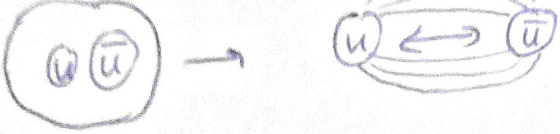
Lecture 2 : Experimental observation and Higher-order corrections

- quarks and gluons are confined.



- $e^+e^- \to$ hadrons are observed in experiments.
  ↳ how to calculate this cross section from the Lagrangian ?

$$\mathcal{L} = \mathcal{L}(e, \gamma, q_i, G_a)$$

$$\to e^+e^- \to \bar{q}_i\, q_i ,\ \bar{q}_i\, q_i + G_a ,\ N(\bar{q}_i\, q_i) + M(G_a), \cdots$$

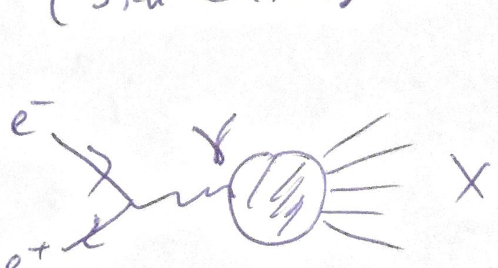
$$i = u, d, c, s, t, b.$$

→ we must go from a system of quarks and gluons to a system of hadron.

We follow here the book of Muta, Foundations of QCD, P. 232. (3rd edition). (2009)

The process of our interest is:

$$e^+ + e^- \to X ;$$



X: a system of hadrons.

(Consider here only the photon exchange)

$$M = {}_{out}\langle X | e^+ e^- \rangle_{in} = \bar{v}_{\lambda_2}(p_2)\, e \gamma^\mu u_{\lambda_1}(p_1) \frac{1}{q^2} \langle X | (-e) j_\mu(q) | 0 \rangle$$

where $q = p_1 + p_2$

Total cross section:

$$\sigma = \frac{1}{2s}\frac{1}{4}\sum_{\lambda_1\lambda_2=1}^{2}\sum_X (2\pi)^4\delta^4(P_X-q)\underbrace{|\cancel{{}_{out}\langle X|e^+e^-\rangle_{in}}|^2}_{|M|^2},$$

where $s = q^2$, m_e has been neglected.

$$\Rightarrow \sigma = \frac{e^4}{2s^3}\, l^{\mu\nu} W_{\mu\nu}$$

$$l^{\mu\nu} = P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - \frac{q^2}{2} g^{\mu\nu},$$

$$W_{\mu\nu} = \sum_X (2\pi)^4\delta^4(P_X-q)\langle 0|j_\mu(q)|X\rangle\langle X|j_\nu(q)|0\rangle .$$

$$= (2\pi)^4\delta^4(P_X-q)\langle 0|j_\mu(q)\, j_\nu(q)|0\rangle$$

$$W_{\mu\nu} = A\, q_\mu q_\nu + B\, g_{\mu\nu}; \quad A, B = f(q^2).$$

Electric current conservation requires: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$

$$\Rightarrow \quad A q^2 + B = 0 \Rightarrow B = -Aq^2$$

$$W_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\frac{1}{6\pi}\,\omega(q^2) \qquad (4.1.55)$$

$$\Rightarrow \sigma = \frac{4\pi\alpha^2}{3s}\,\omega(s); \quad \alpha = \frac{e^2}{4\pi}.$$

For $e^+e^- \to \mu^+\mu^-$: $\sigma^0_{\mu\mu} = \dfrac{4\pi\alpha^2}{3s}$

$$\Rightarrow R = \frac{\sigma_{\text{hadrons}}}{\sigma^0_{\mu\mu}} = \omega(s), \qquad (4.1.59)$$

which is dimensionless.

