Lecture 1. QCD Lagrangian.

1, Reminder of QED Lagrangian:

$$\mathcal{L}_{QED} = \bar{\Psi}_e (i\gamma^\mu D_\mu - m_e)\Psi_e - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu = \partial_\mu - ieA_\mu \; ; \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

→ one is one, simple!

$\boxed{\begin{array}{l} \text{→ Can review gauge invariance} \\ \text{→ Global sym.} \\ \text{→ Local sym} \rightarrow \text{interaction.} \end{array}}$

Now, consider a quark:

$$\Psi_q(x) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} : \text{3 in 1.}$$

$$\Psi_q(x) \rightarrow \Psi'_q(x) = \underbrace{e^{iT_a \alpha_a(x)}}_{U} \Psi_q(x)$$

$$T_a = \frac{\lambda_a}{2}, \quad a = 1, \dots 8 : \text{3×3 Gell-Mann matrices}$$

$[T_a, T_b] = i f_{abc} T_c$; f_{abc} Structure constants (real, totally antisymmetric).

Local Symmetry → interaction.

$$\mathcal{L}_{QCD} = \bar{\Psi}_q (i\gamma^\mu D_\mu - m_q)\Psi_q - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

$$= \bar{\Psi}_q (i\gamma^\mu \partial_\mu - m_q)\Psi_q - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + g_s \bar{\Psi}_q \gamma^\mu T_a \Psi_q G^a_\mu$$

$$D_\mu = \partial_\mu - i g_s T_a G^a_\mu$$

$\boxed{\text{Strong coupling constant } \alpha_s = g_s^2/(4\pi)}$

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f_{abc} G^b_\mu G^c_\nu$$

Lorentz gauge: $\partial_\mu G^\mu_a = 0$

$\bar{\Psi} = \Psi^+ \gamma^0$; $\mathcal{L}_{int} = g_s \bar{\Psi}_q \gamma^\mu T_a \Psi_q G^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + 1/4 F'^a_{\mu\nu} F'^{a\mu\nu}$; $F'^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu$

L1.3

Group theory:

A group = a set of elements

(each element can be a sym transformation, e.g a rotation}, an integer number, ...)

$$G = \{g_1, g_2, g_3, \ldots\}$$

Satisfying 4 requirements.

1, Closure

Any $g_i, g_j \in G$, then $g_i \oplus g_j \in G$

where $\oplus$ is a binary operation. (e.g. addition, multiplication, ...)

2, Associativity

For any $g_i, g_j, g_k \in G$, then

$$(g_i \oplus g_j) \oplus g_k = g_i \oplus (g_j \oplus g_k)$$

3, Identity element

There exists an element $e \in G$, such that

$$a \oplus e = a \text{ and } e \oplus a = a.$$

④ Inverse element

For each $g \in G$, there exists an element g' such that

$$g \oplus g' = e \text{ and } g' \oplus g = e.$$

Representation: $g \to \Lambda_g$ (a $d \times d$ matrix); $e \to I_d$ (unit matrix),

If $\Lambda_{g_1 g_2} = \Lambda_{g_1} \Lambda_{g_2} \Rightarrow \{\Lambda_{g_1}, \Lambda_{g_2}, \Lambda_{g_3}, \ldots\}$

is called a linear representation of the group G. d is the dimension of this representation. Any representation can be expressed in terms of irreducible representations. $SU(2)$: $d = 2j+1$ (j = non-negative integer); $SU(3)$: p, q non-negative integers, $d = \frac{1}{2}(p+1)(q+1)(p+q+2)$.

L1.4

$$\psi_q^i(x) = \sum_{s=1}^{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}^{is}\, u_s(\vec{p})\, e^{-ipx} + b_{\vec{p}}^{is+}\, v_s(\vec{p})\, e^{ipx}\right)$$

$$G_\mu^c(x) = \sum_{\lambda=1}^{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left(a_{\vec{p}}^{c\lambda}\, \varepsilon_\mu^\lambda(\vec{p})\, e^{-ipx} + a_{\vec{p}}^{c\lambda +}\, \varepsilon_\mu^\lambda(\vec{p})^*\, e^{ipx}\right)$$

For observable gluon states, traveling with momentum k^μ:

$$k^\mu \cdot \varepsilon_\mu^\lambda(\vec{k}) = 0 \quad \text{[transverse polarizations]}$$

We can further impose:

$$q^\mu \cdot \varepsilon_\mu^\lambda(\vec{k}) = 0 \quad [q^\mu \text{ is an arbitrary vector}]$$

to have:

$$\sum_{\lambda=1}^{2} \varepsilon_\mu^{\lambda *}\varepsilon_\nu^\lambda = -g_{\mu\nu} + \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q}$$

with $q^2 = 0$, $k\cdot q \neq 0$.

