Journée de Rencontres Jeunes Chercheurs (JRJC) 2019

Leading hadronic contribution to the muon magnetic anomaly from lattice QCD

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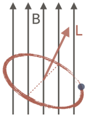
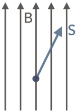
Please welcome...

$$a_\ell = \frac{g_\ell - 2}{2}$$

- Leptons ℓ have magnetic moments $\vec{\mu}_\ell$ due to their spin $\vec{s} = \hbar\frac{\vec{\sigma}}{2}$



$$\vec{\mu}_L = \frac{e}{2mc}\vec{L}$$



$$\vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell c}\vec{s}$$

- **Pauli** equation (g was initially left as free parameter)

$$i\hbar\partial_t\phi(x) = \left[\frac{1}{2m_\ell}\left(-i\hbar\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2 - g\frac{e}{2m_\ell c}\vec{s}\cdot\vec{B} + e_\ell A_0\right]\phi(x)$$

- **Dirac** eq. for a $\frac{1}{2}$−spin particle interacting with electromagnetic field $A_\mu(x)$

$$i\hbar\partial_t\psi(x) = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi(x)$$

- Let $\psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$: **Dirac eq.** $\xrightarrow[v\ll c]{E\sim mc^2}$ **Pauli eq.** $\Leftrightarrow g = 2$

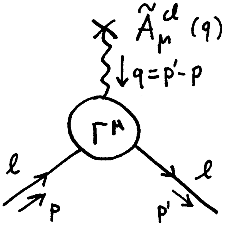
$$g|_{\text{Dirac}} = 2$$

- Kinster and Houston (1934): 1st exp. confirmations of Dirac's prediction $g_e = 2$.

- Kusch and Foley (1948): $g_e = 2.00238(10)$

$$a_e^{\text{exp (1948)}} = \frac{g_e - 2}{2} = 0.00119(5)$$

- Meantime renormalization was developed in QED to make sense of infinite integrals in perturbation theory.



$$i\mathcal{M} = -ie\bar{u}_\ell(p')\Gamma^\mu(p',p)u(p)\tilde{A}^{\text{cl}}_\mu(q)$$

Imposing { Poincaré invariance, Parity conservation, Current conservation }

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

$$g_\ell = 2\left[\underbrace{F_1(0)}_{=1} + F_2(0)\right] \Rightarrow \boxed{a_\ell = F_2(0)}$$

  - a_ℓ is adimensional
  - a_ℓ comes from loops but it's **UV-finite** once fields, charges and masses have been renormalized

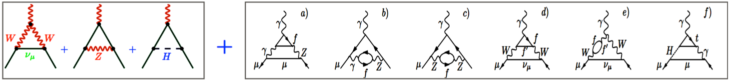
- Schwinger (1948) evaluates the lowest order QED contribution to a_e

$$a_e^{(1)\text{QED}} = \frac{\alpha}{2\pi} \approx 0.0011614 \quad \text{(99\% of the anomaly)}$$

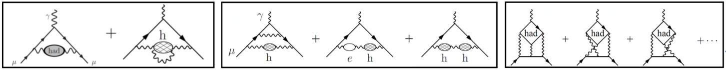
- The full Stadard Model contributes
  - **QED**
    $a_\ell^{QED} = C_\ell^{(2)} \left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + \dots$
    with $C_\ell^{(2n)} \equiv A_1^{(2n)} + A_{2,\ell}^{(2n)}(m_\ell/m_{\ell'}) + A_{3,\ell}^{(2n)}(m_\ell/m_{\ell'}, m_{\ell'}/m_{\ell''})$

$+ O((\alpha/\pi)^3)$

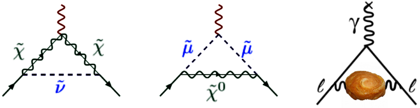
  - **Weak**
    $a_\ell^{weak} = O\left(G_F m_\ell^2\right) + O\left(G_F m_\ell^2 \alpha\right) =$



