Contents

The Universe is expanding

Redshift

- Almost all of galaxy spectra are redshifted

  Doppler effect → Galaxies are moving away from us

Hubble's law

- Galaxy velocity is proportional to its distance

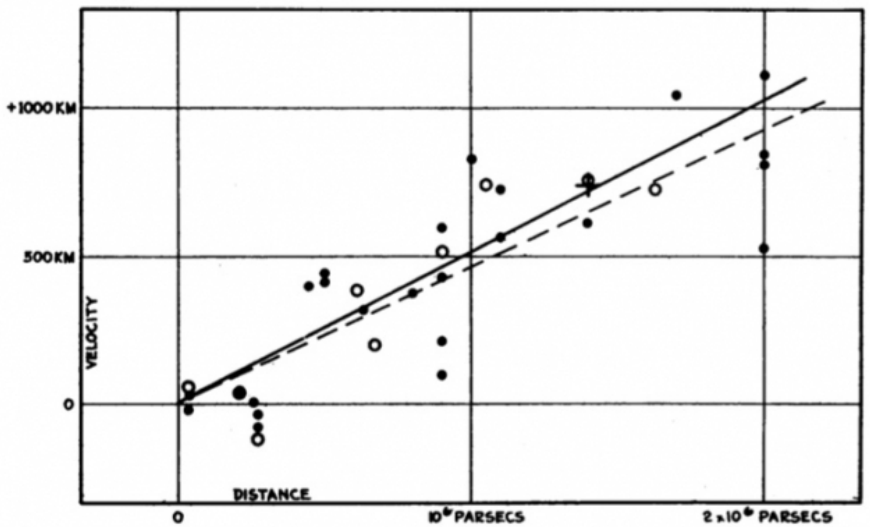
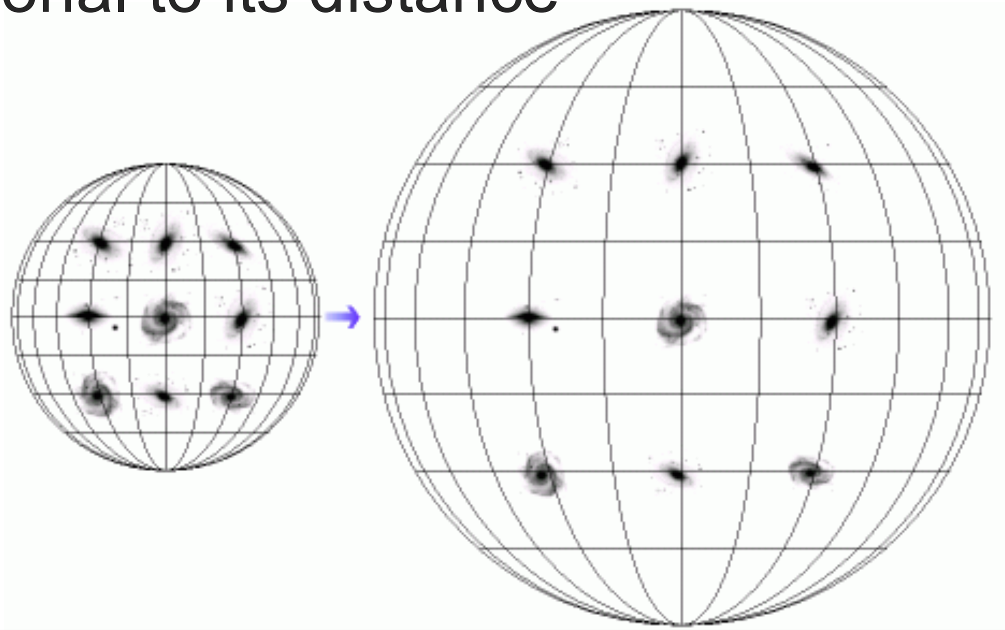


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.



