Self-energy:

$$\Sigma_{qq}(p) \equiv i g_s^2 \mu^{(4-D)} \frac{4}{3} \int_{-\infty}^{+\infty} \frac{d^D q}{(2\pi)^D} \frac{\gamma_\alpha [\not{p} - \not{q} + m] \gamma^\alpha}{q^2 [(p-q)^2 - m^2]}$$

$$\Rightarrow M_1 = -\delta_{im} \cdot \bar{u}_\sigma(p) \Sigma_{qq}(p) \frac{\not{p}+m}{p^2-m^2} A_m \Big|_{p^2 \to m^2}$$

where we have used $(T_a)_{il} (T_a)_{lm} = \frac{4}{3} \delta_{im}$

Write: $\Sigma_{qq}(p) = \not{p}\, \Sigma_V(p^2) + m\, \Sigma_S(p^2)$

Using $\gamma_\alpha \gamma^\alpha = D$; $\gamma_\alpha \not{q} \gamma^\alpha = (2-D)\not{q}$ we get

$$\Sigma_S = i g_s^2 \mu^{(4-D)} \frac{4}{3} D \int_{-\infty}^{+\infty} \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 [(p-q)^2 - m^2]}$$

$$\int_{-\infty}^{+\infty} \frac{d^D q}{(2\pi)^D} \frac{(2-D)\not{q}}{q^2[(p-q)^2 - m^2]} = (2-D)\gamma^\mu \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu}{q^2[(p-q)^2-m^2]}$$

Scalar integral:

$$B_0(p_1, m_0, m_1) \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{(q^2 - m_0^2 + i\epsilon)[(q+p_1)^2 - m_1^2 + i\epsilon]}$$

$$\epsilon \to 0^+$$

Tensor integral:

$$B_\mu(p_1, m_0, m_1) \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_\mu}{(q^2 - m_0^2)[(q+p_1)^2 - m_1^2]}$$

$$\equiv p_{1\mu} \cdot B_1(p_1, m_0, m_1)$$

Note: $\frac{\mu^{4-D}}{(2\pi)^D} = \left(\frac{i}{16\pi^2}\right) \frac{(2\pi\mu)^{4-D}}{i\pi^2}$

$$\Rightarrow \Sigma_S = -g_s^2 \frac{4}{3} D \frac{1}{16\pi^2} B_0(-p, 0, m)$$

$$\Sigma_V = -g_s^2 \frac{4}{3} (2-D) \frac{1}{16\pi^2} \left[B_0(-p,0,m) + B_1(-p,0,m) \right]$$

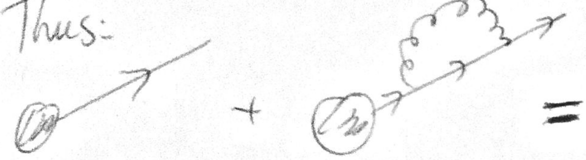
For our "present" case : $m = 0$

$$\Rightarrow \Sigma_{qq}(p) = \not{p}\, \Sigma_V(p^2)$$

$$\Rightarrow M_1 = - \bar{u}_\sigma(p)\, \not{p}\, \Sigma_V(p^2) \frac{\not{p}}{p^2} \cdot A_i \Big|_{p^2 \to m^2}$$

$$= - \Sigma_V(p^2)\, \bar{u}_\sigma(p)\, A_i \Big|_{p^2 \to m^2}$$

Thus:



$$= \bar{u}_\sigma(p) A_i \left[1 - \Sigma_V(p^2) \right] \Big|_{p^2 \to m^2}$$

$$\Rightarrow \sqrt{\tilde{Z}_\psi} = 1 - \Sigma_V(m^2) = 1 - \Sigma_V(0)$$

Using: $B_1(p, m, m) = -\frac{1}{2} B_0(p, m, m)$

$$\Rightarrow \Sigma_V = -g_s^2 \frac{4}{3} (2-D) \frac{1}{16\pi^2} \cdot \frac{1}{2} B_0(0,0,0)$$

