Gauge anomalies and anomaly cancellation

classical symmetry

$$\phi \longrightarrow \phi + \delta\phi$$

$$S(\phi) \longrightarrow S(\phi)$$

quantum symmetry

$$\phi \longrightarrow \phi + \delta\phi$$

$$\int \mathcal{D}\phi \, e^{iS(\phi)} \longrightarrow \int \mathcal{D}\phi \, e^{iS(\phi)}$$

→ the measure $\mathcal{D}\phi$ must be invariant too

* We say that a classical symmetry is anomalous if it cannot be realized in the quantum theory.

* Whether anomalies are good or bad news depends on the type of symmetry:

+ Global symmetries: these symmetries can be considered as accidental symmetries so their breaking (anoamaly) does not spoil the consistency of the theory and can lead to the interesting things.

+ Cgauge ~~tr~~ symmetries: it can be used to remove unphisical states. So their breaking (anomaly) spoils the consistency of the theory: unitarity violation at high energy, non renormalization.

For example:

1) Yang - Mills theory with gauge symmetry $SU(N_c)$

$$S = \int d^4x \, \mathcal{L}_{YM} \quad ; \quad \mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g^2_{YM} f^{abc} A^b_\mu A^c_\nu$$

→ the classical action is invariant under the scale ~~tr~~ transformation (which the physics does not depend on the energy ~~of~~ scale)

$$x^\mu \longrightarrow x'^\mu = \lambda x^\mu$$

$$A^\mu \longrightarrow A'^\mu = \lambda^{-\Delta} A^a_\mu(\lambda^{-1} x)$$

λ: constant , $\Delta = 1$

consequence

$\rightarrow$ $T^\mu{}_\mu = 0$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}_{YM}}{\partial(\partial_\mu A^a_c)} \partial_\nu A^a_e - \delta^\mu_\nu \mathcal{L}_{YM}$$

Now question is what's about when we include the quantum correction?

$\rightarrow$ $\langle T^\mu{}_\mu \rangle \sim \beta(g_{YM}) = -\frac{11 g^3_{YM} N_c}{48\pi^2} \neq 0$

$\rightarrow$ anomaly because we introduce μ-scale in renormalization

Here: $\beta(g_{YM})$ is beta function

$$\mu^2 \frac{\partial g_{YM}}{\partial \mu^2} = \beta(g_{YM})$$

Note that, for $SU(N_c)$ gauge theory with n_f Dirac fermions in the representation R_f and n_s complex scalars in the representation R_s.

Beta function $\beta(g_{YM})$ at one-loop corr.

$$\beta(g_{YM}) = -\left(\frac{11}{3} C(G) - \frac{1}{3} n_s T(R_s) - \frac{4}{3} n_f T(R_f)\right) \times \frac{g_{YM}^3}{16\pi^2}$$

$C(G) = N_c$: quadratic Casimir

$$\overset{\shortparallel}{\sum_{a=1}^{N_c^2-1} (T_a)^2}$$

$$\mathrm{Tr}\left(T^a(R_f)\, T^b(R_f)\right) = T(R_f)\,\delta^{ab}$$

$$\mathrm{Tr}\left(T^a(R_s)\, T^b(R_s)\right) = T(R_s)\,\delta^{ab}$$

2) $\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu)\right]\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

A_μ : gauge field of $U(1)$ local sym.

Axial global $U(1)$ sym.

$$\Psi \longrightarrow e^{-i\alpha\gamma_5}\,\Psi(x) \qquad \{\gamma_5, \gamma^\mu\} = 1$$

$$\bar{\Psi} \longrightarrow \bar{\Psi}\, e^{-i\alpha\gamma_5} \qquad \gamma_5^2 = 1$$

$\rightarrow \partial_\mu J^\mu_A = 0$

But including quantum correction

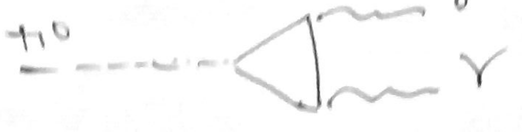
$\rightarrow \langle \partial_\mu J^\mu_A \rangle = -\frac{e^2}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$

Adler - Bell - Jackiw anomaly

* The conservation of axial current is spoiled by quatum correction.