For observable quark states, traveling with momentum p^μ:

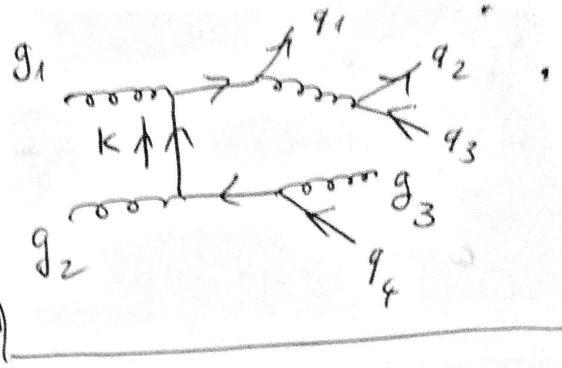
$$\not{p}\, u_s(\vec{p}) = m_q\, u_s(\vec{p}); \quad \not{p}\, v_s(\vec{p}) = -m_q\, v_s(\vec{p})$$

Next topics: Cross section, then Feynman rules.

Home work: Prove that $\mathcal{L}_{QCD}$ is invariant under the local $SU(3)_c$ transformation:

$$\psi'_i = U_{ij}\psi_j; \quad T^a G'^a_\mu = U\left(T^a G^a_\mu - \frac{i}{g_s} U^{-1}\partial_\mu U\right)U^{-1}$$

Feynman rules: (I)



Internal quark line:

$i \xrightarrow{k} j$: $\quad i\,\delta_{ij}\,\dfrac{(\not{k}+m)}{k^2-m^2+i\epsilon}$

[quark propagator]

Internal gluon line:

$a \xrightarrow{k} b$, μ ν $\quad \dfrac{-i\,\delta_{ab}}{k^2+i\epsilon}\left[g_{\mu\nu}-(1-\eta)\dfrac{k_\mu k_\nu}{k^2}\right]$

[gluon propagator] $\ni$ physical and un-physical degrees of freedom

Internal ghost line:

$a \xrightarrow{k} b$ $\quad \dfrac{-i\,\delta_{ab}}{k^2+i\epsilon}$

[ghost propagator] $\ni$ unphysical degrees of freedom

Reason for introducing η (gauge-fixing parameter): ~~later~~ and the ghost degrees of freedom can be seen later.

External fermions: $\xrightarrow{p}$: $\bar{u}_s(p)$; + incoming fermion: $u_s(p)$

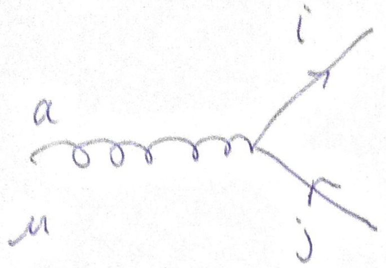
+ outgoing fermion

+, outgoing antifermion: $\xleftarrow{p}$: $v_s(p)$; incoming antifermion: $\bar{v}_s(p)$

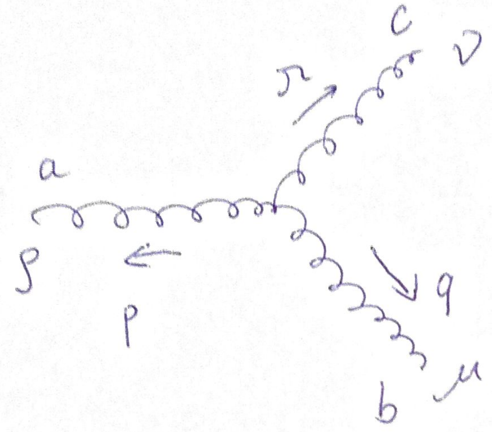
External gauge boson (photon, gluon,):

+, outgoing particle: $\varepsilon^\lambda_\mu(p)^*$; incoming particle: $\varepsilon^\lambda_\mu(p)$