Taking into account the strong interaction among final state particles, we have

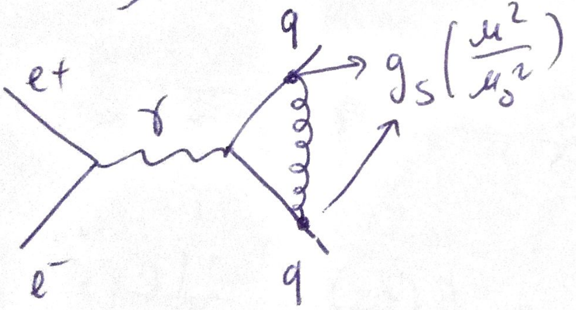
$$R = R(s, g_s) : \text{dimensionless}$$

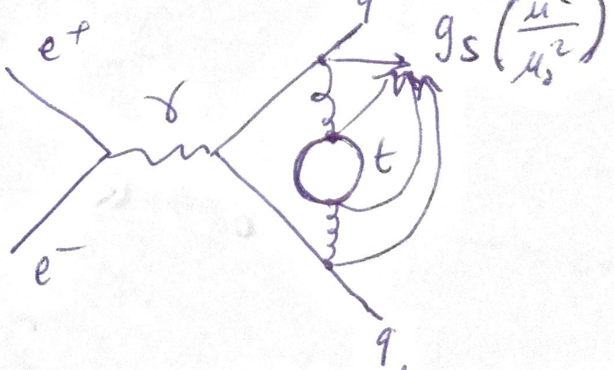
$$\Rightarrow \left[\begin{array}{l} R \notin s, \text{ a constant} \Rightarrow \text{not consistent with exp.} \\ R = R\left(\frac{s}{\mu^2}, g_s\right) \text{ for dimensional reasons.} \end{array}\right.$$

$$g_s = g_s\left(\frac{\mu^2}{\mu_0^2}\right) : \text{dimensionless}$$

where $\mu_0 \simeq 1$ GeV (mass of light hadrons).

$\Rightarrow$ μ is the energy scale at which g_s is defined!





μ is called renormalization scale. It does not occur in the original (bare) Lagrangian. It occurs when quantum corrections are taken into account (more later).

Since the bare Lagrangian, with its bare parameters $g_s = g_s^0$ (here we assume $m = 0$)), is independent of μ (an arbitrary parameter), physical quantities such as R cannot depend on μ.

We must have:

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{s}{\mu^2}, \alpha_s\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] R = 0$$

(renormalization group equation)

if $m_q \neq 0$:

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{s}{\mu^2}, \alpha_s, \frac{m_q}{Q}\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} + \mu^2 \frac{\partial m}{\partial \mu^2} \frac{\partial}{\partial m}\right] R = 0.$$

(with $Q = \sqrt{s}$)

We denote:

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s)$$

$$\mu^2 \frac{\partial m}{\partial \mu^2} = -\gamma_m(\alpha_s) \cdot m$$

~~Define~~ $\cancel{t = \log\left(\frac{Q^2}{\mu^2}\right) \Rightarrow Q^2 = \mu^2 \cdot e^t \Rightarrow \frac{dt}{d\mu^2} = -\frac{1}{\mu^2}}$

$$\cancel{\mu^2 \frac{d}{d\mu^2} R(e^t, \alpha_s) = \left[\mu^2 \frac{\partial t}{\partial \mu^2} \frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right] R(e^t, \alpha_s)}$$

$$\cancel{= \left[-\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right] R(e^t, \alpha_s) = 0.}$$

~~From~~ $\cancel{\frac{\partial \alpha_s(\mu^2)}{\beta(\alpha_s(\mu^2))} = \frac{\partial \mu^2}{\mu^2} \Rightarrow \int \frac{d\alpha_s}{\beta(\alpha_s)}}$

~~Or~~ $$\cancel{\frac{dy}{\beta(y)} = \frac{dx}{x} \Rightarrow \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dy}{\beta(y)} = \int_{\mu^2}^{Q^2} \frac{dx}{x}}$$

~~with $\alpha_s = \alpha_s(\mu^2)$~~

L 2.4

· Running coupling:

