KdV charges ride the $T\overline{T}$ flow (and $J\overline{T}$)

Bruno Le Floch (Philippe Meyer Institute, ENS Paris)

June 4, 2019
ENS, Paris

based on 1903.07606 and 1906.04xxx
with Márk Mezei (SCGP, Stony Brook)

$T\overline{T}$ (Do I need to introduce it at a $T\overline{T}$ workshop?)

$$\text{“}T\overline{T}\text{”} = \det T = T_{00}T_{11} - T_{01}T_{10} = T\overline{T} - \Theta\overline{\Theta} \quad \text{(ignore factors)}$$

Universal irrelevant operator (in translation-invariant 2d QFTs)

Only ambiguous by total derivatives (Zamolodchikov)

$T\overline{T}$ (Do I need to introduce it at a $T\overline{T}$ workshop?)

$$\text{“}T\overline{T}\text{''} = \det T = T_{00}T_{11} - T_{01}T_{10} = T\overline{T} - \Theta\overline{\Theta} \quad \text{(ignore factors)}$$

Universal irrelevant operator (in translation-invariant 2d QFTs)
Only ambiguous by total derivatives (Zamolodchikov)
Deforming by $\partial_{\lambda_{T\overline{T}}} S = \int \mathrm{d}t \int_0^L \mathrm{d}x\, T\overline{T}$ (Smirnov–Zamolodchikov)

- preserves symmetries
- **calculable spectrum** $\partial_{\lambda_{T\overline{T}}} E = \partial_L (E^2 - P^2)/4$ (Burgers eq.)

Related to Jackiw–Teitelboim gravity (Dubovsky, Gorbenko, …), 2d random geometry (Cardy), AdS_3 holography (McGough, Mezei, Verlinde, Giveon, Kutasov, Guica, …)

$T\overline{T}$ (Do I need to introduce it at a $T\overline{T}$ workshop?)

$$\text{“}T\overline{T}\text{”} = \det T = T_{00}T_{11} - T_{01}T_{10} = T\overline{T} - \Theta\overline{\Theta} \quad \text{(ignore factors)}$$

Universal irrelevant operator (in translation-invariant 2d QFTs)
Only ambiguous by total derivatives (Zamolodchikov)
Deforming by $\partial_{\lambda_{T\overline{T}}} S = \int \mathrm{d}t \int_0^L \mathrm{d}x\, T\overline{T}$ (Smirnov–Zamolodchikov)

- preserves symmetries
- **calculable spectrum** $\partial_{\lambda_{T\overline{T}}} E = \partial_L(E^2 - P^2)/4$ (Burgers eq.)

Related to Jackiw–Teitelboim gravity (Dubovsky, Gorbenko, ...), 2d random geometry (Cardy), AdS_3 holography (McGough, Mezei, Verlinde, Giveon, Kutasov, Guica, ...)

1. Generalizations of $T\overline{T}$
2. KdV charges upon $T\overline{T}$ deformation (uses Lorentz-invariance)
3. Energies upon $T\overline{T} + J\overline{T}$ deformation (my April Stony Brook talk)

Burgers equation

Definition

$$\epsilon^{\mu\nu} T_{0\mu}(x) T_{1\nu}(y) = (T\overline{T})(y) + \textbf{derivatives}$$

On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n| T\overline{T} |n\rangle = \epsilon^{\mu\nu} \langle n| T_{0\mu} |n\rangle \langle n| T_{1\nu} |n\rangle$$

Burgers equation

Definition

$$\epsilon^{\mu\nu} T_{0\mu}(x) T_{1\nu}(y) = (T\overline{T})(y) + \textbf{derivatives}$$

On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n | T\overline{T} | n \rangle = \epsilon^{\mu\nu} \langle n | T_{0\mu} | n \rangle \langle n | T_{1\nu} | n \rangle$$

Now deform by $T\overline{T}$:

$$\partial_{\lambda_{T\overline{T}}} E_n = \langle n | \partial_{\lambda_{T\overline{T}}} H | n \rangle = L \underbrace{\langle n | T_{00} | n \rangle}_{-E_n/L} \underbrace{\langle n | T_{11} | n \rangle}_{-\partial_L E_n} - L \underbrace{\langle n | T_{01} | n \rangle}_{iP_n/L} \underbrace{\langle n | T_{10} | n \rangle}_{iP_n/L}$$

In a relativistic theory:

Burgers equation

$$\boxed{\partial_{\lambda_{T\overline{T}}} E_n = E_n \partial_L E_n + \frac{P_n^2}{L}}$$

(if Lorentz-invariance)

Current bilinears

Generalize $T\overline{T}$, $J\overline{T}$, $J\overline{J}$

$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu$ (point-split) defined modulo derivatives

Current bilinears

Generalize $T\overline{T}$, $J\overline{T}$, $J\overline{J}$

$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu$ (point-split) defined modulo derivatives

Proof.

$$\frac{\partial}{\partial x^{\rho}} \epsilon_{\mu\nu} J_a^\mu(x) J_b^\nu(y) = \left(\frac{\partial}{\partial x^{\nu}} + \frac{\partial}{\partial y^{\nu}}\right) \epsilon_{\mu\rho} J_a^\mu(x) J_b^\nu(y)$$

use OPE

$$\epsilon_{\mu\nu} \sum_i \partial_\rho c_i(x-y) O_i^{\mu\nu}(y) = \epsilon_{\mu\rho} \sum_i c_i(x-y) \partial_\nu O_i^{\mu\nu}(y)$$

so any O_i with non-constant $c(x-y)$ must be a total derivative $\partial_\nu(\dots)$ □

Current bilinears

Generalize $T\overline{T}$, $J\overline{T}$, $J\overline{J}$

$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu$ (point-split) defined modulo derivatives

Proof.

$$\frac{\partial}{\partial x^\rho} \epsilon_{\mu\nu} J_a^\mu(x) J_b^\nu(y) = \left(\frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial y^\nu}\right) \epsilon_{\mu\rho} J_a^\mu(x) J_b^\nu(y)$$

use OPE

$$\epsilon_{\mu\nu} \sum_i \partial_\rho c_i(x-y) O_i^{\mu\nu}(y) = \epsilon_{\mu\rho} \sum_i c_i(x-y) \partial_\nu O_i^{\mu\nu}(y)$$

so any O_i with non-constant $c(x-y)$ must be a total derivative $\partial_\nu(\ldots)$ □

$$\partial_{\lambda^{ab}} S = \int \mathrm{d}^2x\, X_{ab} \text{ deformation}$$

Only makes sense if J_a and J_b are still conserved at order $O(\lambda)$ etc.

This happens if and only if $[Q_a, Q_b] = 0$ (see later for “if” direction)

Evolution of energies under deformation by current bilinears

$$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu \longrightarrow \partial_{\lambda^{ab}} S = \int \mathrm{d}^2 x\, X_{ab} \longrightarrow \partial_{\lambda^{ab}} H = \int \mathrm{d}x\, X_{ab}$$

Evolution of energies under deformation by current bilinears

$$X_{ab} \coloneqq \epsilon_{\mu\nu} J_a^\mu J_b^\nu \longrightarrow \partial_{\lambda^{ab}} S = \int \mathrm{d}^2 x\, X_{ab} \longrightarrow \partial_{\lambda^{ab}} H = \int \mathrm{d}x\, X_{ab}$$

On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n | X_{ab} | n \rangle = \epsilon_{\mu\nu} \langle n | J_a^\mu | n \rangle \langle n | J_b^\nu | n \rangle$$

$$\boxed{\partial_{\lambda^{ab}} E_n = L \epsilon_{\mu\nu} \langle n | J_a^\mu | n \rangle \langle n | J_b^\nu | n \rangle}$$

Evolution of energies under deformation by current bilinears

$$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu \longrightarrow \partial_{\lambda^{ab}} S = \int \mathrm{d}^2 x\, X_{ab} \longrightarrow \partial_{\lambda^{ab}} H = \int \mathrm{d}x\, X_{ab}$$

On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n|X_{ab}|n\rangle = \epsilon_{\mu\nu} \langle n|J_a^\mu|n\rangle\langle n|J_b^\nu|n\rangle$$

$$\boxed{\partial_{\lambda^{ab}} E_n = L\epsilon_{\mu\nu} \langle n|J_a^\mu|n\rangle\langle n|J_b^\nu|n\rangle}$$

$$\partial_{\lambda^{ab}} E_n = 2\, \underbrace{L\langle n|J_{[a}^0|n\rangle}_{(Q_a)_n} \langle n|J_{b]}^1|n\rangle$$

- Compact flavour symmetry $\implies$ Q_n quantized
- Spatial translation $\implies$ $Q_n = iP_n \in (2\pi i/L)\mathbb{Z}$
- Time translation $\implies$ $Q_n = -E_n$
- KdV charges $\implies$ $\boxed{\text{need } \partial_\lambda Q_n \text{ equation}}$