  - **QCD**
    $a_\ell^{had} = \underbrace{a_\ell^{\text{LO-HVP}}}_{O(\alpha^2)} + \underbrace{a_\ell^{\text{NLO-HVP}}}_{O(\alpha^3)} + \underbrace{a_\ell^{\text{HLbyL}}}_{O(\alpha^2)} + \underbrace{a_\ell^{\text{EWRC}}}_{O(\alpha G_F m_\ell^2)} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$

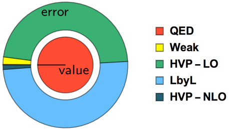


- There could be BSM contributions $a_\ell^{\text{SUSY}} = O(m_\ell^2/M_{\text{SUSY}}^2)$



- $\mathbf{a_e^{exp}} = 11596521.8073(28) \times 10^{-10}$ [D. Hanneke, S. Fogwell, G. Gabrielse (2008)]
  - $\tau_e = \infty$, $m_e = 0.511$ MeV
  - Dominated by QED effects up the 0.66 ppb precision level: sensitivity to hadronic and weak effects as well as to physics beyond SM is tiny.
  - a_e is known 829 more precisely than a_μ
  - Provides best measure of $\alpha = 137.035999046(27)$ [Parker, Yu, Zhong, Estey, Muller (2018)]
  - $a_e^{exp} - a_e^{SM} = -0.0087(28)^{exp}(23)^{\alpha}(2)^{SM} \times 10^{-10} \rightarrow -2.4\sigma$ discrepancy
- $\mathbf{a_\tau^{exp}}$
  - $\tau_\tau = 3 \times 10^{-15}$s, $m_\tau = 1777$ MeV
    Very short lived $\Rightarrow$ no measurements yet
- $\mathbf{a_\mu} = 11659208.9(6.3) \times 10^{-10}$ [BNL '04]
  - latest measurement from experiment **Muon E821** at BNL (final report issued in 2006)
  - $\tau_\mu = 2 \times 10^{-6}$s, $m_\mu = 105$ MeV
  - $m_\mu^2/m_e^2 \approx 205^2$ times more sensitive to physics BSM.
  - $a_\mu^{exp} - a_\mu^{SM} = 27.9(6.3)^{exp}(3.6)^{\alpha}(2)^{SM} \times 10^{-10} \rightarrow 3.6\sigma$ discrepancy

em	$(11658471.895 \pm 0.008) \times 10^{-10}$	[Kinoshita et al., Phys.Rev.Lett. **109**, 111808 (2012)]
weak	$(15.36 \pm 0.10) \times 10^{-10}$	[Gnendinger et al., Phys.Rev. **D88**, 053005 (2013)]
HVP	$(693.26 \pm 2.46) \times 10^{-10}$	[Keshavarzi et al., Phys. Rev. **D97** 114025 (2018)]
HVP(α^3)	$(-9.84 \pm 0.06) \times 10^{-10}$	[Hagiwara et al., J.Phys. **G38**, 085003 (2011)]
LbL	$(10.5 \pm 2.6) \times 10^{-10}$	[Prades et al.,Adv.Ser.Direct.High Energy Phys. 20, 303 (2009)]



To match the future experimental precision:

1. $\Delta a_\mu^{hvp}/a_\mu^{hvp}$ is now 0.5% (R-ratio) or $2-3\%$ (LQCD), must reach 0.2%
2. $\Delta a_\mu^{lbl}/a_\mu^{lbl}$ is now 20%, must reach 10%

Two experiments aim to reduce precision of a_μ^{exp} to 0.14 ppb

- **Muon g-2** at Fermilab (operative since 2017)
- **Muon g-2/EDM** at J-PARC (planned for $\geq$ 2020)

A muon in a $\perp$ magnetic field experiences two frequencies (here $\vec{\omega} \parallel \vec{B}$):

1. $\omega_C = \frac{eB}{m_\mu c \gamma}$ (circular precession)
2. $\omega_S = g_\mu \frac{eB}{2m_\mu c} + \frac{1-\gamma}{\gamma} \frac{eB}{m_\mu c}$ (spin precession)