$$\frac{d\alpha_s}{\beta(\alpha_s)} = \frac{d\mu^2}{\mu^2}$$

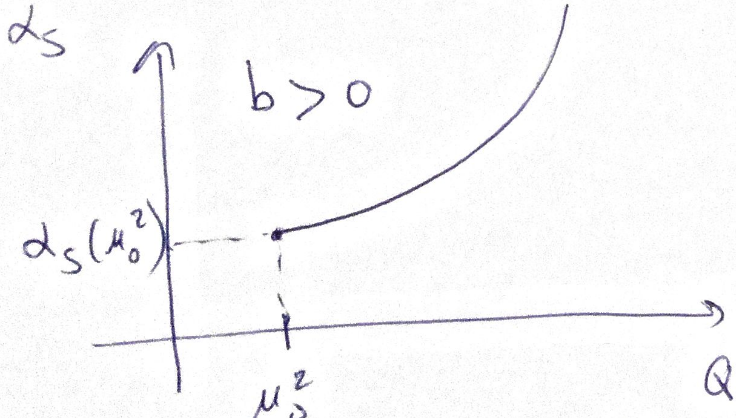
$$\Rightarrow \int_{\alpha_s(\mu_0^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta(\alpha_s)} = \int_{\mu_0^2}^{Q^2} \frac{d\mu^2}{\mu^2}, \quad \alpha_s^{(\mu^2)} = \alpha_s\left(\frac{\mu^2}{\mu_0^2}\right)$$

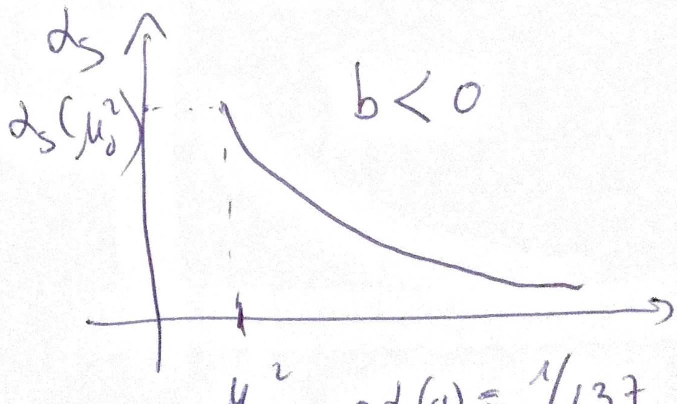
For example: $\beta(\alpha_s) = b\,\alpha_s^2 + O(\alpha_s^3)$

$$\Rightarrow \frac{1}{b}\left(-\frac{1}{\alpha_s}\right)\Bigg|_{\alpha_s(\mu_0^2)}^{\alpha_s(Q^2)} = \ln\left(\frac{Q^2}{\mu_0^2}\right)$$

$$\Rightarrow \frac{1}{b}\left[\frac{1}{\alpha_s(\mu_0^2)} - \frac{1}{\alpha_s(Q^2)}\right] = \ln\left(\frac{Q^2}{\mu_0^2}\right)$$

$$\Rightarrow \alpha_s(Q^2) = \frac{\alpha_s(\mu_0^2)}{1 - b\,\alpha_s(\mu_0^2)\ln\left(\frac{Q^2}{\mu_0^2}\right)}$$





QED: $b = \frac{1}{3\pi} \Rightarrow \alpha = \frac{e^2}{4\pi}$ increases with Q^2. Exp: $\begin{cases} \alpha(0) = 1/137 \\ \alpha(m_W^2) \simeq 1/128 \end{cases}$

QCD: $b < 0 \Rightarrow \alpha_s(Q^2) \to 0$ when $Q^2 \to \infty$
[asymptotic freedom]

Landau pole: For QED-like theories ($b > 0$).

$\alpha(Q^2) = \infty$ when

$$\ln\left(\frac{Q^2}{\mu_0^2}\right) = \frac{1}{b\,\alpha(\mu_0^2)}$$

or $Q^2 = \mu_0^2 \cdot e^{1/(b\,\alpha(\mu_0^2))}$: Landau pole.

However, at large Q^2, the coupling $\alpha(Q^2)$ becomes too large for the perturbation result $\beta(\alpha) = b\,\alpha^2 + O(\alpha^3)$ to be trusted.

QCD is easier than QED at very high energies.