Evolution of energies under deformation by current bilinears

$$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu \longrightarrow \partial_{\lambda^{ab}} S = \int \mathrm{d}^2 x\, X_{ab} \longrightarrow \partial_{\lambda^{ab}} H = \int \mathrm{d}x\, X_{ab}$$

On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n | X_{ab} | n \rangle = \epsilon_{\mu\nu} \langle n | J_a^\mu | n \rangle \langle n | J_b^\nu | n \rangle$$

$$\boxed{\partial_{\lambda^{ab}} E_n = L \epsilon_{\mu\nu} \langle n | J_a^\mu | n \rangle \langle n | J_b^\nu | n \rangle}$$

$$\partial_{\lambda^{ab}} E_n = 2 \underbrace{L \langle n | J_{[a}^0 | n \rangle}_{(Q_a)_n} \boxed{\underbrace{\langle n | J_{b]}^1 | n \rangle}_{?}}$$

- Compact flavour symmetry $\implies$ Q_n quantized
- Spatial translation $\implies$ $Q_n = iP_n \in (2\pi i / L)\mathbb{Z}$
- Time translation $\implies$ $Q_n = -E_n$
- KdV charges $\implies$ $\boxed{\text{need } \partial_\lambda Q_n \text{ equation}}$

Plan

$$\partial_{\lambda^{ab}} E_n = 2\langle n|Q_{[a}|n\rangle\langle n|J^1_{b]}|n\rangle$$

Plan

$$\partial_{\lambda^{ab}} E_n = 2\langle n|Q_{[a}|n\rangle\langle n|J^1_{b]}|n\rangle$$

Part I: Playing with commutators
Similar equation for $\partial_{\lambda^{ab}}(Q_c)_n$

Plan

$$\partial_{\lambda^{ab}} E_n = 2\langle n|Q_{[a}|n\rangle\langle n|J^1_{b]}|n\rangle$$

Part I: Playing with commutators
Similar equation for $\partial_{\lambda^{ab}}(Q_c)_n$

Then two case studies (much shorter).

Part II: $T\overline{T}$ deformation of Lorentz-invariant theory,
KdV charges "ride the Burgers flow"

Part III: $T\overline{T} + J\overline{T} + \dots$ deformation
using background gauge fields (non-rigorous)

Maybe I put too many calculations, sorry

Cartan subalgebra: KdV charges P_s

Focus on **commuting subset** $\{P_s\}$ of all charges $\{Q_a\}$:
translations, Cartan of flavour symmetries, KdV charges

Conserved currents $\bar{\partial} T_{s+1} = \partial \Theta_{s-1}$ of spin $s \in \mathbb{Z}$, charges

$$P_s = \frac{1}{2\pi} \oint \left(T_{s+1} \mathrm{d}z + \Theta_{s-1} \mathrm{d}\bar{z} \right)$$

Cartan subalgebra: KdV charges P_s

Focus on **commuting subset** $\{P_s\}$ of all charges $\{Q_a\}$:
translations, Cartan of flavour symmetries, KdV charges

Conserved currents $\bar{\partial} T_{s+1} = \partial \Theta_{s-1}$ of spin $s \in \mathbb{Z}$, charges

$$P_s = \frac{1}{2\pi} \oint \left(T_{s+1} \mathrm{d}z + \Theta_{s-1} \mathrm{d}\bar{z} \right)$$

with stress-tensor $\begin{pmatrix} T & \Theta \\ \overline{\Theta} & \overline{T} \end{pmatrix} = \begin{pmatrix} T_2 & \Theta_0 \\ T_0 & \Theta_{-2} \end{pmatrix}$

$[P_1, \mathcal{O}] = -i\partial\mathcal{O}$ and $[P_{-1}, \mathcal{O}] = i\bar{\partial}\mathcal{O}$ with $P_{\pm 1} = -\frac{1}{2}(H \pm P)$

Cartan subalgebra: KdV charges P_s

Focus on **commuting subset** $\{P_s\}$ of all charges $\{Q_a\}$:
translations, Cartan of flavour symmetries, KdV charges

Conserved currents $\bar\partial T_{s+1} = \partial\Theta_{s-1}$ of spin $s \in \mathbb{Z}$, charges

$$P_s = \frac{1}{2\pi} \oint \left(T_{s+1} \mathrm{d}z + \Theta_{s-1} \mathrm{d}\bar{z} \right)$$

with stress-tensor $\begin{pmatrix} T & \Theta \\ \overline{\Theta} & \overline{T} \end{pmatrix} = \begin{pmatrix} T_2 & \Theta_0 \\ T_0 & \Theta_{-2} \end{pmatrix}$

$[P_1, \mathcal{O}] = -i\partial\mathcal{O}$ and $[P_{-1}, \mathcal{O}] = i\bar\partial\mathcal{O}$ with $P_{\pm 1} = -\frac{1}{2}(H \pm P)$

Example (CFT): $T_2 = T,\ T_4 = :T^2:,\ T_6 = :T^3: + \frac{c+2}{12} :(\partial T)^2:, \ldots$
$\Theta_{-2k} = \overline{T_{2k}}, \quad \Theta_0 = \Theta_2 = \Theta_4 = \cdots = T_0 = T_{-2} = T_{-4} = \cdots = 0$

KdV currents fixed (up to improvements) by spin and $[P_s, P_t] = 0$

The operators A_s^t

Integrating $[P_s, T_{t+1}\mathrm{d}z + \Theta_{t-1}\mathrm{d}\bar{z}]$ on a contour $\mathcal{C}$ gives $[P_s, P_t^{\mathcal{C}}] = 0$ so the one-form is exact:

$$[P_s, T_{t+1}] = -i\partial A_s^t = [P_1, A_s^t]$$
$$[P_s, \Theta_{t-1}] = -i\bar{\partial} A_s^t = -[P_{-1}, A_s^t]$$

The operators A_s^t

Integrating $[P_s, T_{t+1}\mathrm{d}z + \Theta_{t-1}\mathrm{d}\bar{z}]$ on a contour $\mathcal{C}$ gives $[P_s, P_t^{\mathcal{C}}] = 0$ so the one-form is exact:

$$[P_s, T_{t+1}] = -i\partial A_s^t = [P_1, A_s^t]$$
$$[P_s, \Theta_{t-1}] = -i\bar{\partial} A_s^t = -[P_{-1}, A_s^t]$$

In particular $\boxed{A_1^t = T_{t+1}}$ and $\boxed{A_{-1}^t = -\Theta_{t-1}}$ (up to shifts by identity)

Generic A_s^t are **not** in conserved currents

The operators A_s^t

Integrating $[P_s, T_{t+1}\mathrm{d}z + \Theta_{t-1}\mathrm{d}\bar{z}]$ on a contour $\mathcal{C}$ gives $[P_s, P_t^{\mathcal{C}}] = 0$ so the one-form is exact:

$$
\begin{aligned}
[P_s, T_{t+1}] &= -i\partial A_s^t = [P_1, A_s^t] \\
[P_s, \Theta_{t-1}] &= -i\bar{\partial} A_s^t = -[P_{-1}, A_s^t]
\end{aligned}
$$

In particular $\boxed{A_1^t = T_{t+1}}$ and $\boxed{A_{-1}^t = -\Theta_{t-1}}$ (up to shifts by identity)
Generic A_s^t are **not** in conserved currents

Example (CFT): $T_2 = T,\ T_4 = {:}T^2{:},\ T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}$

$$
\begin{array}{lll}
A_1^1 = T_2 & A_1^3 = T_4 & A_1^5 = T_6 \\
A_3^1 = 3T_4 + \partial(\dots) & A_3^3 = 4{:}T^3{:} - \frac{c+2}{2}{:}(\partial T)^2{:} & \\
A_5^1 = 5T_6 + \partial(\dots) & A_5^3 = \frac{15{:}T^4{:}}{2} - \frac{5(13+2c){:}T(\partial T)^2{:}}{3} + \frac{5(-47+4c+c^2){:}(\partial^2 T)^2{:}}{72} &
\end{array}
$$

The operators A_s^t

Integrating $[P_s, T_{t+1}\mathrm{d}z + \Theta_{t-1}\mathrm{d}\bar{z}]$ on a contour $\mathcal{C}$ gives $[P_s, P_t^{\mathcal{C}}] = 0$ so the one-form is exact:

$$[P_s, T_{t+1}] = -i\partial A_s^t = [P_1, A_s^t]$$
$$[P_s, \Theta_{t-1}] = -i\bar{\partial} A_s^t = -[P_{-1}, A_s^t]$$

In particular $\boxed{A_1^t = T_{t+1}}$ and $\boxed{A_{-1}^t = -\Theta_{t-1}}$ (up to shifts by identity)
Generic A_s^t are **not** in conserved currents