Consequence: π^0 can decay into two photons by anomaly

$\pi^0 \rightarrow 2\gamma$



* In gauge theory, anomaly can appear where the left-handed and right-handed fermions transform in the different representations of gauge symmetry group.

ψ_L: in representation R_L of gauge symmetry G corresponding to the representation of generators T^a_L

ψ_R: in ~~no~~ representation R_R of G corresponding to the representation of generators T^a_R

Classical $\rightarrow$ $\quad D^L_\mu J^{a\mu}_L + D^R_\mu J^{a\mu}_R = 0$

$$J^{a\mu}_L = \bar{\psi}_L T^a_L \gamma^\mu \psi_L \; ; \; J^{a\mu}_R = \bar{\psi}_R T^a_R \gamma^\mu \psi_R$$

quantum
$\longrightarrow$

$$\langle D_\mu^L J_L^{a\mu} \rangle \sim -\mathrm{Tr}\left[T_L^a \{T_L^b, T_L^c\}\right]$$

$$\langle D_\mu^R J_R^{a\mu} \rangle \sim \mathrm{Tr}\left[T_R^a \{T_R^b, T_R^c\}\right]$$

$$\langle D_\mu^L J_L^{a\mu} + D_R^\mu J_R^{a\mu} \rangle \sim \sum_{\text{fermions}}$$

$$\sim \mathrm{Tr}\left[T_R^a \{T_R^b, T_R^c\}\right] - \mathrm{Tr}\left[T_L^a \{T_L^b, T_L^c\}\right]$$

For the vectorlike theory: $T_R^a = T_L^a$

$\rightarrow$ there is no gauge anomaly

Example: SM with gauge symmetry

$$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$$

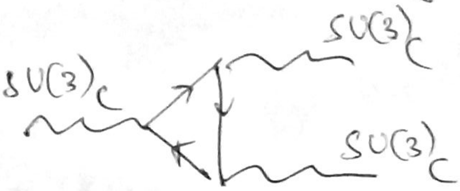
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{aL} \sim (1, 2, -1) \; ; \; e_{aR} \sim (1, 1, -2)$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_{aL} \sim \left(3, 2, \tfrac{1}{3}\right) ; \; u_{aR} \sim \left(3, 1, \tfrac{4}{3}\right)$$

$$d_{aR} \sim \left(3, 1, -\tfrac{2}{3}\right)$$

We have totally ten possible gauge anomalies

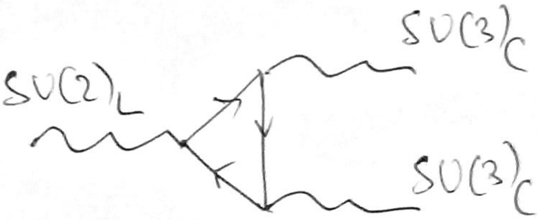
+ 1st anomaly



$$\sim \sum_{\text{quarks}} \left\{ Tr\left[\frac{\lambda^a_R}{2}\left\{\frac{\lambda^b_R}{2}, \frac{\lambda^c_R}{2}\right\}\right] - Tr\left[\frac{\lambda^a_L}{2}\left\{\frac{\lambda^b_L}{2}, \frac{\lambda^c_L}{2}\right\}\right] \right\} = 0$$

$$\lambda^a_R = \lambda^a_L = \lambda^a$$

+ second anomaly

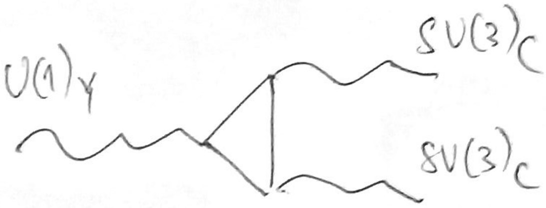


$$\sim \sum_{\substack{\text{left-handed}\\ \text{quarks}}} Tr\left[\frac{\tau_i}{2} \underbrace{\left\{\frac{\lambda^a_L}{2}, \frac{\lambda^b_L}{2}\right\}}_{\frac{1}{3}\delta^{ab} + d^{abc}\frac{\lambda^c}{2}}\right]$$