Similarly, we have the notion of running mass

$$m = m(Q^2)$$

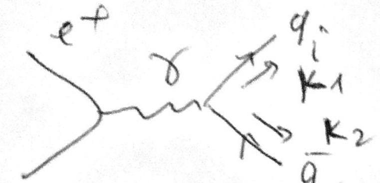
• Back to the calculation of $R = \dfrac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma^0_{e^+e^- \to \mu^+\mu^-}}$,

the result of a perturbative calculation is:

$$R\left(\frac{s}{\mu^2}, \alpha_s\right) = a_0 + a_1\left(\frac{s}{\mu^2}\right) \cdot \alpha_s(\mu^2)$$

$$+ a_2\left(\frac{s}{\mu^2}\right) \alpha_s^2(\mu^2) + \cdots$$

<u>Tree level</u> (leading order)

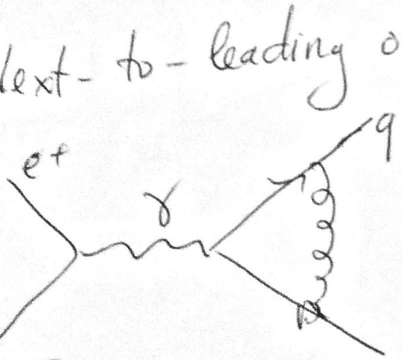


$M_{2,0}$

, $i = u, d, c, s, b = \{q_i\}$ (top quark decays before hadronization).

$$\to \quad a_0 = \left(\sum_i Q_i^2\right) \cdot N_c \quad ; \quad N_c = 3$$

Next-to-leading order (one-loop level):

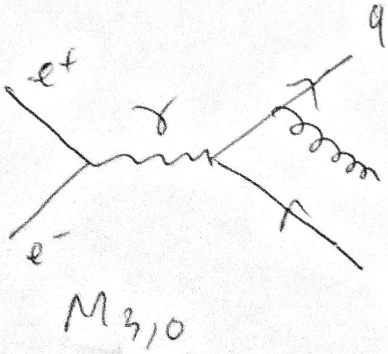


$M_{2,1}$

$$\sigma_2 \sim |M_2|^2$$

$$M_2 = M_{2,0} + M_{2,1} + \cdots$$

$$|M_2|^2 = \underbrace{|M_{2,0}|^2}_{\alpha^2} + \underbrace{2\,\mathrm{Re}(M_{2,1} \cdot M_{2,0}^*)}_{\boxed{\alpha^2 \cdot \alpha_s}} + \cdots + \alpha^2 O(\alpha_s^2)$$



$M_{3,0}$

$$\sigma_3 \sim |M_3|^2$$

$$M_3 = M_{3,0} + M_{3,1} + \cdots$$

$$|M_3|^2 = \underbrace{|M_{3,0}|^2}_{\boxed{\alpha^2 \alpha_s}} + \underbrace{2\,\mathrm{Re}(M_{3,1} \cdot M_{3,0}^*)}_{\alpha^2 \alpha_s^2} + \cdots$$

$$\sigma_{e^+e^- \to hadrons} = \sigma_{2,0} + \left(\sigma_{2,1} + \sigma_{3,0}\right) + \left(\sigma_{2,2} + \sigma_{3,1} + \sigma_{4,0}\right) + \cdots\cdots$$

$$\Rightarrow a_1\left(\frac{s}{\mu^2}\right) \cdot \alpha_s(\mu^2) = \frac{\sigma_{2,1} + \sigma_{3,0}}{\sigma^0_{\mu\mu}}$$

$$\cancel{d\sigma_{2,1} = \frac{1}{2s} \frac{1}{4} \ldots (2\pi)^4} \quad x_1, x_2 = 1$$

$$\sigma_{2,1} = \sum_{i=q_i} \sigma^i_{2,1} + \sum_i \sigma^i_{3,0} \quad , \quad i = \{q_i\}$$

$$\sigma^i_{2,1} = \frac{1}{2s} \overline{|\mathcal{M}^i_{2,1}|^2} \frac{d^3\vec{k_1}}{(2\pi)^3 2k^0_1} \frac{d^3\vec{k_2}}{(2\pi)^3 2k^0_2} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2)$$

$$\sigma^i_{3,0} = \frac{1}{2s} \overline{|\mathcal{M}^i_{3,0}|^2} \frac{d^3\vec{k_1}}{(2\pi)^3 2k^0_1} \frac{d^3\vec{k_2}}{(2\pi)^3 2k^0_2} \frac{d^3\vec{k_3}}{(2\pi)^3 2k^0_3} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2 - k_3)$$

LSZ factors:

$$\widehat{\mathcal{M}}_N^{\text{phys}} = \mathcal{M}_N^{\text{amp.}} \cdot \prod_{i=1}^{N} \sqrt{\tilde{Z}_i}$$

With $\tilde{Z}_i = 1 + \alpha_s a_i + \alpha_s^2 b_i + \dots$

For e^+, e^-: $\tilde{Z}_e = 1$ (QED corrections are ignored).

