\mu p_\nu}{3}\triangle$,
- $C_{\mu\nu}(p, p', m_0, m_1, m_2) = g_{\mu\nu} C_{00}(p, p', m_0, m_1, m_2) = \frac{g_{\mu\nu}}{4}\triangle$,

where $\triangle = \frac{2}{4-D} - \gamma_E + 1$, D is the dimensions of the loop integrals and γ_E is the Euler constant.

Counterterm diagrams

$$i\mathcal{M}_{ct} = \frac{ie^2}{q^2}\bar{u}_e^r(k')\gamma_\alpha u_\mu^{r'}(k)\bar{u}_\mu^s(p')\gamma^\alpha u_\mu^{s'}(p)\left(2\delta_e + \delta_{\psi_e} + \delta_{\psi_\mu}\right), \tag{53}$$

with the counterterm factors are determined by renormalization conditions:

$$\delta_e = \frac{e^2}{8\pi^2}\sum_{j=e,\mu}\text{Re}\left[\frac{-1}{9} + \frac{2}{3}m_j^2\frac{\partial}{\partial q^2}B_0(q^2, m_j, m_j)\bigg|_{q^2=0} + \frac{B_1(0, m_j, m_j)}{3} + \frac{B_0(0, m_j, m_j)}{2}\right], \tag{54}$$

$$\delta_{\psi_j} = \frac{m_j e^2}{4\pi^2}\left[-\frac{B_1(m_j^2, 0, m_j)}{2m_j} - \frac{B_0(m_j^2, 0, m_j)}{2m_j} - m_j\frac{\partial B_1(p^2, 0, m_j)}{\partial p^2}\bigg|_{p^2=m_j^2} + m_j\frac{\partial B_0(p^2, 0, m_j)}{\partial p^2}\bigg|_{p^2=m_j^2} + \frac{1}{4m_j}\right], \tag{55}$$

$$\delta_A = \frac{-e^2}{4\pi^2}\sum_{j=e,\mu}Re\left[-\frac{1}{9} + \frac{2}{3}m_j^2 B_0'(q^2, m_j, m_j)|_{q^2=0} + \frac{B_1(0, m_j, m_j)}{3} + \frac{B_0(0, m_j, m_j)}{2}\right]. \tag{56}$$

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IR divergent values

We can calculate the IR divergent quantities with a regularized photon mass $\lambda \to 0$:

- $B_0'(m_j^2, \lambda, m_j) = \frac{\partial B_0(p^2,\lambda,m_j)}{\partial p^2}|_{p^2=m_j^2} = -\frac{1}{m_j^2}\log\left(\frac{\lambda}{m_j}\right).$
- $C_0(p, p', \lambda, m_j, m_j) = C_0(m_j^2, t, m_j^2, \lambda, m_j, m_j) = \frac{-2x_{tj}}{m^2(1-x_{tj}^2)}\log\left(x_{tj}\right)\log\left(\frac{\lambda}{m_j}\right),$

  with :

$$x_{tj} = \frac{\sqrt{1-\frac{4m_j^2}{t+i\epsilon}}-1}{\sqrt{1-\frac{4m_j^2}{t+i\epsilon}}+1}.$$

- $D_0(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = \frac{2}{q^2}C_0(m_1^2, (p_1-p_2)^2, m_2^2, \lambda, m_1, m_2) = \frac{-2x_{12}\log(x_{12})}{m_1 m_2 q^2(1-x_{12}^2)}\log\left(\frac{\lambda^2}{-q^2-i\epsilon}\right)$ [2],

  with :

$$x_{12} = \frac{\sqrt{1-\frac{4m_1 m_2}{(p_1-p_2)^2+i\epsilon-(m_1-m_2)^2}}-1}{\sqrt{1-\frac{4m_1 m_2}{(p_1-p_2)^2+i\epsilon-(m_1-m_2)^2}}+1}.$$

- $D_\mu(q, p_1, p_2, \lambda, \lambda, m_1, m_2) \sim q_\mu D_1(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = q_\mu \frac{-C_0(m_1^2,(p_1-p_2)^2,m_2^2,\lambda,m_1,m_2)}{q^2}.$
- $D_{\mu\nu}(q, p_1, p_2, \lambda, \lambda, m_1, m_2) \sim q_\mu q_\nu D_{11}(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = q_\mu q_\nu \frac{C_0(m_1^2,(p_1-p_2)^2,m_2^2,\lambda,m_1,m_2)}{q^2},$