Example (CFT): $T_2 = T$, $T_4 = {:}T^2{:}$, $T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}$

$$A_1^1 = T_2 \qquad A_1^3 = T_4 \qquad A_1^5 = T_6$$
$$A_3^1 = 3T_4 + \partial(\dots) \qquad A_3^3 = 4{:}T^3{:} - \frac{c+2}{2}{:}(\partial T)^2{:}$$
$$A_5^1 = 5T_6 + \partial(\dots) \qquad A_5^3 = \frac{15{:}T^4{:}}{2} - \frac{5(13+2c){:}T(\partial T)^2{:}}{3} + \frac{5(-47+4c+c^2){:}(\partial^2 T)^2{:}}{72}$$

The operators A_s^t

$A_1^t = T_{t+1}$ and $A_{-1}^t = -\Theta_{t-1}$
so the definition is equivalent to $[P_{\pm 1}, A_s^t] = [P_s, A_{\pm 1}^t]$

The symmetry generalizes: $\boxed{[P_s, A_t^u] = [P_t, A_s^u]}$

The operators A_s^t

$A_1^t = T_{t+1}$ and $A_{-1}^t = -\Theta_{t-1}$
so the definition is equivalent to $[P_{\pm 1}, A_s^t] = [P_s, A_{\pm 1}^t]$

The symmetry generalizes: $\boxed{[P_s, A_t^u] = [P_t, A_s^u]}$

Proof.

$$\begin{aligned} & [P_1, [P_{[s}, A_{t]}^u]] \\ &= [P_{[s|}, [P_1, A_{|t]}^u]] && \text{(Jacobi)} \\ &= [P_{[s}, [P_{t]}, A_1^u]] && \text{(definition of } A\text{)} \\ &= 0 && \text{(Jacobi)} \end{aligned}$$

Likewise $[P_{-1}, [P_{[s}, A_{t]}^u]] = 0$

so $[P_{[s}, A_{t]}^u] =$ multiple of identity $= 0$ (because traceless) □

Deforming by current bilinears preserves symmetries

For two spins u, v consider $\delta H = \int \mathrm{d}x\, X^{u,v}$

with $X^{u,v} = (T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1})_{\text{reg}}$ current bilinear

To preserve **conservation**, $\delta P_s = ?$

Deforming by current bilinears preserves symmetries

For two spins u, v consider $\delta H = \int \mathrm{d}x\, X^{u,v}$

with $X^{u,v} = (T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1})_{\text{reg}}$ current bilinear

To preserve **conservation**, $\delta P_s = ?$

$$[H, \delta P_s] = [P_s, \delta H] = \int \mathrm{d}x\, \underbrace{[P_s, X^{u,v}(x)]}_{\textbf{total derivative?}}$$

Deforming by current bilinears preserves symmetries

For two spins u, v consider $\delta H = \int \mathrm{d}x\, X^{u,v}$

with $X^{u,v} = (T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1})_{\text{reg}}$ current bilinear

To preserve **conservation**, $\delta P_s = ?$

$$[H, \delta P_s] = [P_s, \delta H] = \int \mathrm{d}x\, \underbrace{[P_s, X^{u,v}(x)]}_{\textbf{total derivative? yes!}}$$

Proof. $[P_s, X^{u,v}] = [P_s, T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1}]$

$= [P_1, A_s^u]\Theta_{v-1} + [P_{-1}, A_s^u]T_{v+1} - (u \leftrightarrow v)$

$= [P_1, A_s^u\Theta_{v-1}] + [P_{-1}, A_s^u T_{v+1}] - (u \leftrightarrow v)$ □

Deforming by current bilinears preserves symmetries

For two spins u, v consider $\delta H = \int \mathrm{d}x\, X^{u,v}$

with $X^{u,v} = (T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1})_{\text{reg}}$ current bilinear

To preserve **conservation**, $\delta P_s = ?$

$$[H, \delta P_s] = [P_s, \delta H] = \int \mathrm{d}x\, \underbrace{[P_s, X^{u,v}(x)]}_{\textbf{total derivative? yes!}}$$

Proof.
$$\begin{aligned}[P_s, X^{u,v}] &= [P_s, T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1}] \\ &= [P_1, A_s^u]\Theta_{v-1} + [P_{-1}, A_s^u]T_{v+1} - (u \leftrightarrow v) \\ &= [P_1, A_s^u\Theta_{v-1}] + [P_{-1}, A_s^u T_{v+1}] - (u \leftrightarrow v) \qquad \square\end{aligned}$$

$$\boxed{\delta P_s = \frac{1}{2}\int \mathrm{d}x\, (X^{u,v}_{s,1} + X^{u,v}_{-1,s})} \quad \text{where } X^{u,v}_{s,t} = (A_s^u A_t^v - A_t^u A_s^v)_{\text{reg}}$$

Toward an evolution equation

Goal: $\partial_\lambda \langle n|P_s|n\rangle = \ldots$ for states $|n\rangle$ on $S^1 \times \mathbb{R}$

We know $\partial_\lambda P_s = \frac{1}{2}\int \mathrm{d}x\,(X_{s,1}^{u,v} + X_{-1,s}^{u,v})$ so we compute

$\langle n|X_{s,t}^{u,v}|n\rangle =$

Toward an evolution equation

Goal: $\partial_\lambda \langle n|P_s|n\rangle = \dots$ for states $|n\rangle$ on $S^1 \times \mathbb{R}$

We know $\partial_\lambda P_s = \frac{1}{2}\int \mathrm{d}x\,(X^{u,v}_{s,1} + X^{u,v}_{-1,s})$ so we compute

$\langle n|X^{u,v}_{s,t}|n\rangle = \langle n|A^u_s|n\rangle\langle n|A^v_t|n\rangle - \langle n|A^u_t|n\rangle\langle n|A^v_s|n\rangle$ **(factorization)**

Toward an evolution equation

Goal: $\partial_\lambda \langle n|P_s|n\rangle = \dots$ for states $|n\rangle$ on $S^1 \times \mathbb{R}$

We know $\partial_\lambda P_s = \frac{1}{2}\int \mathrm{d}x\,(X^{u,v}_{s,1} + X^{u,v}_{-1,s})$ so we compute

$\langle n|X^{u,v}_{s,t}|n\rangle = \langle n|A^u_s|n\rangle\langle n|A^v_t|n\rangle - \langle n|A^u_t|n\rangle\langle n|A^v_s|n\rangle$ **(factorization)**

Proof summary. Insert complete set of states (eigenstates of all $P_\bullet$)

$$\langle n|X^{u,v}_{s,t}|n\rangle = \sum_{|m\rangle}\Big(\langle n|A^u_s|m\rangle\langle m|A^v_t|n\rangle - \langle n|A^u_t|m\rangle\langle m|A^v_s|n\rangle\Big)$$

For any spin r, compute a bit to show

$$\langle n|[P_r, A^u_s]|m\rangle\langle m|A^v_t|n\rangle - \langle n|[P_r, A^u_t]|m\rangle\langle m|A^v_s|n\rangle = 0$$

This is $\langle m|P_r|m\rangle - \langle n|P_r|n\rangle$ times the summand,
so summand $= 0$ except for $|m\rangle = |n\rangle$ **(assumes nondegenerate spectrum)** $\square$

Side comment on collisions

In fact we can define more **general collisions**

$$k!A^{t_1}_{[s_1}(x_1)\dots A^{t_k}_{s_k]}(x_k) = X^{t_1,\dots,t_k}_{s_1,\dots,s_k}(x) + \sum_i [P_{s_i}, \dots]$$

- defined up to commutators $\sum_i [P_{s_i}, \dots]$
  (like $X^{u,v}$ is defined up to derivatives)
- obey factorization

$$\langle n|X^{t_1,\dots,t_k}_{s_1,\dots,s_k}|n\rangle = k!\langle n|A^{t_1}_{[s_1}|n\rangle \dots \langle n|A^{t_k}_{s_k]}|n\rangle$$

- obey

$$[P_{[s_0}, X^{t_1,\dots,t_k}_{s_1,\dots,s_k]}] = 0$$

(but deforming by these operators breaks all symmetries, so they are most likely not that useful)

Main evolution equation

Denoting $\langle\mathcal{O}\rangle := \langle n|\mathcal{O}|n\rangle$, we end up with

$$\boxed{2\partial_{\lambda_{u,v}}\langle P_s\rangle = \langle P_u\rangle\langle A_s^v\rangle - \langle P_v\rangle\langle A_s^u\rangle}$$

Sadly, $\partial_{\lambda_{u,v}}\langle A_s^t\rangle =$ nothing in general

Main evolution equation

Denoting $\langle \mathcal{O} \rangle := \langle n | \mathcal{O} | n \rangle$, we end up with

$$\boxed{2\partial_{\lambda_{u,v}} \langle P_s \rangle = \langle P_u \rangle \langle A_s^v \rangle - \langle P_v \rangle \langle A_s^u \rangle}$$

Sadly, $\partial_{\lambda_{u,v}} \langle A_s^t \rangle =$ nothing in general

Part II: $(u, v) = (1, -1)$, arbitrary s
$T\overline{T}$ deformation of Lorentz-invariant theory,
KdV charges “ride the Burgers flow”

Main evolution equation

Denoting $\langle \mathcal{O} \rangle := \langle n | \mathcal{O} | n \rangle$, we end up with

$$\boxed{2\partial_{\lambda_{u,v}} \langle P_s \rangle = \langle P_u \rangle \langle A_s^v \rangle - \langle P_v \rangle \langle A_s^u \rangle}$$

Sadly, $\partial_{\lambda_{u,v}} \langle A_s^t \rangle =$ nothing in general

Part II: $(u,v) = (1,-1)$, arbitrary s
$T\overline{T}$ deformation of Lorentz-invariant theory,
KdV charges "ride the Burgers flow"

Part III: $|u|, |v|, |s| \leq 1$, combine different deformations
$T\overline{T} + J\overline{T} + \ldots$ deformation
using background gauge fields (non-rigorous)

Part II: Deforming by $T\overline{T}$

$$\boxed{2\partial_{\lambda_{T\overline{T}}}\langle P_s\rangle = \langle P_1\rangle\langle A_s^{-1}\rangle - \langle P_{-1}\rangle\langle A_s^{1}\rangle}$$

Need to understand $A_s^{\pm 1}$. Two steps.