For $q_i, \bar{q}_i$: $\tilde{Z}_q = 1 + \alpha_s a_q + O(\alpha_s^2)$

$$\tilde{Z}_{\bar{q}} = \tilde{Z}_q, \quad a_q \text{ Real.}$$

$$\Rightarrow \widehat{\mathcal{M}}_N^{\text{phys}} = \mathcal{M}_N^{\text{amp}} (1 + \alpha_s a_q)$$

$$= (\mathcal{M}_{N,0} + \mathcal{M}_{N,1})(1 + \alpha_s a_q)$$

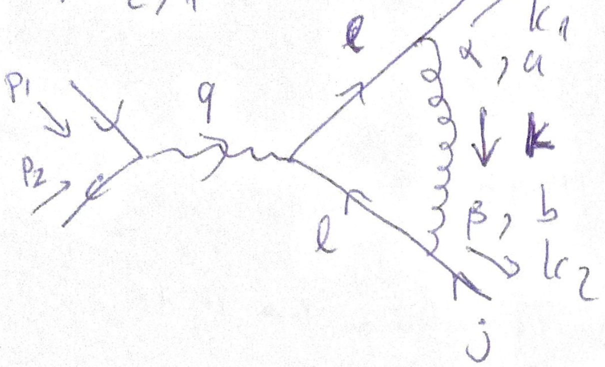
$$|\widehat{\mathcal{M}}_N^{\text{phys}}|^2 = (1 + 2\alpha_s a_q) |\mathcal{M}_{N,0} + \mathcal{M}_{N,1}|^2$$

$$= \underbrace{|\mathcal{M}_{N,0}|^2}_{O(1)} + \underbrace{2 \operatorname{Re}(\mathcal{M}_{N,1} \cdot \mathcal{M}_{N,0}^*) + 2\alpha_s a_q \cdot |\mathcal{M}_{N,0}|^2}_{O(\alpha_s)} + \dots$$

$$\Rightarrow |\widehat{\mathcal{M}}_{2,1}^{\text{phys}}|^2 = |\mathcal{M}_{2,1}|^2 + \underbrace{2\alpha_s a_q |\mathcal{M}_{2,0}|^2}_{\text{from LSZ factor}}$$

$$|\widehat{\mathcal{M}}_{N,0}^{\text{phys}}|^2 = |\mathcal{M}_{N,0}|^2$$

$M_{2,1} = ?$



$$M_{2,1} = \bar{v}_{\lambda_2}(p_2)\, e\gamma^\mu\, u_{\lambda_1}(p_1)\cdot\frac{1}{q^2}\cdot$$

$$\times\ \bar{u}_{s_1}(k_1)\, i g_s \gamma_\alpha T^a_{il} \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\, \frac{i(\not{k}+\not{k}_1)}{(k+k_1)^2}\,(i\overset{Q_i}{e}\gamma_\mu)\,\frac{i(\not{k}-\not{k}_2)}{(k-k_2)^2}$$

$$\times\ i g_s \gamma_\beta T^b_{lj}\cdot\frac{-i\delta_{ab}\,g^{\alpha\beta}}{k^2}\, v_{s_2}(k_2)$$

Where we have ~~used~~ chosen $\eta = 1$ (Feynman gauge).

$$\equiv \bar{v}_{\lambda_2}(p_2)\, e\gamma^\mu\, u_{\lambda_1}(p_1)\,\frac{1}{q^2}\,(+e)\cdot\bar{u}_{s_1}(k_1)\, g_s^2\,(T^aT^a)_{ij}$$

$$\times\ \Lambda_\mu\, v_{s_2}(k_2)$$

$$g_s^2\,\Lambda_\mu \equiv g_s^2\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\alpha(\not{k}+\not{k}_1)\gamma_\mu(\not{k}-\not{k}_2)\gamma^\alpha}{(k+k_1)^2\,(k-k_2)^2\,k^2}$$

$$\rightarrow \begin{cases} \infty \text{ when } k^\mu \to \pm\infty & (\text{UV divergence}) \sim \frac{k^6}{k^6} \\ \infty \text{ when } k^\mu \to 0 & (\text{IR divergence}) \sim \frac{1}{k^2} \end{cases}$$

Dimension: $[g_s^2\,\Lambda_\mu] = 0$

Homework: Prove that $(T^aT^a)_{ij} = C_F\,\delta_{ij}$ with $C_F = 4/3$.