Scaless integrals:

$$A_0(0) \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int_{-\infty}^{+\infty} d^D q \frac{1}{q^2} = 0$$

$$B(0,0,0) \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int_{-\infty}^{+\infty} d^D q \frac{1}{q^4} = \infty_{UV} - \infty_{IR} = \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$$

thus:

$$\tilde{Z}_\psi = 1 + g_s^2 \frac{1}{24\pi^2}(2-D)\left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}\right); \quad D = 4-2\varepsilon$$

$$= 1 + g_s^2 \frac{1}{12\pi^2}\left(\frac{1}{\varepsilon_{IR}} - \frac{1}{\varepsilon_{UV}}\right)$$

Back to $M_{2,1}$:

$$\Lambda_\mu = \mu^{(4-D)} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma_\alpha (\not k + \not k_1) \gamma_\mu (\not k - \not k_2) \gamma^\alpha}{k^2 (k+k_1)^2 (k-k_2)^2}$$

$$\bar{u}_{\rho_1}(k_1) \Lambda_\mu v_{\rho_2}(k_2) = \frac{i}{16\pi^2} \cdot \bar{u}_{\rho_1}(k_1) \Big\{ -2\left[2C_{00} - B_0(q^2)\right.$$

$$\left. + 2 k_1 k_2 (C_1 + C_2 + C_0)\right] + (4-D)\left[2C_{00} - B_0(q^2)\right] \Big\} \times \gamma_\mu v_{\rho_2}(k_2)$$

$$V_3 = -2\left[2C_{00} + 2k_1k_2(C_1 + C_2 + C_0) - B_0(q^2)\right]$$

$$+ (4-D)\left[2C_{00} - B_0(q^2)\right]$$

with $B_0(q) = B_0(q, 0, 0)$

$$\underset{(r_1, r_2)}{C_0} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{q^2 (q+r_1)^2 (q+r_2)^2}$$

$$C^\mu(r_1, r_2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q^\mu}{q^2 (q+r_1)^2 (q+r_2)^2}$$

$$\equiv r_1^\mu C_1(r_1, r_2) + r_2^\mu C_2(q, r_1, r_2)$$

$$C^{\mu\nu}(r_1, r_2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q^\mu q^\nu}{q^2 (q+r_1)^2 (q+r_2)^2}$$

$$\equiv g^{\mu\nu} C_{00} + \sum_{i,j=1}^{2} r_i^\mu r_j^\nu C_{ij} \quad (C_{21} = C_{12})$$

L2.17

We have: $C_1 = \frac{1}{2 r_1 r_2}\left[B_0(r_1) - B_0(k)\right]$; $k = r_1 + r_2$.

$$C_2 = \frac{1}{2 r_1 r_2}\left[B_0(r_2) - B_0(k)\right]$$

$$(D-2)\, C_{00} = \frac{1}{2} B_0(k)$$

$$\Rightarrow V_3 = -2 q^2 C_0 + 4 B_0(0) + \frac{\left[(4-D)(3-D) - 2\overset{(D-1)}{\cancel{(7-3D)}}\right]}{D-2} \cdot B_0(q)$$

with $C_0 = C_0(k_1, -k_2)$: UV finite

$$B_0(q) = B_0(q, 0, 0) = \Delta_{UV} - \log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) + 2$$

$\log(-q^2 - i\epsilon) = \log q^2 - i\pi$ for $q^2 > 0$.

$$\Delta_{UV} = \frac{2}{4-D} - \gamma_E + \log 4\pi = \frac{1}{\varepsilon_{UV}} - \gamma_E + \log 4\pi$$