[Hubble, PNAS15 (3), 168 (1929)]

Discovery of universal expansion

Big bang cosmology

Expansion of the Universe

- The early Universe should be extremely small

➡ Extremely high temperature and density : Big bang

Bing bang nucleosynthesis (BBN)

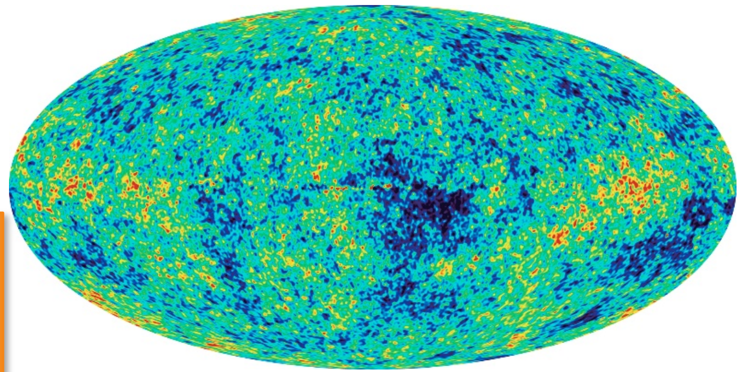
- Light elements are produced in the first few minutes

Comic microwave background (CMB) radiation

- Electrons are captured by nuclears and neutral atoms are formed: Recombination

➡ Photons can travel freely and be observed

Penzias and Wilson discovered the CMB of 3K (1965)



[apod.nasa.gov]

Homogeneous and isotropic Universe

Cosmological principle

- Galaxies, clusters, etc. exist at small scales
- At scales larger than $100\ \mathrm{Mpc}$,
  the Universe is homogeneous and isotropic

Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right)$$

(t, r, θ, ϕ) : Comoving coordinates

$R(t)$: Scale factor; $a(t) = \frac{R(t)}{R_0}$: Dimensionless Scale factor

Curvature constant: $k = \begin{cases} 1 & \text{Closed} \\ 0 & \text{Flat} \\ -1 & \text{Open} \end{cases}$

Flat ← LambdaCDM

Equation of motion

Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

- Homogeneity and isotropy → Perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \qquad \text{w/} \qquad u^\mu = (1,0,0,0)$$

- (0,0) component: $H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3}$ Freidmann eq.

  ↳ Hubble parameter

- (i,i) component: $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$
- Energy conservation ($T^{\mu\nu}{}_{;\nu} = 0$): $\frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}R^3$

  1st law of thermodynamics

- Two of the above three equations are independent

Cosmological parameters

Scaled Hubble parameter h

$$H = 100\ h\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$$

- Observed value: $h \simeq 0.67$

Critical density

- Density such that the Universe is flat

$$\rho_c \equiv \frac{3H^2}{8\pi G} = \frac{3H^2 M_{\mathrm{Pl}}^2}{8\pi} = 1.05 \times 10^{-5} h^2\ \mathrm{GeV\ cm^{-3}}$$

Density parameters

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \qquad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} \equiv \frac{\Lambda/8\pi G}{\rho_c}$$

Solution for equations of state

Equation of state

- Radiation: $p_R = \frac{1}{3}\rho_R$

$\rho_R \propto R^{-4}$ $\qquad R(t) \propto t^{1/2}, \quad H = \frac{1}{2t}$

- Matter: $p_M = 0$

$\rho_M \propto R^{-3}$ $\qquad R(t) \propto t^{2/3}, \quad H = \frac{2}{3t}$

- Vacuum: $p_\Lambda = -\rho_\Lambda$

$\rho_\Lambda = \text{const.}$ $\qquad R(t) \propto e^{\sqrt{\Lambda/3}t}, \quad H = \text{const.}$

- General: $p = w\rho$

$\rho_M \propto R^{-3(1+w)}$ $\qquad R(t) \propto t^{2/[3(1+w)]}$

Thermodynamics

- In the early Universe, particles are in thermal equilibrium in many cases

Distribution function

$$f(p) = \frac{1}{\exp\left[(E-\mu)T\right] \pm 1}$$ w/ $E^2 = p^2 + m^2$

+: Fermion
-: Boson

Number density:

