Stochastic Gravitational Waves from spin-3/2 fields

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- Today, we describe Nature using fundamental particles of spin 0, $\frac{1}{2}$, 1 and 2. What about a spin $\frac{3}{2}$ fundamental particle ?

- In Minkowski space, it is well known that a minimally coupled spin $\frac{3}{2}$ field suffers from the "Velo and Zwanziger" problem : The wave-front propagates faster than light ; loss of causality.

- This issue is not simply solved by adding non minimal couplings.

- In a consistent theoretical frame, we are lead to assume that spin $\frac{3}{2}$ fundamental fields have only gravitational interactions.

- Particles with only gravitational couplings are challenging to detect.

- In 2016 we witnessed a major breakthrough in gravitational astronomy with a first direct detection of Gravitational Waves (GW).
- Can gravitational astronomy help us to learn about the existence of elementary spin $\frac{3}{2}$ particles ?
- GW are produced by quadrupole moment of mass distribution: binary system, out of equilibrium gases,...
- Can we imagine a set-up where GWs can be produced by spin $\frac{3}{2}$ states ? Would these GW carry a peculiar signature of their origin ?

- We use FLRW metric in conformal time τ

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]$$

- Gravitational Waves (GW) production are governed by the equation

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} - \nabla h_{ij} = 16\pi G\Pi_{ij}^{TT}$$

- We use the Transverse Traceless (TT) gauge, which is defined by $\partial^i h_{ij} = 0$ and $h^i{}_i = 0$.
- To have GW we need an out of equilibrium source: non-adiabatically varying fields during preheating can produce stochastic GW.
  This is what we will consider here.

- It is more convenient to work in the Fourier space. Using the comoving wave-number $\mathbf{k}$, we can define the TT projector :

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}}) \qquad P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j$$

where $\hat{\mathbf{k}} \equiv \frac{\mathbf{k}}{|\mathbf{k}|}$

- Then the anisotropic stress energy tensor is

$$\Pi_{ij}^{TT}(\mathbf{k}, t) = \Lambda_{ij,lm}(\hat{\mathbf{k}})(T^{lm}(\mathbf{k}, t) - \mathcal{P}g^{lm})$$

where $\mathcal{P}$ is the background pressure

We will concentrate on the sub-horizon scale, ie $k \gg \mathcal{H}$

$$h_{ij} = \frac{16\pi G}{a(t)k} \int_{t_I}^{t} dt' \sin\left(k(t-t')\right) a(t') \Pi_{ij}^{TT}(k,t')$$

The energy density is given by

$$\rho = \frac{1}{32\pi G} \left\langle \dot{h}_{ij}(\mathbf{x},t) \dot{h}_{ij}(\mathbf{x},t) \right\rangle$$

$$\frac{d\rho_{GW}}{d\log k} = \frac{2Gk^3}{\pi a^4(t)} \int_{t_I}^{t} dt' \int_{t_I}^{t} dt'' a(t') a(t'') \cos[k(t'-t'')] \Pi^2(k,t',t'')$$

$\Pi^2(k,t',t'')$ is the unequal-time correlator of Π_{ij}^{TT} defined as

$$\langle \Pi_{ij}^{TT}(\mathbf{k},t) \Pi^{TTij}(\mathbf{k}',t') \rangle \equiv (2\pi)^3 \Pi^2(k,t,t') \delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

- The simplest way to build a spin $\frac{3}{2}$ is to use a tensorial product ψ_μ^α of spin 1 and $\frac{1}{2}$
- we have the following decomposition

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right)=\frac{1}{2}\oplus\left(1\otimes\frac{1}{2}\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$$

- the two extra-spinors are eliminated by imposing the constraints

$$\gamma_\mu\psi^\mu=0$$
$$\partial_\mu\psi^\mu=0$$

- We consider here a Majorana spin $\frac{3}{2}$

- A Lagrangian describing this field is the Rarita-Schwinger Lagrangian

$$\mathcal{L} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \frac{1}{4}m_{3/2}\bar{\psi}_\mu\left[\gamma^\mu,\gamma^\nu\right]\psi_\nu$$

- The corresponding stress-energy tensor is

$$T_{ij} = \frac{i}{4}\bar{\psi}_\mu\gamma_{(i}\partial_{j)}\psi^\mu - \frac{i}{4}\bar{\psi}_\mu\gamma_{(i}\partial^\mu\psi_{j)} + h.c$$

- A well motivated fundamental spin $\frac{3}{2}$ particle is the superpartner of the graviton, the gravitino. It is also a good candidate for Dark Matter.