· Dimensional regularization: ['t Hooft and Veltman 1972]

$4 \rightarrow D$ - dimension spacetime

$k^{\mu} (\mu = 0, \dots, 3) \rightarrow k^{\mu}$ with $\mu = 0, \dots D-1$.

$\gamma_{\mu} (\mu = 0, \dots, 3) \rightarrow \gamma_{\mu}$ with $\mu = 0, \dots, D-1$.

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2 g_{\mu\nu} I \rightarrow \{\gamma_{\mu}, \gamma_{\nu}\} = 2 g_{\mu\nu} I$$

$$g_{\mu\nu} g^{\mu\nu} = \delta^{\mu}_{\mu} = 4 \longrightarrow g_{\mu\nu} g^{\mu\nu} = \delta^{\mu}_{\mu} = D.$$

$$\hookrightarrow \gamma_{\mu} \gamma^{\mu} = 4 . I \longrightarrow \gamma_{\mu} \gamma^{\mu} = D . I$$

Convention for the unit matrix I of the gamma matrices: $\mathrm{Tr}\,(I) = 4$ also in D dimensions.

We can also choose $\mathrm{Tr}\,(I) = D \Rightarrow$ same final result.

From the above rules, we will have:

$$\gamma_{\lambda} \gamma_{\mu} \gamma^{\lambda} = (2 - D) \gamma_{\mu}$$

$$\gamma_{\lambda} \gamma_{\mu} \gamma_{\nu} \gamma^{\lambda} = 4 g_{\mu\nu} + (D-4) \gamma_{\mu} \gamma_{\nu}$$

Now:

$$g_s^2 \Lambda_{\mu} \rightarrow g_s^2 \int_{-\infty}^{+\infty} \frac{d^D k}{(2\pi)^D} \frac{\gamma_{\alpha} (\not{k} + \not{k}_1) \gamma_{\mu} (\not{k} - \not{k}_2) \gamma^{\alpha}}{(k+k_1)^2 (k-k_2)^2 k^2}$$

which has a dimension of $[E]^{D-4}$. To have the correct dimension:

$$\rightarrow g_s^2 \cdot \mu^{4-D} \int_{-\infty}^{+\infty} \frac{d^D k}{(2\pi)^D} (\dots), \text{ with } [\mu] = [E].$$

μ: arbitrary parameter.

No worry, when taking the limit $D \to 4$ then the factor $\mu^{4-D} \to 1$ and the μ will disappear.

The problem is: $\int_{-\infty}^{+\infty} \frac{d^D k}{(2\pi)^D} (\ldots)$ is divergent.

$$D = 4 - 2\varepsilon \Rightarrow \int_{-\infty}^{+\infty} \frac{d^D k}{(2\pi)^D} (\ldots) = \underbrace{\frac{d}{\varepsilon}}_{\text{div part}} + \underbrace{f + O(\varepsilon)}_{\text{finite part}}$$

$$\Rightarrow g_s^2 \mu^{2\varepsilon} \left(\frac{d}{\varepsilon} + f\right) = g_s^2 [1 + \varepsilon \ln \mu^2] \left(\frac{d}{\varepsilon} + f\right)$$

$$\left[\mu^{2\varepsilon} = e^{\ln(\mu^{2\varepsilon})} = e^{\varepsilon \ln(\mu^2)} = 1 + \varepsilon \ln(\mu^2) + O(\varepsilon^2)\right]$$

$$= g_s^2 \left[\frac{d}{\varepsilon} + f + d \ln \mu^2 + O(\varepsilon)\right]$$

$\Rightarrow$ the finite part now becomes $(f + d \ln \mu^2)$, which depends on μ.