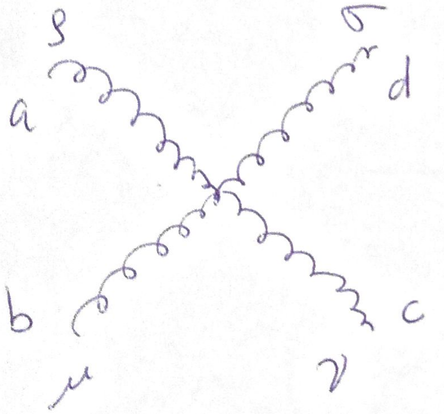
Feynman Rules (II):



$$i g_s \gamma_\mu T^a_{ij}$$



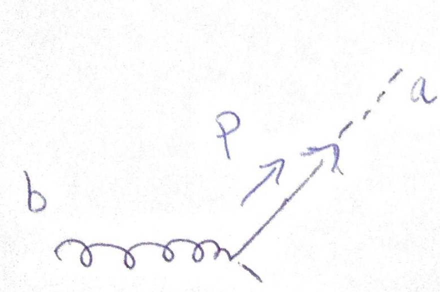
$$-g_s f^{abc}\left[(p-q)_\nu g_{\rho\mu} + (q-r)_\rho g_{\mu\nu} + (r-p)_\mu g_{\nu\rho}\right]$$



$$-i g_s^2 f^{abe} f^{cde}(g_{\rho\nu} g_{\mu\sigma} - g_{\rho\sigma} g_{\mu\nu})$$

$$-i g_s^2 f^{ace} f^{bde}(g_{\rho\mu} g_{\nu\sigma} - g_{\rho\sigma} g_{\mu\nu})$$

$$-i g_s^2 f^{ade} f^{cbe}(g_{\rho\nu} g_{\mu\sigma} - g_{\rho\mu} g_{\sigma\nu})$$

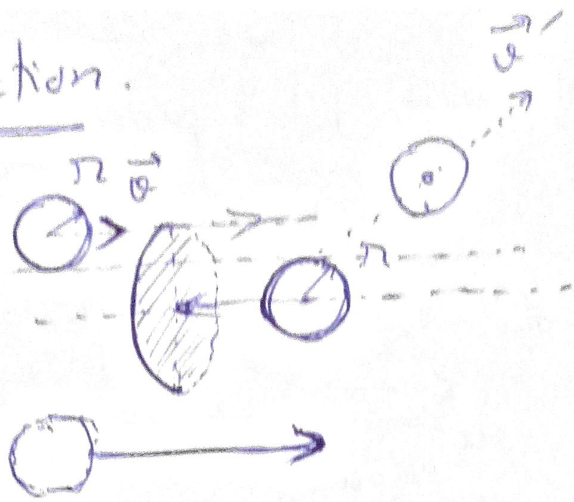


$$g_s f^{abc} p_\mu$$

Note: $\mathcal{L}_{FP\text{-}ghost} = i(\partial^\mu \chi_1^a) D_\mu^{ab} \chi_2^b$; $D_\mu^{ab} = \delta^{ab}\partial_\mu - g_s f^{abc} A_\mu^c$

χ_1^a, χ_2^b are real fields.

Cross section.



Classical physics:

For hard inelastic balls of radius r. The cross section of two colliding balls is: $\sigma = \pi (2r)^2 = 4\pi r^2$.

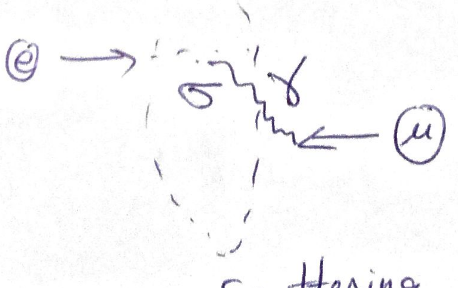
$$\Rightarrow [\sigma] = m^2.$$

In the natural unit where $\hbar = c = 1$

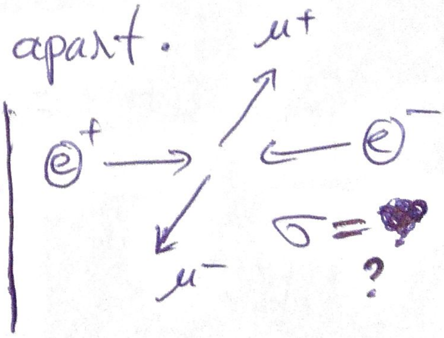
$$\Rightarrow [\sigma] = \frac{1}{GeV^2}$$

Quantum particles:

+) $r = 0$.
+) We do not know where ~~is~~ the particles are.
+) Particles interact when they are far apart.