We consider the canonical quantization of spin $\frac{3}{2}$ field

$$\psi^\mu(\mathbf{x},t) = \sum_{\lambda=\pm\frac{3}{2},\pm\frac{1}{2}} \int \frac{d\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \{\hat{a}_{\mathbf{p},\lambda}\tilde{\psi}^\mu_{\mathbf{p},\lambda}(t) + \hat{a}^\dagger_{-\mathbf{p},\lambda}\tilde{\psi}^{\mu C}_{\mathbf{p},\lambda}(t)\}$$

$$\tilde{\psi}^\mu_{\mathbf{p},\lambda}(t) = \sum_{s=\pm1, l=\pm1,0} \langle 1, \frac{1}{2}, l, \frac{s}{2} | \frac{3}{2}, \lambda \rangle \epsilon^\mu_{\mathbf{p},l} \mathbf{u}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t)$$

$\epsilon^\mu_{\mathbf{p},l}$ are the polarizations, $u^{(|\lambda|)}_{\mathbf{p},\pm}(t)$ wave functions, $\chi_s(\mathbf{p})$ two-component normalized eigenvectors of the helicity operator. All the time dependence is in the wave functions.

$$\mathbf{u}^{(|\lambda|)T}_{\mathbf{p},\frac{s}{2}}(t) = (u^{(|\lambda|)}_{\mathbf{p},+}(t)\chi^T_s(\mathbf{p}),\ s\, u^{(|\lambda|)}_{\mathbf{p},-}(t)\chi^T_s(\mathbf{p}))$$

$$\frac{d\rho_{GW}}{d\log k} \sim \frac{2Gk^3}{\pi a^4(t)} \int \int \cdots \langle \Pi^{TT}_{ij}(\mathbf{k}, t')\Pi^{TTij}(\mathbf{k}', t'')\rangle$$

- We first expand the product $\hat{\Pi}^{lm}(\mathbf{p}, t)\hat{\Pi}^{ij}(\mathbf{q}, t')$
- An average must be then taken on the product of creation and annihilation operators

Among the 16 products, the only non-0 average is

$$\langle 0|\hat{a}_{-\mathbf{p},\lambda}\hat{a}_{\mathbf{k}+\mathbf{p},\kappa}\hat{a}^{\dagger}_{\mathbf{q},\lambda'}\hat{a}^{\dagger}_{\mathbf{k}'-\mathbf{q},\kappa'}|0\rangle =$$

$$(2\pi)^6\delta^{(3)}(\mathbf{k}-\mathbf{k}')\{\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{q})\delta_{\lambda,\kappa'}\delta_{\kappa,\lambda'} - \delta^{(3)}(\mathbf{p}+\mathbf{q})\delta_{\lambda,\lambda'}\delta_{\kappa,\kappa'}\}$$

$$\mathbf{p}' = \mathbf{p} + \mathbf{k}$$

We expect the $\pm 3/2$ and $\pm 1/2$ helicity to be produced differently. Therefore, they come with different wave-functions

- helicity $\pm 3/2$ wave-function

$$\tilde{\psi}^{\mu}_{\mathbf{p},\pm\frac{3}{2}}(t) = \epsilon^{\mu}_{\mathbf{p},\pm 1}\,\mathbf{u}^{(3/2)}_{\mathbf{p},\pm\frac{1}{2}}(t)$$