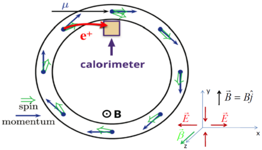
$\Rightarrow \omega_a = \omega_S - \omega_C = a_\mu \frac{eB}{m_\mu c}$

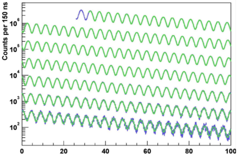
Also $\vec{E}$ contributes to $\vec{\omega}$:

$$\vec{\omega} = a_\mu \frac{e\vec{B}}{m_\mu} - a_\mu \frac{\gamma}{\gamma+1}(\vec{\beta} \cdot \frac{e\vec{B}}{m_\mu})\vec{\beta} + \underbrace{\left(-a_\mu + \frac{1}{\gamma^2+1}\right)}_{0 \text{ at } \gamma_{\text{magic}} \text{ (FNAL)}} \underbrace{\frac{\vec{\beta} \times \vec{E}}{c}}_{\vec{E}=0 \text{ (J-PARC)}} \frac{e}{m_\mu}$$

Muons decay preferentially in the spin direction: each detector will measure

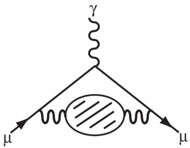
$$N(E,t) = N_0(E) e^{-t/\gamma\tau_\mu} \left[1 + A(E) \cos(\omega_a t + \phi)\right]$$





$$a_\mu^{\text{LO HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w\left(Q^2/m_\mu^2\right) \underbrace{\left(\Pi(Q^2) - \Pi(0)\right)}_{\hat{\Pi}(Q^2)}$$

$$w\left(Q^2/m_\mu^2\right) = \pi\left(r + 2 - \sqrt{r(r+4)}\right)^2 / \sqrt{r(r+4)}$$



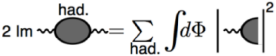
– How can I get $\hat{\Pi}(Q^2)$? –

For now the **R-ratio** gives the best extimation of a_μ^{hvp}:

1. Using a dispertion relation to rewrite

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$

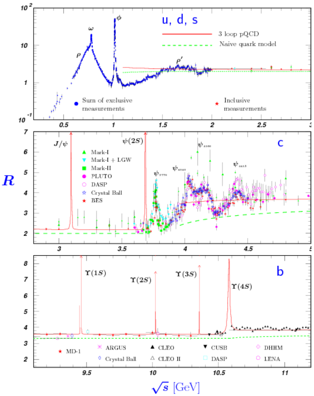
2. replacing $\mathrm{Im}\Pi$ with $\sigma(\gamma \to \text{had})$ via optical theorem:



3. taking from experiments (BaBar, KLOE,...) the R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{had}, s)}{\sigma(e^+e^- \to e^+e^-)} = \frac{\sigma(e^+e^- \to \gamma^* \to \text{had}, s)}{\frac{4\pi\alpha^2}{3S}}$$

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} R(s)$$



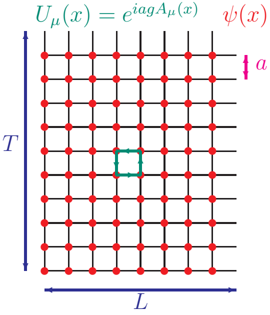
Lattice QCD

1974: K. Wilson needed a framework to explore the confined phase of QCD.
Recipe:

1. Lattice = euclidean tool ⇒ $t \to -ix_4$
2. Discretize spacetime on lattice Λ of size $L^3 \times T$ and spacing a
   ⇒ IR and UV divergences are now regularized
3. Define discretized equivalents of continuum fields
   - $\phi(t, \vec{x}) \to \phi(x)$ with $x = a(n_1, n_2, n_3, n_4)$
   - $A_\mu(x) \to U_\mu(x) = P\{\exp \int_x^{x+a\hat{e}_\mu} ds A_\mu(s)\}$
4. Define lattice action so that $\mathcal{S}_{\text{lat}} \xrightarrow{a\to 0} \mathcal{S}_E$
   Example: the Wilson action

$$S_W = \frac{\beta}{2N} \sum_{x\in\Lambda,\mu\nu} \mathrm{Re}\,\mathrm{Tr}\{U_{\mu\nu}(x)\} \xrightarrow{a\to 0} -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$$