Then:

$$V_3 = -2q^2 C_0 + 4 B_0(0) + \left[-3 - 2\varepsilon + O(\varepsilon^2)\right] \cdot B_0(q)$$

$$= -2q^2 C_0 + 4 B_0(0) - 3 B_0(q) - 2 + O(\varepsilon)$$

$$M_{2,1} = \bar{v}_{\lambda_2}(p_2)\, e \gamma^\mu u_{\lambda_1}(p_1) \frac{1}{q^2} (+e) \left[g_s^2 \cdot \frac{4}{3}\, \delta_{ij}\right]$$

$$\times\ \bar{u}_{\rho_1}(k_1)\, \gamma_\mu v_{\rho_2}(k_2) \cdot V_3 \cdot \frac{i}{16\pi^2}$$

$$M_{2,0} = \bar{v}_{\lambda_2}(p_2)\, e \gamma^\mu u_{\lambda_1}(p_1) \frac{1}{q^2} (ie)\, \bar{u}_{\rho_1}(k_1) \gamma_\mu v_{\rho_2}(k_2)\, \delta_{ij}$$

$$\Rightarrow M_{2,1} = \frac{4}{3} \frac{\alpha_s}{4\pi} V_3 \cdot M_{2,0}$$

· Summary of the $2 \to 2$ part:

$$\sigma_2 \sim |\hat{M}_2|^2$$

$$\hat{M}_2 = \left(\sqrt{\tilde{Z}_q}\right)^2 M_2 = \tilde{Z}_q \left(M_{2,0} + M_{2,1} + \dots\right)$$

$$\tilde{Z}_q = 1 + \frac{\alpha_S}{3\pi}\left(\frac{1}{\varepsilon_{IR}} - \frac{1}{\varepsilon_{UV}}\right)$$

$$M_{2,1} = \frac{\alpha_S}{3\pi} V_3 \, M_{2,0}$$

$$\Rightarrow \hat{M}_2 = \left[1 + \frac{\alpha_S}{3\pi}\left(\frac{1}{\varepsilon_{IR}} - \frac{1}{\varepsilon_{UV}}\right)\right]\left(1 + \frac{\alpha_S}{3\pi} V_3\right) M_{2,0}$$

$$= \left[1 + \frac{\alpha_S}{3\pi}\left(\frac{1}{\varepsilon_{IR}} - \frac{1}{\varepsilon_{UV}} + V_3\right)\right] M_{2,0}$$

$$V_3 = -2q^2 C_0 + 4 B_0(0) - 3 B_0(q) - 2$$

$$C_0 = C_0(k_1, -k_2) = \frac{1}{q^2}\left\{\frac{\Gamma(1+\varepsilon_{IR})}{\varepsilon_{IR}^2}\left(\frac{4\pi\mu^2}{-(q^2+i\epsilon)}\right)^{\varepsilon_{IR}} - \frac{\pi^2}{6}\right\}$$

[Ref: Dittmaier hep-ph/0308246]

$$V_3 = -\frac{2\Gamma(1+\varepsilon_{IR})}{\varepsilon_{IR}^2}\left(\frac{4\pi\mu^2}{-(q^2+i\epsilon)}\right)^{\varepsilon_{IR}} + \frac{\pi^2}{3}$$

$$+ \frac{1}{\varepsilon_{UV}} - \frac{4}{\varepsilon_{IR}} + 3\gamma_E + 3\ln\left(\frac{q^2}{4\pi\mu^2}\right) - 8 - 3i\pi$$

$$\widehat{M}_2 = M_{2,0}\left\{1+\frac{\alpha_s}{3\pi}\left[-\frac{2\Gamma(1+\varepsilon_{IR})}{\varepsilon_{IR}^2}\left(\frac{4\pi\mu^2}{-(q^2+i\epsilon)}\right)^{\varepsilon_{IR}}\right.\right.$$

$$\left.\left.+\frac{\pi^2}{3}-\frac{3}{\varepsilon_{IR}}+3\gamma_E+3\ln\left(\frac{q^2}{4\pi\mu^2}\right)-8-3i\pi\right]\right\}$$

which is UV finite, but IR divergent.