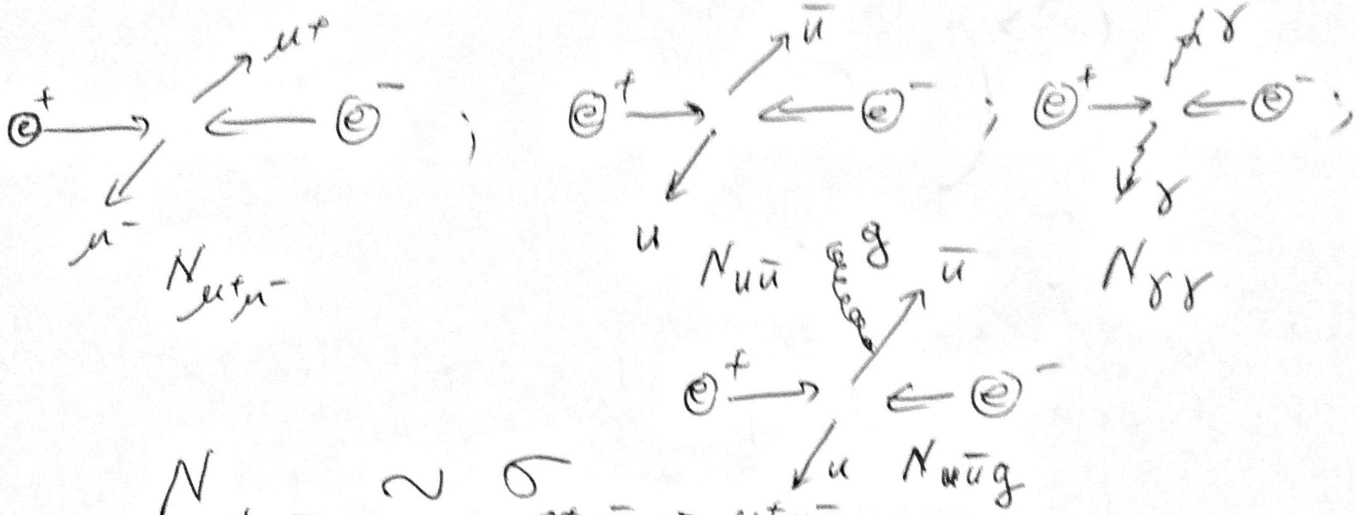


$: \sigma_{e\mu} = \infty$



$\Rightarrow$ the notion of a scattering cross section has to be extended. In quantum mechanics, ~~ex~~ even when the 2 colliding particles are aimed to be head-to-head, we can never say for sure that a scattering is going to happen.

We therefore define the cross section as the probability that. a specific scattering process will take place. a measure of



$$N_{\mu^+\mu^-} \sim \sigma_{e^+e^- \to \mu^+\mu^-}$$

$\Rightarrow$ experimentalists can measure the cross section by counting the number of events.

Theorists have to calculate the cross section from the probability of the scattering process:

$$q_a + q_b \to p_1 + \cdots + p_N$$

$$\text{Probability} = |M|^2 = {}_{out}\langle p_1, p_2, \ldots p_N | q_a, q_b \rangle_{in}^{\,2}$$

Differential cross section:

$$d\sigma = \frac{J}{\text{flux}} \cdot |M|^2 \, d\phi_N ,$$

$$\text{flux} = 4\sqrt{(q_a \cdot q_b)^2 - m_a^2 m_b^2} ,$$

$J = \frac{1}{j!}$ is a statistical factor to be included for each group of j identical particles in the final state.

• $d\phi_N$ is the differential phase space volume.

$$d\phi_N = \prod_{i=1}^{N} \frac{d^3\vec{p}_i}{(2\pi)^3 2p_i^0} \cdot (2\pi)^4 \delta^4\left(p_a + p_b - \sum_{i=1}^{N} p_i\right)$$

When experimentalists count the number of events in a certain phase-space volume that their detector covers (measured in the spherical angles θ, φ and the minimum energy E_{min}). Call this volume $V_{meas.}$

Theorists have to do the same to calculate the cross sections:

$$\sigma_{th} = \frac{J}{flux} \int_{V_{meas.}} |M|^2 d\phi_N$$

Do tree-level here first. Include quantum corrections later.

$$M = {}_{out}\langle p_1, p_2, \cdots p_N | q_a, q_b \rangle_{in}$$

$$= \langle p_1, p_2, \cdots p_N | (S-1) | q_a, q_b \rangle$$

$$= \lim_{T \to \infty(1-i\epsilon)} \left({}_0\langle p_1, p_2, \cdots p_N | T\left(\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right) | q_a, q_b \rangle_0 \right)^{-1}$$

Connected, amputated

$$\times \sqrt{\tilde{Z}_a}\sqrt{\tilde{Z}_b} \cdot \prod_{i=1}^{N} \sqrt{\tilde{Z}_i}$$

[Ref: Peskin and Schroeder book]
Muta, Foundations of QCD

$$H_I(t) = -\int d^3x\, \mathcal{L}_{int}(x)$$

$\tilde{Z}_i$: Lehmann–Symanzik–Zimmermann (LSZ) factors, having the form: $\tilde{Z}_i = 1 + \alpha_s a_i + \alpha_s^2 b_i + \cdots$