- Understand ∂_L
- Relate $A_s^{\pm 1}$ to $A_{\pm 1}^{s}$ in Lorentz-invariant theories

Changing the length

We know $\partial_L H = \int \mathrm{d}x\, T_{xx} = \dfrac{1}{2\pi} \int \mathrm{d}x \left(A_1^1 - A_1^{-1} + A_{-1}^1 - A_{-1}^{-1}\right)$

Changing the length

We know $\partial_L H = \int \mathrm{d}x\, T_{xx} = \frac{1}{2\pi} \int \mathrm{d}x\, (A_1^1 - A_1^{-1} + A_{-1}^1 - A_{-1}^{-1})$
Use conservation $[H, \partial_L P_s] = [P_s, \partial_L H]$ to deduce

$$\partial_L P_s = \frac{1}{2\pi} \int \mathrm{d}x\, (A_s^1 - A_s^{-1})$$

Changing the length

We know $\partial_L H = \int \mathrm{d}x\, T_{xx} = \frac{1}{2\pi} \int \mathrm{d}x\, (A_1^1 - A_1^{-1} + A_{-1}^1 - A_{-1}^{-1})$
Use conservation $[H, \partial_L P_s] = [P_s, \partial_L H]$ to deduce

$$\partial_L P_s = \frac{1}{2\pi} \int \mathrm{d}x\, (A_s^1 - A_s^{-1})$$

For states with zero momentum ($\langle P_1 - P_{-1} \rangle = 0$), we're done:

$$2\partial_{\lambda_{T\overline{T}}} \langle P_s \rangle = \langle H \rangle \partial_L \langle P_s \rangle$$

Changing the length

We know $\partial_L H = \int \mathrm{d}x\, T_{xx} = \frac{1}{2\pi} \int \mathrm{d}x\, (A_1^1 - A_1^{-1} + A_{-1}^1 - A_{-1}^{-1})$
Use conservation $[H, \partial_L P_s] = [P_s, \partial_L H]$ to deduce

$$\partial_L P_s = \frac{1}{2\pi} \int \mathrm{d}x\, (A_s^1 - A_s^{-1})$$

For states with zero momentum ($\langle P_1 - P_{-1} \rangle = 0$), we're done:

$$2\partial_{\lambda_{T\overline{T}}} \langle P_s \rangle = \langle H \rangle \partial_L \langle P_s \rangle$$

In fact, for zero momentum ($\langle P_1 - P_{-1} \rangle = 0$),

$$\partial_\lambda \langle P_s \rangle = \langle Q \rangle \partial_L \langle P_s \rangle \quad \text{under } \epsilon^{\mu\nu} J_\mu T_{x\nu} \text{ deformation}$$

The deformation "scales space according to $\langle Q \rangle$"

Relating A_s^t and A_t^s

Relating A_s^t and A_t^s

Example (CFT): $T_2 = T,\ T_4 = {:}T^2{:},\ T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}$

$$\begin{array}{lll} A_1^1 = T_2 & A_1^3 = T_4 & A_1^5 = T_6 \\ A_3^1 = 3T_4 + \partial(\dots) & A_3^3 = \dots & A_3^5 = \frac{3}{5}A_5^3 + \dots \\ A_5^1 = 5T_6 + \partial(\dots) & A_5^3 = \dots & \end{array}$$

Relating A_s^t and A_t^s

Example (CFT): $T_2 = T,\ T_4 = {:}T^2{:},\ T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}$

$$\begin{aligned}
A_1^1 &= T_2 & A_1^3 &= T_4 & A_1^5 &= T_6 \\
A_3^1 &= 3T_4 + \partial(\dots) & A_3^3 &= \dots & A_3^5 &= \tfrac{3}{5}A_5^3 + \dots \\
A_5^1 &= 5T_6 + \partial(\dots) & A_5^3 &= \dots
\end{aligned}$$

Observe $t\,A_s^t = s\,A_t^s$ up to improvements of currents $T_4,\ T_6,\ \dots$
This selects preferred improvements of higher-spin currents:
$T_{s+1} = \frac{1}{s}A_s^1$ is uniquely defined (up to shifts by the identity)

More generally true in Lorentz-invariant theories

Evolution of KdV charges under $T\overline{T}$ deformation

Combining (up to factors)

$$\langle n|A_s^1 - A_s^{-1}|n\rangle = \partial_L \langle n|P_s|n\rangle$$
$$\langle n|A_s^1 + A_s^{-1}|n\rangle = \frac{s}{L}\langle n|P_s|n\rangle$$

Evolution of KdV charges under $T\overline{T}$ deformation

Combining (up to factors)

$$\langle n|A_s^1 - A_s^{-1}|n\rangle = \partial_L \langle n|P_s|n\rangle$$
$$\langle n|A_s^1 + A_s^{-1}|n\rangle = \frac{s}{L}\langle n|P_s|n\rangle$$

we get

$$\partial_\lambda \langle P_s\rangle = \langle H\rangle \partial_L \langle P_s\rangle + \frac{s}{L}\langle P\rangle\langle P_s\rangle$$

All charges propagate along the same characteristics

Starting from a CFT we can solve

$$\langle P_s\rangle = \begin{cases} \#\langle P_1\rangle^s & \text{for holomorphic currents} \\ \#\langle P_1\rangle^{-s} & \text{for antiholomorphic currents} \end{cases}$$

Energy levels of CFT $+ J\overline{T} + T\overline{T} + \dots$

$$
\begin{aligned}
T\overline{T} &= \epsilon^{\mu\nu} T_{0\mu} T_{1\nu}, \\
J\overline{T} &= \epsilon^{\mu\nu} J_\mu T_{\bar{z}\nu}, \\
J\overline{J} &= \epsilon^{\mu\nu} J_\mu \overline{J}_\nu
\end{aligned}
$$

Let's do all of them

Energy levels of CFT $+ J\overline{T} + T\overline{T} + \dots$

$$
\begin{aligned}
T\overline{T} &= \epsilon^{\mu\nu} T_{0\mu} T_{1\nu}, \\
J\overline{T} &= \epsilon^{\mu\nu} J_\mu T_{\bar{z}\nu}, \\
J\overline{J} &= \epsilon^{\mu\nu} J_\mu \overline{J}_\nu
\end{aligned}
$$

Let's do all of them

Why?

- To explore possible UV behaviours
- To find if we get new deformations by commuting these

$$\partial_{\lambda_{AB}} E_n = 2 \underbrace{L\langle n|J_0^{[A}|n\rangle}_{Q_n^A} \langle n|J_1^{B]}|n\rangle$$

- Compact flavour symmetry $\implies$ Q_n quantized
- Spatial translation $\implies$ $Q_n = iP_n \in (2\pi i/L)\mathbb{Z}$
- Time translation $\implies$ $Q_n = -E_n$
- KdV charges $\implies$ need $\partial_\lambda Q_n$ equation

$$\partial_{\lambda_{AB}} E_n = 2\,\underbrace{L\langle n|J_0^{[A}|n\rangle}_{Q_n^A}\boxed{\underbrace{\langle n|J_1^{B]}|n\rangle}_{?}}$$

- Compact flavour symmetry $\Longrightarrow$ Q_n quantized
- Spatial translation $\Longrightarrow$ $Q_n = iP_n \in (2\pi i/L)\mathbb{Z}$
- Time translation $\Longrightarrow$ $Q_n = -E_n$
- KdV charges $\Longrightarrow$ need $\partial_\lambda Q_n$ equation

Space component of current $\langle n | J_1 | n \rangle$

- **Failed attempt:** write

$$\partial_\lambda \langle n|J_1|n\rangle = \underbrace{\Big(\partial_\lambda \langle n|\Big) J_1 |n\rangle}_{\substack{\text{nonzero because}\\ J_1|n\rangle \not\propto |n\rangle}} + \langle n| \underbrace{\Big(\partial_\lambda J_1\Big)}_{\substack{\text{non-universal}\\ \text{(see later)}}} |n\rangle + \langle n| J_1 \Big(\partial_\lambda |n\rangle\Big)$$

Space component of current $\langle n | J_1 | n \rangle$

- **Failed attempt:** write

$$\partial_\lambda \langle n | J_1 | n \rangle = \underbrace{\Big(\partial_\lambda \langle n|\Big) J_1 |n\rangle}_{\substack{\text{nonzero because} \\ J_1|n\rangle \not\propto |n\rangle}} + \langle n| \underbrace{\Big(\partial_\lambda J_1\Big)}_{\substack{\text{non-universal} \\ \text{(see later)}}} |n\rangle + \langle n| J_1 \Big(\partial_\lambda |n\rangle\Big)$$

- **Successful method:** turn on background gauge fields

$$H \to H + a_A \int \mathrm{d}x\, J_1^A \qquad \Longrightarrow \langle n | J_1^A | n \rangle = \frac{1}{L} \partial_{a_A} E_n$$

In particular $\langle n | T_{11} | n \rangle = -\partial_L E_n$ and $\langle n | T_{01} | n \rangle = i \partial_b E_n$ where $b :=$ background for time translation.