- helicity $\pm 1/2$ wave-function

$$\tilde{\psi}^{\mu}_{\mathbf{p},\pm\frac{1}{2}}(t) = \sqrt{\frac{2}{3}}\epsilon^{\mu}_{\mathbf{p},0}\,\mathbf{u}^{(\frac{1}{2})}_{\mathbf{p},\pm\frac{1}{2}}(t) + \sqrt{\frac{1}{3}}\epsilon^{\mu}_{\mathbf{p},\pm 1}\,\mathbf{u}^{(\frac{1}{2})}_{\mathbf{p},\mp\frac{1}{2}}(t)$$

Thus, we separate the calculation into two parts, helicities $\lambda, \lambda' = \pm\frac{3}{2}$ and $\lambda, \lambda' = \pm\frac{1}{2}$

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Thus, we separate the calculation into two parts, helicities $\lambda, \lambda' = \pm\frac{3}{2}$ and $\lambda, \lambda' = \pm\frac{1}{2}$

Note that the longitudinal polarization $\epsilon^{\mu}_{\mathbf{p},0}$ appears in the helicity $\pm\frac{1}{2}$.

- In the relativistic regime we have $\epsilon^{\mu}_{\mathbf{p},0} \propto \frac{p^{\mu}}{m_{3/2}} + \dots$
- This implies that the production of the helicity $\pm\frac{1}{2}$ is enhanced compare to the $\pm\frac{3}{2}$.

We focus on the leading contribution

$$\Pi^2_{\frac{1}{2}}(k,t,t') \simeq \frac{1}{2\pi^2}\int_{p,p'\gg m_{3/2}} dp\, d\theta\, K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) W^{(\frac{1}{2})}_{\mathbf{k},\mathbf{p}}(t) W^{(\frac{1}{2})*}_{\mathbf{k},\mathbf{p}}(t')$$

With a kinematic factor ($\mathbf{p}' = \mathbf{p} + \mathbf{k}$)

$$K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) = \frac{1}{36 m^2_{3/2}} p^4 p'^2 \sin\theta \{(\cos\theta - \cos\theta')^2 + 4\sin^4(\frac{\theta-\theta'}{2})(1+\sin\theta\sin\theta')\} + \dots$$

and a wave-function factor

$$W^{(|\lambda|)}_{\mathbf{k},\mathbf{p}}(t) = u^{(|\lambda|)}_{\mathbf{p},+}(t) u^{(|\lambda|)}_{\mathbf{p}',+}(t) - u^{(|\lambda|)}_{\mathbf{p},-}(t) u^{(|\lambda|)}_{\mathbf{p}',-}(t)$$

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Note the k^2 enhancement factor from the p'^2, leading to an overall k^5 dependence of the spectrum density per logarithm of frequency of the GW energy.

- In the relativistic regime we have $\epsilon^{\mu}_{\mathbf{p},0} \propto \frac{p^{\mu}}{m_{3/2}} + \dots$
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$$\Pi^2_{\frac{1}{2}}(k,t,t') \simeq \frac{1}{2\pi^2}\int_{p,p'\gg m_{3/2}} dp\, d\theta\, K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) W^{(\frac{1}{2})}_{\mathbf{k},\mathbf{p}}(t) W^{(\frac{1}{2})*}_{\mathbf{k},\mathbf{p}}(t')$$

Note the wave-function factor

$$W^{(|\lambda|)}_{\mathbf{k},\mathbf{p}}(t) = u^{(|\lambda|)}_{\mathbf{p},+}(t)u^{(|\lambda|)}_{\mathbf{p}',+}(t) - u^{(|\lambda|)}_{\mathbf{p},-}(t)u^{(|\lambda|)}_{\mathbf{p}',-}(t)$$

This wave-function factor can only be computed in specific model.