$Tr\left(\frac{\tau_i}{2}\right) = 0$

$= 0$

+ third anomaly



$$\sim \left\{ \sum_{q_L} Tr\left[Y_L \left\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right\}\right] - \sum_{q_R} Tr\left[Y_R \left\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right\}\right] \right\}$$

$= 0$

using

$$Y = 2(Q - T_3), \quad T_{3,L} \to \frac{\tau_3}{2}, \quad T_{3,R} \to 0$$

$$\sim \sum_{\substack{\text{left-haded}\\ \text{quarks}}} Q - \sum_{\substack{\text{right-handed}\\ \text{quarks}}} Q$$

$= 0$

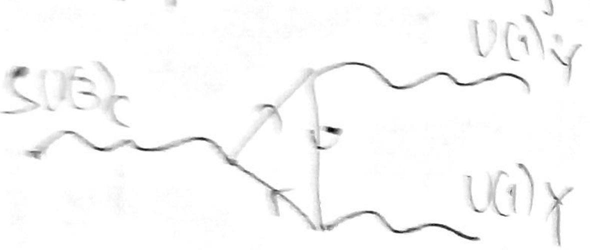
+ fourth anomaly

$$\sim \sum_{\substack{\text{left-handed}\\ \text{quarks}}} \mathrm{Tr}\left[\frac{\lambda^a}{2}\underbrace{\left\{\frac{\tau^i}{2},\frac{\tau^j}{2}\right\}}_{\epsilon^{ijk}\frac{\tau^k}{2}}\right]$$

$$\underline{\underline{\mathrm{Tr}\left(\frac{\lambda^a}{2}\right)=0}} \qquad 0$$

or $\mathrm{Tr}\left(\frac{\tau^i}{2}\right)=0$

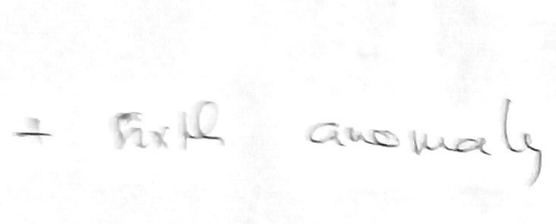
+ fifth anomaly



$$\sim \sum_{q_R} \mathrm{Tr}\left[\frac{\lambda^a}{2} Y_R^2\right] - \sum_{q_L} \mathrm{Tr}\left[\frac{\lambda^a}{2} Y_L^2\right]$$

$$\underline{\underline{\mathrm{Tr}\left[\frac{\lambda^a}{2}\right]=0}} \qquad 0$$

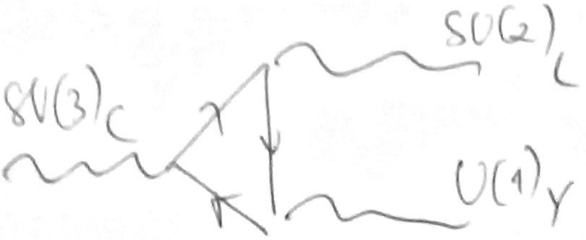
+ sixth anomaly



$$\sim \sum_{\substack{\text{left-handed}\\ \text{fermion}}} \mathrm{Tr}\left[\frac{\tau^i}{2}\underbrace{\left\{\frac{\tau^j}{2},\frac{\tau^k}{2}\right\}}_{\delta^{jk}/2}\right]$$