Naive transport equation

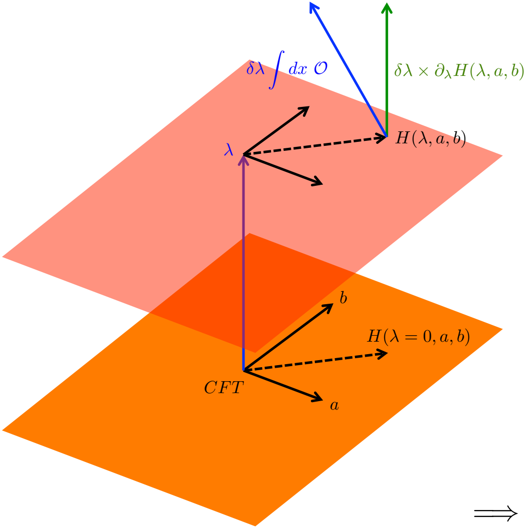
$$\frac{\partial E_n}{\partial \lambda_{AB}} \overset{\text{naive}}{=} 2\underbrace{L\langle n|J_0^{[A}|n\rangle}_{Q_n^A}\underbrace{\langle n|J_1^{B]}|n\rangle}_{\partial E_n/\partial a_B} \overset{\text{naive}}{=} Q_n^A \frac{\partial E_n}{\partial a_B} - Q_n^B \frac{\partial E_n}{\partial a_A}$$

More generally all charges **would** obey

$$\left(\frac{\partial}{\partial \lambda_{AB}} - Q_n^A \frac{\partial}{\partial a_B} + Q_n^B \frac{\partial}{\partial a_A}\right) Q_n^C \overset{\text{naive}}{=} 0$$

solved by the **method of characteristics**

Deformations don't commute



We want

$$\partial_a E_n \simeq \langle J_1 \rangle$$
$$\partial_b E_n \simeq \langle T_{01} \rangle$$

everywhere so define

$$\text{Theory}(\lambda, a, b)$$

by deforming
first by λ
then by a and b

$$\implies \partial_\lambda T_{00} = J^A_\mu J^B_\nu \epsilon^{\mu\nu} + \dots$$

Example: $J\overline{T}$

$$
\begin{aligned}
\partial_\lambda T_{00}(\lambda, a, b) = & -2\pi i J_{[0|} T_{\bar{z}|1]} - \pi b J_{[0|} T_{0|1]} \\
& -2\pi^2 i a T_{\bar{z}1} - \pi^2 ab T_{01} + \frac{\pi^2 a^2}{2} J_1 + \text{derivatives}
\end{aligned}
$$

(Origin explained later)

Example: $J\overline{T}$

$$\begin{aligned}\partial_\lambda T_{00}(\lambda, a, b) = &-2\pi i J_{[0|} T_{\bar{z}|1]} - \pi b J_{[0|} T_{0|1]} \\ &-2\pi^2 i a T_{\bar{z}1} - \pi^2 ab T_{01} + \frac{\pi^2 a^2}{2} J_1 + \text{derivatives}\end{aligned}$$

(Origin explained later)

- For $a = b = 0$ get original deformation
- No λ on RHS
- All terms bilinear (antisymmetric) or linear in currents
- Finitely many terms because $[a] = 1$, $[b] = 0$
  Analogue with KdV charges has infinitely many terms?

Example: $J\overline{T}$

$$\begin{aligned}\partial_\lambda T_{00}(\lambda, a, b) = & -2\pi i J_{[0|} T_{\bar{z}|1]} - \pi b J_{[0|} T_{0|1]} \\ & -2\pi^2 i a T_{\bar{z}1} - \pi^2 ab T_{01} + \frac{\pi^2 a^2}{2} J_1 + \text{derivatives}\end{aligned}$$

(Origin explained later)

- For $a = b = 0$ get original deformation
- No λ on RHS
- All terms bilinear (antisymmetric) or linear in currents
- Finitely many terms because $[a] = 1$, $[b] = 0$
  Analogue with KdV charges has infinitely many terms?

Deduce evolution of energies

$$\partial_\lambda E_n = (\cdots)\partial_a E_n + (\cdots)\partial_b E_n + (\cdots)\partial_L E_n$$

To solve, need initial data $E_n(\lambda = 0|a, b, L)$

Strategy

- Get initial data $E_n(\lambda = 0|a, b, L)$

  Doable for chiral flavour currents and stress tensor of CFT

  Doable for some currents of free scalars/fermions
- Write evolution equation $\partial_\lambda E_n$

  Doable for flavour symmetries and stress tensor ($J\overline{J}$, $J\overline{T}$, $J\overline{\Theta}$, $T\overline{J}$, $\Theta\overline{J}$, $T\overline{T}$)
- Solve using method of characteristics
  (actually just used Ansatz and checked)

Initial data

Goal: deform CFT by $\partial_b T_{00} = iT_{01}$
Recall **we keep momentum and charges fixed** so $T_{10} = T_{10}^{\mathrm{CFT}}$

Initial data

Goal: deform CFT by $\partial_b T_{00} = iT_{01}$
Recall **we keep momentum and charges fixed** so $T_{10} = T_{10}^{\text{CFT}}$
Ansatz

$$T_{00} = f(b)\, T_{00}^{\text{CFT}} + ig(b)\, T_{10}^{\text{CFT}}$$
$$iT_{01} = f'(b)\, T_{00}^{\text{CFT}} + ig'(b)\, T_{10}^{\text{CFT}}$$

Conservation $[\int T_{00}, T_{00}] = \partial_1 T_{01}$ translates to differential equations $f'(b) = 2f(b)g(b)$ and $g'(b) = f(b)^2 + g(b)^2$:

$$T_{00} = \frac{1}{1-b^2}\, T_{00}^{\text{CFT}} + \frac{b}{1-b^2} iT_{10}^{\text{CFT}}$$
$$iT_{01} = \frac{2b}{(1-b^2)^2}\, T_{00}^{\text{CFT}} + \frac{1+b^2}{(1-b^2)^2} iT_{10}^{\text{CFT}}$$

Initial data

Recall **we keep momentum and charges fixed** so

$$T_{10} = T_{10}^{\text{CFT}} , \qquad J_0 = J_0^{\text{CFT}} , \qquad \overline{J}_0 = \overline{J}_0^{\text{CFT}}$$

We want the deformation

$$T_{01} = -i\partial_b T_{00} , \qquad J_1 = i\partial_a T_{00} , \qquad \overline{J}_1 = -i\partial_{\overline{a}} T_{00}$$

Conservation of $T_{\mu\nu}$, J_ν, $\overline{J}_\nu$ is solved by

$$T_{00} = \frac{1}{1-b^2} T_{00}^{\text{CFT}} + \frac{ib}{1-b^2} T_{10}^{\text{CFT}} - \frac{1}{1-b} a J_0^{\text{CFT}} - \frac{1}{1+b} \overline{a}\overline{J}_0^{\text{CFT}}$$

$$\left\langle n \middle| \boxed{H = -\frac{P_+^{\text{CFT}} + aQ}{1-b} - \frac{P_-^{\text{CFT}} + \overline{a}\overline{Q}}{1+b}} \middle| n \right\rangle$$

Ambiguities

- Improvements $J_\mu \to J_\mu + \epsilon_{\mu\nu}\partial^\nu\phi$ (with ϕ local)
- Mixing $J^A_\mu \to \Lambda^A{}_B J^B_\mu$ (linear combinations)
- Shifts $J^A_\mu \to J^A_\mu + (\# \times 1)$

Fixing time components (J_0, $\overline{J}_0$, T_{10} and the evolution of T_{00})
$\implies \partial_1 J_1$ known and J_1 local $\implies J_1$ fixed up to shifts by 1