- A model to give an example of the wave-function factor.
- Polonyi Model inflaton plus a scalar field z

$$\mathcal{K} = |z|^2 - \frac{|z|^4}{\Lambda^2} \qquad \mathcal{W} = \mu^2 z + \mathcal{W}_0,$$

- An estimate of the mass order near the minimum is

$$m_{3/2} \simeq \frac{\mu^2}{\sqrt{3}M_{Pl}} \qquad m_z \simeq 2\sqrt{3}\frac{m_{3/2}M_{Pl}}{\Lambda}$$

- we require $\Lambda < M_{Pl}$ and $m_z > m_{3/2}$.
- We assume that the $F = \sqrt{3}m_{3/2}M_{Pl}$ term of z does not contribute to the Hubble expansion but is large enough to lead to a gravitino mass that satisfies

$$\mathcal{H} \ll m_{3/2} < m_z$$

- The wave-function satisfies the Dirac equation

$$[i\gamma^0\partial_0 - a\, m_{3/2} + (A + iB\gamma^0)\mathbf{p}\cdot\gamma] \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0$$

- We assume as an initial condition that the occupation number vanishes.
- This equation is similar to the production of the spin $1/2$ from a Yukawa coupling to an oscillating scalar with a quadratic potential. The effective Yukawa coupling is $\tilde{y} = \frac{m_z^2}{2F}$ and the oscillation is described by a source term $\Theta(t) = -\frac{a m_z^2 \delta z}{2\sqrt{3} m_{3/2} M_{Pl}} = -\frac{a m_z^2 \delta z}{2F}$
- The fermion production in this case fill up a Fermi sphere with comoving radius

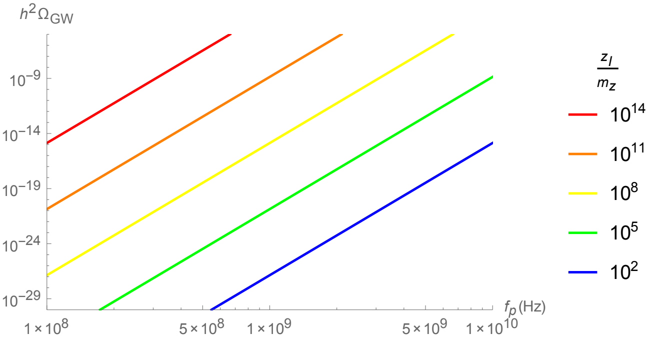
$$k_F \sim (a/a_I)^{1/4} q^{1/4} m_z \qquad q \equiv \frac{\tilde{y}^2 z_I^2}{m_z^2}$$

- The peak of the GW is reached at the Fermi sphere radius, and has a frequency given by

$$f_p \simeq 6 \cdot 10^{10} \tilde{y}^{\frac{1}{2}} \, \mathrm{Hz}$$

- The amplitude is given by

$$h^2 \Omega_{GW}(f_p) \simeq 3 \cdot 10^{-10} \left(\frac{f_p}{6 \cdot 10^{10} \mathrm{Hz}}\right)^{12} \left(\frac{z_l}{m_z}\right)^2$$



- GW might be produced during preheating of the Universe from a non adiabatic gas of spin $\frac{3}{2}$ fields. They will appear as a bump on top of the stochastic GW signal with a peculiar frequency dependence.
- Though they share some features with GW produced by fields of different spins, they exhibit some important differences. For instance, if the GW spectrum produced by the spin $3/2$ is close to the spin $1/2$ case, there is two main differences :
  - An enhancement by a factor $\frac{k_F}{m_{3/2}^2}$
  - A k^5 dependence near the peak, due to the apparition of k in the Π^{TT}

- A bump shows up at high frequency. Improvements of the sensitivity of experiments at such frequencies, searching mainly for axions for example, is needed to look for these GW.

