Excercises : Problem Sheet 1 (QCD)

(1) Work out the Feynman rule of three-gluon vertex from the Lagrangian. Try also the 4-gluon vertex if you can.

(2) Calculate the gluon propagator after the introduction of the gauge-fixing Lagrangian.

(3) Calculate the amplitude squared of the following processes

a) $e^+e^- \to \mu^+\mu^-$ in QED

b) $e^+e^- \to \bar{u}u$ in QED

c) $\bar{u}u \to e^+e^-$ in QED

d) $\bar{u}u \to \bar{d}d$ in QCD

Where all the masses are neglected. For the final-state particles, sum over all colors and polarizations. For the initial-state particles, average over colors and polarizations.

e) Calculate the total cross section when the colliding energy is $E = 2, 10$ GeV, for the case $e^+e^- \to \mu^+\mu^-$.

④ Prove the result

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2 m_a}\sqrt{1-\frac{4m^2}{m_a^2}}\,|M|^2$$

for the decay $a \to P_1 + P_2$; $m_1 = m_2 = m$.

⑤ Prove the result

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{P}_1|}{|\vec{P}_a|}|M|^2$$

for the scattering $P_a + P_b \to P_1 + P_2$ in the center-of-mass system. Note, $s = (P_a + P_b)^2$;

$d\Omega = \sin\theta \, d\theta \, d\varphi$ as usual with $\theta = (\vec{P}_a, \vec{P}_1)$.

⑥ Prove that

$$B_0(p, 0, m) = \Delta - \log\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2 - p^2}{p^2}\log\left(\frac{m^2 - p^2 - i\epsilon}{m^2}\right)$$

with $\Delta = \frac{2}{4-D} - \gamma_E + \log(4\pi)$, γ_E is the Euler constant.

Calculate also $B_1(p, 0, m)$.

⑦ Prove that $(T^a T^a)_{ij} = C_F \delta_{ij}$ with $C_F = \frac{4}{3}$.

Problem Sheet 2 (QCD)

1) We have

$$C_0 = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \, \frac{1}{q^2(q+k_1)^2(q-k_2)^2}$$

Prove that

$$C_0 = \frac{1}{q^2}\left\{\frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\left(\frac{4\pi\mu^2}{-(q^2+i\epsilon)}\right)^{\varepsilon} - \frac{\pi^2}{6} + O(\varepsilon)\right\}$$

2) We define

$$\int d\phi_2^{(D)} \equiv \int \prod_{i=1}^{2}\frac{d^{D-1}\vec{k}_i}{(2\pi)^{D-1}\,2k_{i0}}\,(2\pi)^D\,\delta^D(k_1+k_2-q)$$

Work in the center of mass frame of k_1 and k_2, where $q_\mu = k_{1\mu}+k_{2\mu} = (\sqrt{s},0,0,0)$; $k_1^2=k_2^2=0$

And use $d^{D-1}k_1 = |\vec{k}_1|^{D-2}\, d|\vec{k}_1|\, d\Omega_{D-2})$

with $d\Omega_{n-1} = d\phi\, d\theta_1 \sin\theta_1 \ldots d\theta_{n-2}\sin^{n-2}\theta_{n-2}$

and use $\int d\Omega_{n-1} = \int d\Omega_{n-2}\, d\theta_{n-2}\sin^{n-2}\theta_{n-2}$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int d\theta_{n-2}\sin^{n-2}\theta_{n-2}$$

[with $y=(1+\cos\theta)/2$]

to prove: $\int d\phi_2^{(D)} = \frac{1}{8\pi}\left[(4\pi)^{\varepsilon}/\Gamma(1-\varepsilon)\right] s^{-\varepsilon}\int_0^1 dy\,[y(1-y)]^{-\varepsilon}$.

3, Prove this result

$$V_3 = -2\left[2C_{00} + 2k_1k_2(C_1 + C_2 + C_0) - B_0(q)\right] + (4-D)\left[2C_{00} - B_0(q)\right]$$

in the lecture note.

4, Prove this identity

$$q^\mu I_{\mu\nu} = q^\nu I_{\mu\nu} = 0$$ in the lecture note.

5, Prove this identity

$$\int \prod_{i=1}^{3} \frac{d^{D-1}\vec{k}_i}{2k_{i0}} \delta^D\left(\sum_{i=1}^{3} k_i - q\right) g^{\mu\nu} G_{\mu\nu}$$

$$= \frac{\pi(\pi s)^{1-2\varepsilon}}{4\Gamma(2-2\varepsilon)} \int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i \, \delta\left(2 - \sum_{i=1}^{3} x_i\right) \times g^{\mu\nu} G_{\mu\nu}$$

with $x_i = \dfrac{2k_i \cdot q}{q^2}$; $g^{\mu\nu} G_{\mu\nu} = -8(1-\varepsilon)\dfrac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$

as in the lecture note.

6, Prove this

$$K = \int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i \, \delta\left(2 - \sum_{i=1}^{3} x_i\right) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \left(\frac{4}{\varepsilon^2} - \frac{12}{\varepsilon} + 10 - 4\varepsilon\right) B(1-\varepsilon, 2-2\varepsilon) \cdot B(1-\varepsilon, 1-\varepsilon) + O(\varepsilon)$$

with $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.