Ambiguities

- Improvements $J_\mu \to J_\mu + \epsilon_{\mu\nu}\partial^\nu \phi$ (with ϕ local)
- Mixing $J^A_\mu \to \Lambda^A{}_B J^B_\mu$ (linear combinations)
- Shifts $J^A_\mu \to J^A_\mu + (\# \times 1)$

Fixing time components (J_0, $\overline{J}_0$, T_{10} and the evolution of T_{00})
$\implies$ $\partial_1 J_1$ known and J_1 local $\implies$ J_1 fixed up to shifts by 1

Boils down to ambiguous f_1, f_2, f_3 in

$$T_{00} = \frac{1}{1-b^2} T_{00}^{\text{CFT}} + \frac{ib}{1-b^2} T_{10}^{\text{CFT}} - \frac{1}{1-b} a J_0^{\text{CFT}} - \frac{1}{1+b} \overline{a} \overline{J}_0^{\text{CFT}} \\ + f_1(b) a^2 + f_2(b) a\overline{a} + f_3(b)\overline{a}^2$$

Evolution equation

Goal: find universal evolution equation

$$\partial_\lambda T_{00}(\lambda|a,b,L) = \underbrace{\mathcal{O}_1(\lambda|a,b,L)}_{\text{e.g. } J\overline{T}} + b\mathcal{O}_2(\lambda|a,b,L) + a\mathcal{O}_3(\lambda|a,b,L) + \dots$$

Universal $\implies$ holds classically so **pick a classical theory**

Classical scalar with shift symmetry

Hamiltonian $H = \int \mathrm{d}x\, \mathcal{H}(\partial_x \phi, \Pi)$

- Translation symmetry $t \to t + \dots$ and $x \to x + \dots$
- Shift symmetry $\phi(t,x) \to \phi(t,x) + \dots$ splits into two symmetry currents J_μ and $\overline{J}_\mu$

Classical scalar with shift symmetry

Hamiltonian $H = \int \mathrm{d}x \, \mathcal{H}(\partial_x\phi, \Pi)$

- Translation symmetry $t \to t + \dots$ and $x \to x + \dots$
- Shift symmetry $\phi(t,x) \to \phi(t,x) + \dots$ splits into two symmetry currents J_μ and $\overline{J}_\mu$

Noether currents

$$J_0 = -\frac{1}{2}(\partial_x\phi - 4\pi\Pi)\,, \qquad J_1 = 2\pi i\left(\frac{\partial\mathcal{H}}{\partial(\partial_x\phi)} - \frac{1}{4\pi}\frac{\partial\mathcal{H}}{\partial\Pi}\right),$$

$$\overline{J}_0 = -\frac{1}{2}(\partial_x\phi + 4\pi\Pi)\,, \qquad \overline{J}_1 = -2\pi i\left(\frac{\partial\mathcal{H}}{\partial(\partial_x\phi)} + \frac{1}{4\pi}\frac{\partial\mathcal{H}}{\partial\Pi}\right),$$

$$T_{00} = -\mathcal{H}\,, \qquad T_{01} = -i\frac{\partial\mathcal{H}}{\partial\Pi}\frac{\partial\mathcal{H}}{\partial(\partial_x\phi)}\,,$$

$$T_{10} = -i\,\Pi\partial_x\phi\,, \qquad T_{11} = \Pi\frac{\partial\mathcal{H}}{\partial\Pi} + \partial_x\phi\frac{\partial\mathcal{H}}{\partial(\partial_x\phi)} - \mathcal{H}\,.$$

Input at $a = b = 0$**:** $\partial_\lambda \mathcal{H} = F\left(\mathcal{H}, \frac{\partial \mathcal{H}}{\partial(\partial_x \phi)}, \frac{\partial \mathcal{H}}{\partial \Pi}, \partial_x \phi, \Pi\right)$

Wanted output: $\partial_\lambda \mathcal{H} = F\left(a, b, \mathcal{H}, \frac{\partial \mathcal{H}}{\partial(\partial_x \phi)}, \frac{\partial \mathcal{H}}{\partial \Pi}, \partial_x \phi, \Pi\right)$

Input at $a = b = 0$: $\quad \partial_\lambda \mathcal{H} = F\Big(\mathcal{H}, \frac{\partial \mathcal{H}}{\partial(\partial_x\phi)}, \frac{\partial \mathcal{H}}{\partial \Pi}, \partial_x\phi, \Pi\Big)$

Wanted output: $\quad \partial_\lambda \mathcal{H} = F\Big(a, b, \mathcal{H}, \frac{\partial \mathcal{H}}{\partial(\partial_x\phi)}, \frac{\partial \mathcal{H}}{\partial \Pi}, \partial_x\phi, \Pi\Big)$

Recall $\mathcal{H}$ is defined by first turning on λ then a, b so we know nothing about $\partial_\lambda \mathcal{H}$ at non-zero a, b. But

$$\partial_a \partial_\lambda \mathcal{H} = \partial_\lambda \partial_a \mathcal{H} = \partial_\lambda i J_1 = -2\pi \left(\frac{\partial}{\partial(\partial_x\phi)} - \frac{1}{4\pi} \frac{\partial}{\partial \Pi} \right) \partial_\lambda \mathcal{H}$$

Namely $D_1 \partial_\lambda \mathcal{H} = 0$ (likewise, $D_2 \partial_\lambda \mathcal{H} = 0$) where

$$D_1 := -\partial_a - 2\pi \left(\frac{\partial}{\partial(\partial_x\phi)} - \frac{1}{4\pi} \frac{\partial}{\partial \Pi} \right)$$

$$D_2 := -\partial_b - \left(\frac{\partial \mathcal{H}}{\partial(\partial_x\phi)} \frac{\partial}{\partial \Pi} + \frac{\partial \mathcal{H}}{\partial \Pi} \frac{\partial}{\partial(\partial_x\phi)} \right)$$

$\partial_\lambda \mathcal{H}$ known at $a = b = 0$, and $D_1 \partial_\lambda \mathcal{H} = 0$ and $D_2 \partial_\lambda \mathcal{H} = 0$
$\implies$ unique solution

$$\partial_\lambda \mathcal{H} = \sum_{m,n\geq 0} \frac{1}{m!n!} a^m b^n D_1^m D_2^n F\left(\mathcal{H}, \frac{\partial \mathcal{H}}{\partial(\partial_x \phi)}, \frac{\partial \mathcal{H}}{\partial \Pi}, \partial_x \phi, \Pi\right)$$

where ∂_a, ∂_b, $\frac{\partial}{\partial(\partial_x\phi)}$, $\frac{\partial}{\partial \Pi}$ inside D_1, D_2 act on the $\mathcal{H}$ arguments too

$\partial_\lambda\mathcal{H}$ known at $a = b = 0$, and $D_1\partial_\lambda\mathcal{H} = 0$ and $D_2\partial_\lambda\mathcal{H} = 0$
$\implies$ unique solution

$$\partial_\lambda\mathcal{H} = \sum_{m,n\geq 0} \frac{1}{m!n!} a^m b^n D_1^m D_2^n F\Big(\mathcal{H}, \frac{\partial\mathcal{H}}{\partial(\partial_x\phi)}, \frac{\partial\mathcal{H}}{\partial\Pi}, \partial_x\phi, \Pi\Big)$$

where ∂_a, ∂_b, $\frac{\partial}{\partial(\partial_x\phi)}$, $\frac{\partial}{\partial\Pi}$ inside D_1, D_2 act on the $\mathcal{H}$ arguments too

Explicit calculations show D_1 and D_2 map currents to currents

	J_t	J_x	$\overline{J}_t$	$\overline{J}_x$	T_{tt}	T_{tx}	T_{xt}	T_{xx}
D_1	2π	0	0	0	0	0	iJ_t	iJ_x
$\overline{D}_1$	0	0	2π	0	0	0	$-i\overline{J}_t$	$-i\overline{J}_x$
D_2	iJ_x	0	$i\overline{J}_x$	0	iT_{tx}	0	$-i(T_{tt} - T_{xx})$	$-iT_{tx}$

and (bi)linears to (bi)linears
(We also found this for more general classical scalars)

Example of $J\overline{T}$:

$$\partial_\lambda \mathcal{H} = 2\pi i J_{[t|} T_{\bar{z}|x]} + \pi b J_{[t|} T_{t|x]} + 2\pi^2 i a T_{\bar{z}x} + \pi^2 ab T_{tx} - \frac{\pi^2 a^2}{2} J_x$$