$$\underline{\underline{\mathrm{Tr}\left(\frac{\tau^i}{2}\right)=0}} \qquad 0$$

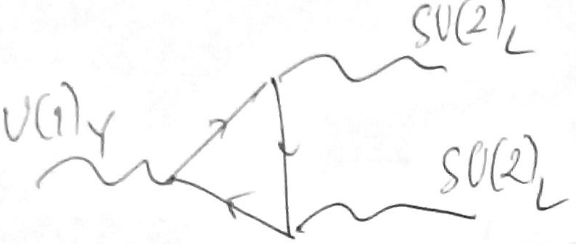
* seventh anomaly



$$\sim \sum_{q_L} \mathrm{Tr}\left[\frac{\lambda^a}{2}\frac{\sigma^i}{2}Y_L\right]$$

$$= 0$$

* eighth anomaly



$$\sim \sum_{\text{left-handed fermions}} \mathrm{Tr}\left[Y\left\{\frac{\sigma^i}{2},\frac{\sigma^j}{2}\right\}\right]$$

$$\sim \sum_{\text{left-haded fermions}} \mathrm{Tr}\left[\frac{\delta^{ij}}{2}\,\alpha(Q-T_3)\right]$$

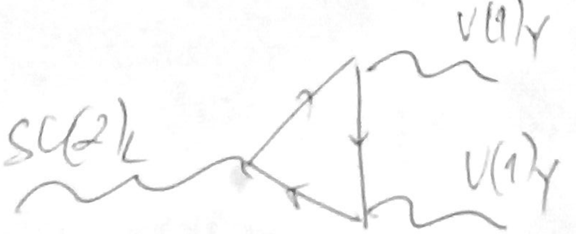
#(color) of quark

charged lepton $\quad Q_u \quad Q_d \quad$ #(generation)

$$\sim \sum_{\text{left-handed fermions}} Q = [-1+3(\tfrac{2}{3}-\tfrac{1}{3})]\times 3 = 0$$

$\cancel{=0}$

* night anomaly



$$\sim \sum_{\text{left-haded fermions}} \mathrm{Tr}\left[\frac{\sigma^i}{2}Y^2\right]$$

$$\mathrm{Tr}\left(\frac{\sigma^i}{2}\right)=0$$

$$= 0$$

+ tenth anomaly

$$\sim \sum_{\substack{\text{right-handed}\\ \text{fermions } (\equiv f_R)}} \mathrm{Tr}(Y_R^3) - \sum_{\substack{\text{left-handed}\\ \text{fermions } (\equiv f_L)}} \mathrm{Tr}(Y_L^3)$$

$$= \sum_{f_R} \mathrm{Tr}(8Q^3) - \sum_{f_L} \mathrm{Tr}\left[8\left(Q - \frac{\tau^3}{2}\right)^3\right]$$

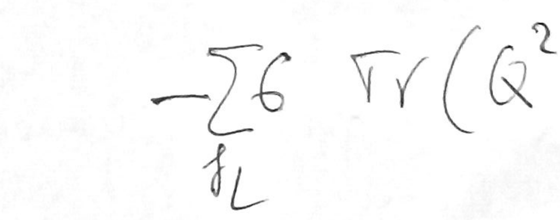
$$= \sum_{f_R} 8\,\mathrm{Tr}(\cancel{8}Q^3) - \sum_{f_L} 8\,\mathrm{Tr}(Q^3) - \sum_{f_L} 24\,\mathrm{Tr}\left[Q\underbrace{\left(\frac{\tau^3}{2}\right)^2}_{=1}\right]$$

$$-\sum_{f_L} 6\,\mathrm{Tr}(Q^2\tau_3) - \sum_{f_L} \mathrm{Tr}(\tau^3)$$

$$= -6\sum_{f_L} \mathrm{Tr}\,Q = \left[-1 + 3\left(\frac{2}{3} - \frac{1}{3}\right)\right] \times 3$$

$$= 0$$

+ mixed gauge-gravity anomaly



$$\sim \sum_{\substack{\text{charged}\\ \text{fermions}}} \mathrm{Tr}\,Q = 0$$