$\Rightarrow$ after removing the divergent part $\frac{d}{\varepsilon}$ the final result a_1 will depend on μ

$$\to a_1 = a_1\left(\frac{s}{\mu^2}\right)$$

How to remove the divergent part ?

$\to$ renormalization ?

No need in this case.

Definition of S-matrix element:

Scalar fields $\quad q_a + q_b \rightarrow p_1 + p_2$

Tree-level:

$$\mathcal{M}_{\text{Scalar}} = \langle p_1, p_2 | (S-1) | q_a, q_b \rangle$$

$$= \underbrace{\tilde{G}_4 (p_1, p_2, q_a, q_b)}_{\text{Green function}} \Big|_{\text{connected}}$$

Quantum corrections:

two-point function

$$\tilde{G}_2 (p) \sim T(\phi_1(x)\, \phi_2(y))$$

Tree level: $\quad \tilde{G}_2 (p) = \dfrac{i}{p^2 - m^2}$

Loop corrections: $\quad \tilde{G}_2 (p) = \underbrace{\dfrac{i \tilde{Z}}{p^2 - m^2}}_{\text{pole part}} + (\ldots)$

$$\Rightarrow \phi_{loop} = \sqrt{\tilde{Z}}\, \phi_{tree}$$

with $\sqrt{\tilde{Z}} = 1 + \alpha a_1 + \alpha^2 a_2 + \ldots$

To all-orders:

$$\mathcal{M}_{\text{scalar}} = \left(\sqrt{\tilde{Z}}\right)^4 \tilde{G}_4(p_1, p_2, q_a, q_b)$$ connected amputated.

Vector fields:

$$\mathcal{M}_{\text{vector}} = \left(\sqrt{\tilde{Z}}\right)^4 \varepsilon^{\mu_1}_{\lambda_1}(p_1)^* \, \varepsilon^{\mu_2}_{\lambda_2}(p_2)^* \, \tilde{G}^{ac}_{4\,\mu_1\mu_2\nu_1\nu_2}(p_1, p_2, q_a, q_b)$$

$$\times \, \varepsilon^{\nu_1}_{\lambda_a}(q_a) \, \varepsilon^{\nu_2}_{\lambda_b}(q_b).$$

where $\tilde{G}_{2,\mu\nu}(p) = -\dfrac{i g_{\mu\nu}}{p^2 - m^2} \cdot \tilde{Z} + (\text{non-pole part}) + \underbrace{p_\mu p_\nu \cdot B}_{p_\mu \cdot \varepsilon^\mu = 0!}$

Spinor fields:

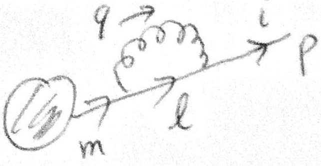
$$\mathcal{M}_{\text{spinor}} = \left(\sqrt{\tilde{Z}_\psi}\right)^4 \bar{u}^{\alpha_1}_{\lambda_1}(p_1) \, \bar{u}^{\alpha_2}_{\lambda_2}(p_2) \cdot \tilde{G}^{ac}_{4\,\alpha_1\alpha_2\beta_1\beta_2}$$

$$\times \, u^{\beta_1}_{\lambda_a}(q_a) \, u^{\beta_2}_{\lambda_b}(q_b).$$

Where $$\tilde{G}_2(p)^\alpha{}_\beta \, u^\beta_\lambda(p) = \frac{i(\not{p} + m)^\alpha{}_\beta}{p^2 - m^2} \cdot \tilde{Z}_\psi \, u^\alpha_\lambda(p) + (\dots)$$

in the limit $p^2 \to m^2$.

How to calculate $\tilde{Z}_\psi$:



$$\mathcal{M}_1 = \int_{-\infty}^{+\infty} \frac{d^D q}{(2\pi)^D} \times \mu^{(4-D)} \, \bar{u}_\sigma(p) \left[i g_s (T_a)_{il} \gamma^\beta \right] \frac{i(\not{p} - \not{q} + m)}{(p-q)^2 - m^2}$$

$$\times \frac{-i g_{\alpha\beta}}{q^2} \left[i g_s (T_a)_{lm} \gamma^\alpha \right] \frac{i(\not{p} + m)}{p^2 - m^2} \cdot A_m \Big|_{p^2 \to m^2}$$