More generally, given in the table

	$J\overline{J}$	$J\overline{T}$	$J\Theta$	$\overline{J}T$	$\overline{J\Theta}$	$T\overline{T}$	J_t	J_x	$\overline{J}_t$	$\overline{J}_x$	T_{tt}	T_{tx}	T_{xt}	T_{xx}
$J\overline{J}$	1	0	0	0	0	0	0	$i\pi\bar{a}$	0	$-i\pi a$	0	0	0	0
$J\overline{T}$	$i\pi\bar{a}$	$1-\frac{b}{2}$	$-\frac{b}{2}$	0	0	0	0	$-\frac{\pi^2}{2}\left(a^2+\bar{a}^2\right)$	0	$\pi^2 a\bar{a}$	0	$-\pi^2 a(1-b)$	0	$i\pi^2 a$
$J\Theta$	$-i\pi\bar{a}$	$\frac{b}{2}$	$1+\frac{b}{2}$	0	0	0	0	$\frac{\pi^2}{2}\left(a^2+\bar{a}^2\right)$	0	$-\pi^2 a\bar{a}$	0	$-\pi^2 a(1+b)$	0	$-i\pi^2 a$
$\overline{J}T$	$-i\pi a$	0	0	$1+\frac{b}{2}$	$\frac{b}{2}$	0	0	$\pi^2 a\bar{a}$	0	$-\frac{\pi^2}{2}\left(a^2+\bar{a}^2\right)$	0	$-\pi^2\bar{a}(1+b)$	0	$-i\pi^2\bar{a}$
$\overline{J\Theta}$	$i\pi a$	0	0	$-\frac{b}{2}$	$1-\frac{b}{2}$	0	0	$-\pi^2 a\bar{a}$	0	$\frac{\pi^2}{2}\left(a^2+\bar{a}^2\right)$	0	$-\pi^2\bar{a}(1-b)$	0	$i\pi^2\bar{a}$
$T\overline{T}$	0	$-i\pi a$	$-i\pi a$	$i\pi\bar{a}$	$i\pi\bar{a}$	1	0	0	0	0	0	$i\pi^3\left(a^2-\bar{a}^2\right)$	0	0

RHS has no λ. Finitely-many terms, all (bi)linears of currents

$$\text{“}J\overline{J}\text{”} \equiv -iJ_{[t}\overline{J}_{x]}, \qquad \text{“}J\overline{T}\text{”} \equiv 2\pi i J_{[t|} T_{\bar{z}|x]}, \qquad \text{“}J\Theta\text{”} \equiv -2\pi i J_{[t|} T_{z|x]}$$

$$\text{“}\overline{J}T\text{”} \equiv -2\pi i \overline{J}_{[t|} T_{z|x]}, \qquad \text{“}\overline{J\Theta}\text{”} \equiv 2\pi i \overline{J}_{[t|} T_{\bar{z}|x]}, \qquad \text{“}T\overline{T}\text{”} \equiv -2\pi^2 T_{t[t|} T_{x|x]}$$

Back to quantum

Conjecture that **universal** classical equation holds quantumly

$$\partial_{\lambda_{J\overline{T}}} \mathcal{H} = 2\pi i J_{[t|} T_{\bar{z}|x]} + \pi b J_{[t|} T_{t|x]} + 2\pi^2 i a T_{\bar{z}x} + \pi^2 ab T_{tx} - \frac{\pi^2 a^2}{2} J_x + \text{derivatives}$$

Back to quantum

Conjecture that **universal** classical equation holds quantumly

$$\partial_{\lambda_{J\overline{T}}}\mathcal{H} = 2\pi i J_{[t|}T_{\overline{z}|x]} + \pi b J_{[t|}T_{t|x]} + 2\pi^2 i a T_{\overline{z}x} + \pi^2 ab T_{tx} - \frac{\pi^2 a^2}{2} J_x + \text{derivatives}$$

Use factorization $\langle n|J_{[t|}T_{\overline{z}|x]}|n\rangle = \langle n|J_{[t|}|n\rangle\langle n|T_{\overline{z}|x]}|n\rangle$ then write time-components as charges and space-components as a, b, L derivatives to get

$$0 = \frac{2L}{i\pi}\frac{\partial}{\partial\lambda_{J\overline{T}}}E_n + \Big(-\overline{a}\hat{\overline{Q}}_n - \pi(a^2 - \overline{a}^2)L - (1-b)E_n + P_n\Big)\partial_a E_n - \overline{a}\hat{Q}_n\partial_{\overline{a}}E_n + (1-b)\hat{Q}_n\partial_b E_n + L\hat{Q}_n\partial_L E_n\,,$$

where $\hat{Q} \equiv Q + 2\pi a L$ and $\hat{\overline{Q}} \equiv \overline{Q} + 2\pi\overline{a}L$

Solution

Turn on $J\overline{T}$, $J\Theta$, $\overline{J}T$, $\overline{J\Theta}$, $T\overline{T}$ with a single coupling μ

$$\boxed{\epsilon_n \equiv E_n \hat{L} = \frac{1+b}{2}\epsilon + \frac{1-b}{2}\overline{\epsilon} + \frac{-B-\sqrt{B^2-4AC}}{2A}},$$

$$\epsilon \equiv \epsilon_0 + p + 2aL\left(\hat{Q} - \pi a L\right), \quad \overline{\epsilon} \equiv \epsilon_0 - p + 2\overline{a}L\left(\hat{\overline{Q}} - \pi\overline{a}L\right),$$

$$\hat{L} \equiv (1-b^2)L, \quad \hat{Q} \equiv Q + 2\pi a L, \quad \hat{\overline{Q}} \equiv \overline{Q} + 2\pi\overline{a}L,$$

$$A = \left(\frac{\pi}{2}\left(G_{J\overline{T}}^2 + G_{\overline{J}T}^2\right) + \hat{G}_{T\overline{T}}\right)\mu^2,$$

$$B = -1 - \left(G_{J\overline{T}}\hat{Q} + G_{\overline{J}T}\hat{\overline{Q}}\right)\mu + \left(\left(\pi G_{\overline{J}T}^2 + \hat{G}_{T\overline{T}}\right)\epsilon + \left(\pi G_{J\overline{T}}^2 + \hat{G}_{T\overline{T}}\right)\overline{\epsilon}\right)\mu^2,$$

$$C = -\left(G_{\overline{J}T}\hat{\overline{Q}}\epsilon + G_{J\overline{T}}\hat{Q}\overline{\epsilon}\right)\mu + \left(\frac{\pi}{2}G_{\overline{J}T}^2\epsilon^2 + \hat{G}_{T\overline{T}}\epsilon\overline{\epsilon} + \frac{\pi}{2}G_{J\overline{T}}^2\overline{\epsilon}^2\right)\mu^2,$$

$$G_{J\overline{T}} \equiv (1-b)g_{J\overline{T}}, \quad G_{\overline{J}T} \equiv (1+b)g_{\overline{J}T},$$

$$\hat{G}_{T\overline{T}} \equiv (1-b^2)\left(g_{T\overline{T}} + \frac{\pi}{2}\left(g_{J\overline{T}}g_{J\Theta} + g_{\overline{J}T}g_{\overline{J\Theta}}\right)\right).$$

Square-root singularity

$$\epsilon_n = \dots - \frac{1}{2A}\sqrt{B^2 - 4AC}$$

$$\begin{aligned} B^2 &- 4AC \\ &= \Big(1 + (G_{J\overline{T}}\hat{Q} + G_{\overline{J}T}\hat{\overline{Q}})\mu\Big)^2 - \mu^2(\epsilon-\overline{\epsilon})(\pi G_{\overline{J}T}^2 - \pi G_{J\overline{T}}^2) \\ &\quad + 2\mu^3(\epsilon-\overline{\epsilon})\Big(G_{\overline{J}T}\hat{\overline{Q}}\big(\pi G_{J\overline{T}}^2 + \hat{G}_{T\overline{T}}\big) - G_{J\overline{T}}\hat{Q}(\pi G_{\overline{J}T}^2 + \hat{G}_{T\overline{T}})\Big) \\ &\quad + \mu^4(\epsilon-\overline{\epsilon})^2(\hat{G}_{T\overline{T}}^2 - \pi^2 G_{J\overline{T}}^2 G_{\overline{J}T}^2) - 4A\frac{\epsilon+\overline{\epsilon}}{2} \end{aligned}$$

Here $\epsilon - \overline{\epsilon} \sim$ momentum while $\epsilon + \overline{\epsilon} \sim$ energy
Whether low-lying or high-energy modes are lost (become complex) is controlled by

$$A = \Big(\frac{\pi}{2}(G_{J\overline{T}}^2 + G_{\overline{J}T}^2) + \hat{G}_{T\overline{T}}\Big)\mu^2 \gtrless 0\,,$$

Matching with "holomorphic" $J\overline{T}$

In other works, $J\overline{T}$ is solved by preserving holomorphy of J

Instead we keep J_t fixed

We determine $\langle n|J_x|n\rangle$ using background fields rather than holomorphy