$$\Rightarrow \sigma_2 = (1+A_V)\,\sigma_{2,0}$$

$$A_V = \frac{\alpha_s}{\pi}\cdot\frac{4}{3}\left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon}\frac{\cos\pi\varepsilon}{\Gamma(1-\varepsilon)}\left(-\frac{1}{\varepsilon^2}-\frac{3}{2\varepsilon}-4+O(\varepsilon)\right)$$

with $\varepsilon = \varepsilon_{IR}$

Phase space integration in D-dimensions:

$$\sigma_{2,0} = \frac{1}{8s}\int\prod_{i=1}^{2}\frac{d^{D-1}k_i}{(2\pi)^{D-1}\,2k_{i0}}(2\pi)^D\delta^D(k_1+k_2-p_1-p_2)\times|M_{2,0}|^2$$

$$= \frac{1}{8s}\int d\phi_2^{(D)}\cdot|M_{2,0}|^2$$

Use: $\int d\phi_2^{(D)} = \frac{1}{8\pi}\frac{(4\pi)^{\varepsilon}}{\Gamma(1-\varepsilon)}\left(s^{-\varepsilon}\right)\int_0^1 dy\,[y(1-y)]^{-\varepsilon}$

with $y = (1+\cos\theta)/2$ in the c.m.s of (k_1,k_2).

Then:

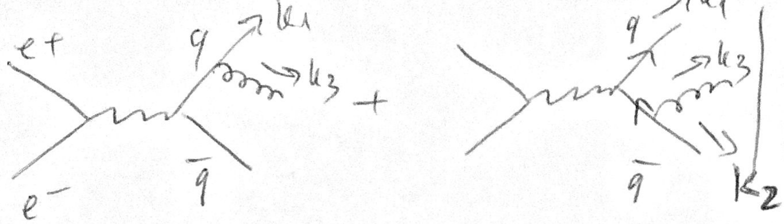
$$\sigma_{2,0} = \frac{4\pi\alpha^2}{3s}\left(3\sum_{i=1}^{n_f}Q_i^2\right)\left(\frac{4\pi}{s}\right)^{\varepsilon}\frac{3(1-\varepsilon)\Gamma(2-\varepsilon)}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}$$

- Real corrections, with $D = 4 - 2\varepsilon$:

$$\sigma_{3,0} = \frac{1}{8s} \int \prod_{i=1}^{3} \frac{d^{D-1}\vec{k}_i}{(2\pi)^{D-1}\, 2k_{i0}} (2\pi)^D \delta^D\left(\sum_{i=1}^{3} k_i - P_1 - P_2\right) \cdot F_R$$

$$\times (\mu^{4\varepsilon})$$

$$F_R \sim \Bigg| \quad + \quad \Bigg|^2$$



$$F_R = -\left(3 \sum_{i=1}^{n_f} Q_i^2\right) \frac{e^4}{q^4} g_s^2 \cdot \frac{4}{3} \mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right]$$
$$\times \mathrm{Tr}\left[\not{k}_1 S_{\lambda\mu} \not{k}_2 S^\lambda{}_\nu\right]$$

$$S_{\mu\nu} = \gamma_\mu \frac{-1}{\not{k}_1 + \not{k}_3} \gamma_\nu + \gamma_\nu \frac{1}{\not{k}_2 + \not{k}_3} \gamma_\mu$$

$$G_{\mu\nu} = \mathrm{Tr}\left[\not{k}_1 S_{\lambda\mu} \not{k}_2 S^\lambda{}_\nu\right]$$

$$L^{\mu\nu} = \mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right] = 4\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2} g^{\mu\nu}\right)$$

$$I_{\mu\nu} = \int \prod_{i=1}^{3} \frac{d^{D-1}\vec{k}_i}{2k_{i0}} \delta^D\left(\sum_{i=1}^{3} k_i - q\right) G_{\mu\nu}$$

$$\Rightarrow \sigma_{3,0} = \frac{-e^4 g_s^2 \mu^{4\varepsilon}}{8s (2\pi)^{2D-3} q^4} \left(3 \sum_{i=1}^{n_f} Q_i^2\right) \frac{4}{3} L^{\mu\nu} I_{\mu\nu}$$