$\mathrm{Tr}\{U_{\mu\nu}(x)\} = \mathrm{Tr}\{U_\mu(x)U_\nu(x+a\hat{e}_\mu)U_\mu^\dagger(x+a\hat{e}_\nu)U_\nu^\dagger(x)\}$
is the elementary plaquette.
The equivalence holds iff $\beta = \frac{2N}{g^2}$



Lattice QCD

5 The QFT partition function

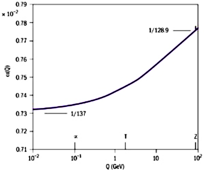
$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi e^{i\left[S_G + \int \bar\psi D[M]\psi\right]} = \int \mathcal{D}A_\mu \, \mathrm{det}(D[M]) e^{iS_G}$$

becomes on the lattice

$$\mathcal{Z} = \prod_{\rho,x} \int dU_\rho(x) \, \mathrm{det}(D_x[M]) e^{-\frac{\beta}{2N}\sum \mathrm{ReTr} U_{\mu\nu}}$$

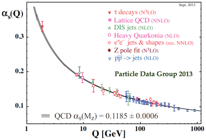
Looks like the partition function of a statistical system in the **canonical ensemble**

$\Rightarrow$ new tools from Statistical Mechanics, like stochastic methods (MC, ...) to perform numerical evaluations.



6 Asymptotic freedom implies $a \xrightarrow{g\to 0} 0$!

$$\begin{cases} \alpha_s(a) = \frac{g^2(a)}{4\pi} = -\frac{4\pi}{b_0 \log(a^2\Lambda^2)} \\ b_0 = 11 - \frac{2}{3} n_f \end{cases} \Rightarrow a(g) \sim \frac{1}{\Lambda} \exp\left(-\frac{8\pi^2}{b_0 g^2}\right)$$



7 Fix QCD parameters using $1 + n_f$ physical inputs.

8 Restore ∞-volume by extrapolation from simulations in different volumes.

Lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

1. Form R-ratio we know that $\Pi(Q)$ is peaked on $m_\mu/2 \sim 50\text{MeV} \Rightarrow$ NP regime
2. In Euclidean space the polarization tensor is

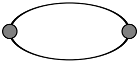
$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x}\langle J_\mu(x)J_\nu(0)\rangle \quad \leftarrow \quad \text{measurable on the lattice}$$
$$= \underbrace{(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\,\Pi(Q^2)}_{\text{O(4) inv. and current conservation}} \leftarrow \quad \text{what we need}$$

with $J_\mu/e = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

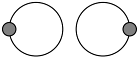
3. We define

$$C_L(t) = \frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(x)J_i(0)\rangle$$
$$= C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\text{disc}}(t) = C_L^{I=0}(t) + C_L^{I=1}(t)$$

where $C_L^{ud}(t), \ldots$ correspond to different Wick contractions:



quark-connected (qc)

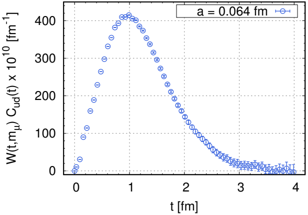


quark-disconnected (qd)

Lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

4. Performing a Fourier transformation and subtracting $\Pi^L_{\mu\nu}(Q=0) \neq 0$ (which gives a FV contribution $\propto L^4 \exp(-EL/2)$) , we get the connection between $\hat{\Pi}^f_L(Q^2)$ and $C_L(t)$:

$$\hat{\Pi}^f_L(Q^2) \equiv \Pi^f_L(Q^2) - \Pi^f_L(0) = \frac{1}{3}\sum_{i=1}^{3} \frac{\Pi^f_{ii,L}(0) - \Pi^f_{ii,L}(Q)}{Q^2} - \Pi^f_L(0) = 2a \sum_{t=0}^{T/2} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2}\right] \mathrm{Re}C^f_L(t)$$