Are the deformations the same? Yes,

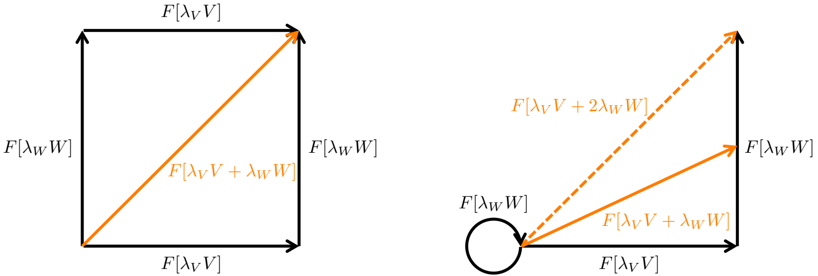
$$J_\mu^{\text{holomorphic}} = J_\mu^{\text{us}} - 2\pi^2 \lambda T_{\bar{z}\mu}$$

so

$$J^{\text{hol}}\overline{T} = \epsilon^{\mu\nu} J_\mu^{\text{hol}} T_{\bar{z}\nu} = \epsilon^{\mu\nu} J_\mu^{\text{us}} T_{\bar{z}\nu} - 2\pi^2 \lambda \underbrace{\epsilon^{\mu\nu} T_{\bar{z}\mu} T_{\bar{z}\nu}}_{\text{0 by antisymmetry}}$$

Deformations one after another

Doing the deformations one after the other typically commutes, except for $V = J\overline{T}$ and $W = J\Theta$ (or complex conjugates)



Nothing too exciting: stay in the same space of deformations

Work in progress

- Interpret of $\partial_\lambda E_n = \dots$ as transport along characteristics, compare with Turino group (Tateo, Negro, Conti, ...)
- Commutator of $J\overline{J}$, $J\overline{T}$, $J\overline{\Theta}$, $T\overline{J}$, $\Theta\overline{J}$, $T\overline{T}$ deformations
- Prove evolution equation, generalize to KdV charges to get control of $\langle n|A_s^t|n\rangle$
- Massive free scalar: get initial data, solve evolution equation

Thank you!

Hamiltonian quantum perturbation theory ($T\overline{T}$ case)

Set $\ell_n = L_n - \delta_{n,0}\frac{c}{24}$ and $\overline{\ell}_n = \overline{L}_n - \delta_{n,0}\frac{c}{24}$. Using conservation,

$$P = \ell_0 - \overline{\ell}_0$$

$$H = \ell_0 + \overline{\ell}_0 + \mu \sum_m \ell_m \overline{\ell}_m + O(\mu^2)$$

$$T_{00} = \frac{-1}{2\pi} \sum_k e^{ikx} \left(\ell_k + \overline{\ell}_{-k} + \mu \sum_m \ell_{k+m} \overline{\ell}_m + O(\mu^2) \right)$$

$$T_{01} = \frac{i}{2\pi} \sum_k e^{ikx} \left(1 - \mu \frac{c}{12} k^2 \right) (\ell_k - \overline{\ell}_{-k}) + O(\mu^2)$$

$$T_{10} = \frac{i}{2\pi} \sum_k e^{ikx} (\ell_k - \overline{\ell}_{-k}) \qquad \text{exactly (by definition)}$$

$$T_{11} = \frac{1}{2\pi} \sum_k e^{ikx} \left(\left(1 + \mu \frac{c}{12} k^2 \right) (\ell_k + \overline{\ell}_{-k}) + 3\mu \sum_m \ell_{m+k} \overline{\ell}_m + O(\mu^2) \right)$$

Next want $T_{00}(x) T_{11}(y) - T_{01}(x) T_{10}(y)$ OPE

Hamiltonian quantum perturbation theory ($T\overline{T}$ case)

Practice OPE's in CFT:

$$T(y+\epsilon)T(y) = \sum_{k,m} e^{ik\epsilon + i(k+m)y} \ell_k \ell_m = \sum_s e^{isy} \sum_k e^{ik\epsilon} \ell_k \ell_{s-k}$$

Notice that $\sum_k \ell_k \ell_{s-k}$ is singular: e.g. $\langle 0| \sum_k \ell_k \ell_{-k} |0\rangle = \infty$.
Need to commute ℓ's to put lowering operators right so when acting on $|0\rangle$ only finitely many terms remain

Hamiltonian quantum perturbation theory ($T\overline{T}$ case)

Practice OPE's in CFT:

$$T(y+\epsilon)T(y) = \sum_{k,m} e^{ik\epsilon + i(k+m)y} \ell_k \ell_m = \sum_s e^{isy} \sum_k e^{ik\epsilon} \ell_k \ell_{s-k}$$

Notice that $\sum_k \ell_k \ell_{s-k}$ is singular: e.g. $\langle 0| \sum_k \ell_k \ell_{-k} |0\rangle = \infty$. Need to commute ℓ's to put lowering operators right so when acting on $|0\rangle$ only finitely many terms remain

After deformation, ℓ_k, $k > 0$ don't kill the new vacuum but

$$\Lambda_k = \ell_k + \mu \left(\frac{c}{24} k^2 \overline{\ell}_{-k} + \sum_{n\neq 0} \frac{n-k}{2n} \ell_{k+n} \overline{\ell}_n \right) + O(\mu^2)$$

$$\overline{\Lambda}_k = \overline{\ell}_k + \mu \left(\frac{c}{24} k^2 \ell_{-k} + \sum_{n\neq 0} \frac{n-k}{2n} \ell_n \overline{\ell}_{k+n} \right) + O(\mu^2)$$

do! And $[\Lambda_k, \Lambda_m] = (k-m)\Lambda_{k+m} + \frac{c}{12} k^3 \delta_{k+m} + O(\mu^3)$, ...

Deformed Virasoro algebra ($T\overline{T}$ case)

Both $\ell_k, \bar{\ell}_k$ and $\Lambda_k, \overline{\Lambda}_k$ obey the same algebra
but $\Lambda_k, \overline{\Lambda}_k$ map energy eigenstates to eigenstates, and

$$H = \Lambda_0 + \overline{\Lambda}_0 + \mu\Lambda_0\overline{\Lambda}_0 + \mu^2\Lambda_0\overline{\Lambda}_0(\Lambda_0 + \overline{\Lambda}_0) + O(\mu^3)$$

Define Λ_k, $\overline{\Lambda}_k$ as ℓ_k, $\bar{\ell}_k$ "conjugated" by the deformation:

$$\Lambda_k|n\rangle_\mu = (\ell_k|n\rangle)_\mu \text{ and } \overline{\Lambda}_k|n\rangle_\mu = (\bar{\ell}_k|n\rangle)_\mu$$

so $\Lambda_0 \pm \overline{\Lambda}_0$ measure eigenvalues of $\ell_0 \pm \bar{\ell}_0$ acting on $|n\rangle$,
i.e. initial energy and momentum of $|n\rangle$, so expect

$$H = \frac{1 - \sqrt{1 - 2\mu(\Lambda_0 + \overline{\Lambda}_0) + \mu^2(\Lambda_0 - \overline{\Lambda}_0)^2}}{\mu}$$

Spectrum-generating operators

Define Λ_k, Υ_k, $\overline{\Lambda}_k$, $\overline{\Upsilon}_k$ as ℓ_k, j_k, $\bar{\ell}_k$, $\bar{j}_k$ conjugated by deformation

- $\Lambda_0|n\rangle_\lambda = (\ell_0|n\rangle)_\lambda = h|n\rangle_\lambda$ and so on so $\Lambda_0 \pm \overline{\Lambda}_0$, Υ_0, $\overline{\Upsilon}_0$ acting on $|n\rangle_\lambda$ measure initial energy, momentum, charges
- Charges fixed $\Longrightarrow \Upsilon_0 = j_0$, $\overline{\Upsilon}_0 = \bar{j}_0$, $\Lambda_0 - \overline{\Lambda}_0 = \ell_0 - \bar{\ell}_0$
- Same Virasoro and Kač–Moody algebra as ℓ_k, j_k, $\bar{\ell}_k$, $\bar{j}_k$, e.g.,

$$[\Lambda_k, \Lambda_m] = (k-m)\Lambda_{k+m} + \frac{c}{12}k^3\delta_{k+m,0}$$

- Λ_k, Υ_k, $\overline{\Lambda}_k$, $\overline{\Upsilon}_k$ times eigenstate $|n\rangle_\lambda$ gives eigenstate. "Spectrum generating" or "raising and lowering" operators

Expect energy of $|n\rangle_\lambda$ only depends on initial energy, momentum, charges so

$$H|n\rangle_\lambda = \mathrm{H}(\lambda; h^0, q^0, \overline{h}^0, \overline{q}^0)|n\rangle_\lambda = \mathrm{H}(\lambda; \Lambda_0, \Upsilon_0, \overline{\Lambda}_0, \overline{\Upsilon}_0)|n\rangle_\lambda$$

e.g. for the $J\overline{T}$-deformed CFT we expect

$$H \overset{\text{prediction}}{=} \frac{2\pi}{L}\left(\Lambda_0 - \overline{\Lambda}_0 - \frac{L^2}{2\pi^4\lambda^2}\left(1 - \frac{2\pi^2 i\lambda}{L}\Upsilon_0 - \sqrt{\left(1 - 2\pi^2 i(\lambda/L)\Upsilon_0\right)^2 - 2\left(2\pi^2 i\lambda/L\right)^2\overline{\Lambda}_0}\right)\right)$$