$I_{\mu\nu}$ depends only on q_μ and satisfies $q^\mu I_{\mu\nu} = q^\nu I_{\mu\nu} = 0$

$$\Rightarrow I_{\mu\nu} = I(q^2)\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)$$

With $I(q^2) = -\ g^{\mu\nu} I_{\mu\nu}/(D-1)$.

$$\Rightarrow L^{\mu\nu} I_{\mu\nu} = \frac{D-2}{D-1} q^2 g^{\mu\nu} \cdot I_{\mu\nu}$$

And we have, after some calculation,

$$g^{\mu\nu} G_{\mu\nu} = -8(1-\varepsilon) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)},$$

with $x_i = \dfrac{2k_i \cdot q}{q^2}$ $(i = 1, 2, 3)$.

$$\Rightarrow g^{\mu\nu} I_{\mu\nu} = \frac{\pi (\pi s)^{1-2\varepsilon}}{4\Gamma(2-2\varepsilon)} \int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i$$

$(\times \mu^{4\varepsilon})$ for dimensional reason

$$\times \delta(2 - \sum_{i=1}^{3} x_i)\, g^{\mu\nu} G_{\mu\nu}$$

Then:

$$\sigma_R = \left(3 \sum_{i=1}^{n_f} Q_i^2\right) \alpha^2 \alpha_s \frac{4}{3} \frac{2}{s} \left(\frac{4\pi\mu^2}{s}\right)^{2\varepsilon} \frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} K$$

$$K = \int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i\, \delta(2 - \sum_{i=1}^{3} x_i) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \left(\frac{4}{\varepsilon^2} - \frac{12}{\varepsilon} + 10 - 4\varepsilon\right) B(1-\varepsilon, 2-2\varepsilon)$$

$$\times B(1-\varepsilon, 1-\varepsilon) + O(\varepsilon)$$

Note, $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \dots$$

(L2.22)

$\Rightarrow \sigma_{3,0} = A_R \cdot \sigma_{2,0}$

with $A_R = \frac{\alpha_s}{\pi}\frac{4}{3}\left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon}\frac{\cos\pi\varepsilon}{\Gamma(1-\varepsilon)}$

$$\times\left(\frac{1}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{19}{4}+O(\varepsilon)\right)$$

Thus: $\sigma_{hadrons} = \sigma_2 + \sigma_3$

$$= (1 + A_V + A_R)\,\sigma_{2,0}$$

$$= \left(1 + \frac{\alpha_s}{\pi}\right)\sigma_{2,0}$$

$\Rightarrow$ IR divergences have been cancelled out.
[Kinoshita theorem].

$$\Rightarrow R\left(\frac{s}{\mu^2}, g_s\right) = \left(3\sum_{i=1}^{n_f} Q_i^2\right)\left(1 + \frac{\alpha_s}{\pi}\right).$$

Higher order terms:

$$R\left(\frac{s}{\mu^2}, g_s\right) = \left(3\sum_{i=1}^{n_f} Q_i^2\right)\left(1 + \frac{\alpha_s}{\pi} + a_2\left(\frac{s}{\mu^2}\right)\alpha_s^2 + \cdots\right),$$

where $a_2\left(\frac{s}{\mu^2}\right)$ does depends on s. Via term $\log\left(\frac{s}{\mu^2}\right)$.
R is independent of μ because $\alpha_s = \alpha_s(\mu)$.
$\Rightarrow$ the best choice is $\mu^2 = s \Rightarrow$ better convergent series because $\log\left(\frac{s}{\mu^2}\right) = 0$ and $\alpha_s(s)$ is small.