$$n = \frac{g}{(2\pi)^3}\int d^3p f(p)$$

Energy density

$$\rho = \frac{g}{(2\pi)^3}\int d^3p E(p) f(p)$$

Pressure

$$p = \frac{g}{(2\pi)^3}\int d^3p \frac{p^2}{3E} f(p)$$

Thermodynamics

Relativistic particles

$$T \gg m, \mu$$

$$n = \begin{cases} (\zeta(3)/\pi^2) g T^3 & \text{(boson)} \\ (3/4)(\zeta(3)/\pi^2) g T^3 & \text{(fermion)} \end{cases}$$

$$\rho = \begin{cases} (\pi^2/30) g T^4 & \text{(boson)} \\ (7/8)(\pi^2/30) g T^4 & \text{(fermion)} \end{cases}$$

$$p = \frac{\rho}{3}$$

$$\langle E \rangle \equiv \frac{\rho}{n} = \begin{cases} [\pi^4/30\zeta(3)] T \sim \\ [7\pi^4/180\zeta(3)]\, T \end{cases}$$

$$\zeta(3) = 1.20206 \cdots$$

Non-relativistic particles

$$T \ll m$$

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} \exp\left[-\frac{m-\mu}{T} \right]$$

$$\rho = mn$$

$$p = nT \ll \rho$$

$$\langle E \rangle = m + \frac{3}{2} T$$

Thermodynamics

The energy density of all relativistic particles

$$\rho = \frac{\pi^2}{30} g_* T^4$$

w/ the relativistic degrees of freedom

$$g_* = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

$$T \gtrsim 300\text{GeV} \qquad g_* = 106.75$$

$$100\text{MeV} \gtrsim T \gtrsim 1\text{MeV} \qquad g_* = 10.75$$

$$T \ll \text{MeV} \qquad g_* = 3.36$$

Radiation dominated era

$$H^2 = \frac{8\pi G}{3}\rho_{\text{tot}} = \frac{8\pi G}{3}\rho_{\text{rad}} = \frac{4\pi^3 g_* T^4}{45 M_{\text{Pl}}^2}$$

$$H = \sqrt{\frac{4\pi^3}{45}} g_*^{1/2} \frac{T^2}{M_{\text{Pl}}} = 1.66 g_*^{1/2} \quad t = \frac{1}{2}\sqrt{\frac{45}{4\pi^3}} g_*^{-1/2} \frac{M_{\text{Pl}}}{T^2} = \frac{2.4}{g_*^{1/2}} \left(\frac{\text{MeV}}{T}\right)^2 \text{sec}$$

Entropy

2nd law of thermodynamics

$$dS = \frac{dU + pdV}{T} = \frac{d(\rho V) + pdV}{T} = \frac{V}{T}\frac{\partial \rho}{\partial T}dT + \frac{\rho + p}{T}dV \qquad V = R^3$$

Integrability condition

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \Rightarrow T\frac{dp}{\partial T} = \rho + p \Rightarrow dS = d\left[\frac{(\rho + p)V}{T} + \text{const.}\right]$$

Entropy density

$$s = \frac{S}{V} = \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_s T^3$$

- In many cases, entropy is conserved: $sR^3 = \text{const.}$

  ➡ Roughly $T \propto R^{-1}$

Limit on the mass of dark matter

Lower bound on fermionic dark matter

- Pauli exclusion principle

→ The phase space density is bounded by the internal d.o.f

$f(x,p) < g$ → $m_{\rm DM} \gtrsim 70 \ {\rm eV}$ (aka. Tremaine-Gunn limit)

Lower bound on bosonic dark matter

- The de Broglie wavelength of dark matter should be smaller than the core scale

$$\lambda_{\rm dB}^{\rm DM} = \frac{2\pi}{m_{\rm DM} v} \lesssim 1 \ {\rm kpc} \quad \rightarrow \quad m_{\rm DM} \gtrsim 10^{-22} \ {\rm eV}$$

Upper bound on particle dark matter

- The Compton wavelength of dark matter should be larger than the Schwarzschild radius

$$\lambda_{\rm C}^{\rm DM} = \frac{2\pi}{m_{\rm DM}} \gtrsim r_S = 2G m_{\rm DM} \quad \rightarrow \quad m_{\rm DM} \lesssim M_{\rm Pl} \simeq 10^{19} \ {\rm GeV}$$