$W(tm_\ell, Q^2_{\max}/m^2_\ell)\ \mathrm{Re}C^{ud}_L(t)$

Therefore $a_{\ell,f}^{\text{LO-HVP}}$ from $C^f_L(t)$ is

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \le Q^2_{\max}) = \alpha^2 \int_0^{Q^2_{\max}} \frac{dQ^2}{Q^2} w\left(Q^2/m^2_\mu\right) \hat{\Pi}(Q^2)$$

$$= \lim_{\substack{a\to 0\\ L\to\infty\\ T\to\infty}} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m^2_\ell}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\max}/m^2_\ell)\ \mathrm{Re}C^f_L(t)$$

$$W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$

5. Finally, adding using pQCD for $Q > Q_{\max}$ (blue=measurable on the lattice)

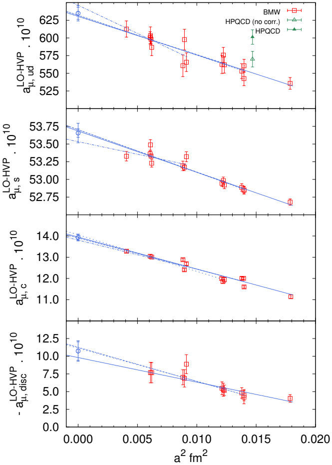
$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{\max}) + \gamma_\ell(Q_{\max})\, \hat{\Pi}^f(Q^2_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$$

Continuum extrapolation: fit, systematic and statistical errors

- Large t $\leftrightarrow$ large statistical error
  Have implemented many improvements to reduce statistical error.

  Ex: signal is lost for $t \gtrsim 3\,\mathrm{fm}$ for $C^{ud/\mathrm{disc}}(t)$
  $\Rightarrow C_L^f(t > t_c)$ is replaced by the average of its upper and lower bounds.

- Many possibilities for interpolation to ϕ_{m_q} and for continuum extrapolation (adjustments in the fit model)
  $\Rightarrow$ systematic error



1. Vary t_c ($t_0 = (3 \pm 0.134)$fm, $t_0 = (2.600 \pm 0.134)$fm).
2. a-cuts (no cut, 0.118 fm, 0.111 fm, 0.095 fm).
3. Q-cuts ($Q^2_{\max} = 1, \cdots, 5\,\mathrm{GeV}^2$).
4. There are 4 external inputs: w_0 , m_π, m_ρ, m_{η_c}.
   Each ensemble has a different set of inputs: the masses are chosen to be around their physical value. In the fit we can choose $i_\pi, i_\pi, i_\eta = 0, 1$.
5. Choice of Padé approximant.

$$y(a^2, m_\pi^2, m_K^2, m_\eta^2) = \mathrm{cnt}\frac{1 + \sum_{i=1}^{\mathrm{np}} \mathrm{cp}_i a^{2i}}{1 + \sum_{i=1}^{\mathrm{nq}} \mathrm{cq}_i a^{2i}} + i_\pi c_\pi \Delta m_\pi^2 + i_K c_K \Delta m_K^2 + i_\eta c_\eta \Delta m_\eta^2$$

Comparison of $a_\mu^{\rm hvp}$ obtained by between different collaborations

- "No New Physics" scenario: $= (720 \pm 7) \times 10^{-10}$
- BMWc '17 consistent with "No new physics" scenario & pheno.
- Total uncertainty of 2.7% is $\sim 6\times$ pheno. err.
- Need to reduce our error by 10!
  - → Increase statistics by $\times 50 \div 100$ (need new methods)
  - → Understand and control FV effects much better
  - → Compute QED and $m_d \neq m_u$ corrections (see RBC/UKQCD '17-'18, ETM '17)
  - → Need high precision scale setting
  - → Detailed comparison to phenomenology to understand where we agree and why if we don't
  - → Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18), only if the two agree statistically with comparable errors