Back-up slides

- we can compute the TT Stress-Energy tensor

$$\Pi_{ij}^{TT}(\mathbf{k},t) = \frac{1}{4}\Lambda_{ij,lm}(\hat{\mathbf{k}}) \int \frac{d\mathbf{p}}{(2\pi)^3} \{\hat{\Pi}^{lm}(\mathbf{p},t) + h.c.\},$$

- $\mathbf{k}$ is the momentum mode of the GW
- Π^{lm} is expressed in terms of $\hat{a}$ and ψ

$$\begin{aligned}\hat{\Pi}^{lm}(\mathbf{p},t) = &\left[\hat{a}_{-\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\mathbf{p},\lambda} + \hat{a}^{\dagger}_{\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\mathbf{p},\lambda}\right]\gamma^{(l}\partial^{m)}\times \\ &\left[\hat{a}_{\mathbf{p}+\mathbf{k},\lambda'}\tilde{\psi}_{\mu\mathbf{p}+\mathbf{k},\lambda'} + \hat{a}^{\dagger}_{-\mathbf{p}-\mathbf{k},\lambda'}\tilde{\psi}^{C}_{\mu\mathbf{p}+\mathbf{k},\lambda'}\right] \\ &-\left[\hat{a}_{-\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\mathbf{p},\lambda} + \hat{a}^{\dagger}_{\mathbf{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\mathbf{p},\lambda}\right]\gamma^{((l}\partial_{\mu}\times \\ &\left[\hat{a}_{\mathbf{p}+\mathbf{k},\lambda'}\tilde{\psi}^{m)}_{\mathbf{p}+\mathbf{k},\lambda'} + \hat{a}^{\dagger}_{-\mathbf{p}-\mathbf{k},\lambda'}\tilde{\psi}^{m)C}_{\mathbf{p}+\mathbf{k},\lambda'}\right].\end{aligned}$$

- Putting the non-0 average in the product gives

$$\Pi^2(k,t,t') = 2\int \frac{d\mathbf{p}}{(2\pi)^3} \left[\bar{\mathbf{v}}^{(|\lambda|)}_{\mathbf{p},\frac{s}{2}}(t)\Delta^{\lambda s,\lambda' s'}_{ij}(t)\mathbf{u}^{(|\lambda'|)}_{\mathbf{p}',\frac{s'}{2}}(t)\right]\times \left[\bar{\mathbf{u}}^{(|\lambda'|)}_{\mathbf{p}',\frac{r'}{2}}(t')\Delta^{\lambda r,\lambda' r'}_{ij}(t')^{*}\mathbf{v}^{(|\lambda|)}_{\mathbf{p},\frac{r}{2}}(t')\right],$$

- $\mathbf{v}^{(|\lambda|)}_{\mathbf{p},\frac{r}{2}} = i\gamma^0\gamma^2\bar{\mathbf{u}}^{|\lambda|T}_{\mathbf{p},\frac{r}{2}}$
- Δ is defined by

$$\Delta^{\lambda s,\lambda' s'}_{ij}(t) = \frac{1}{4}\Lambda_{ij,lm}\langle 1,\frac{1}{2},r,\frac{s}{2}|\frac{3}{2},\lambda\rangle\langle 1,\frac{1}{2},r',\frac{s'}{2}|\frac{3}{2},\lambda'\rangle\times \{2\epsilon_{\mu\mathbf{p},r}\epsilon^{\mu}_{\mathbf{p}',r'}\ p^{(l}\gamma^{m)} - \epsilon_{\mu\mathbf{p},r}p'^{\mu}\epsilon^{(l}_{\mathbf{p}',r'}\gamma^{m)} - \epsilon_{\mu\mathbf{p}',r'}p^{\mu}\epsilon^{(l}_{\mathbf{p},r}\gamma^{m)}\}$$

helicity $\pm 3/2$

- Using the decomposition of helicity $\pm 3/2$ we get

$$\Pi^2_{\frac{3}{2}}(k,t,t') = \frac{1}{32\pi^2}\int dp\, d\theta\ K^{(\frac{3}{2})}(p,k,\theta,m_{3/2})\ W^{(\frac{3}{2})}_{\mathbf{k},\mathbf{p}}(t) W^{(\frac{3}{2})*}_{\mathbf{k},\mathbf{p}}(t'$$

- $\theta(\theta')$ is the angle between $\mathbf{k}$ and $\mathbf{p}(\mathbf{p}')$

$$K^{(\frac{3}{2})}(p,k,\theta,m_{3/2}) = p^2k^2\{5\sin^3\theta\sin^2\theta' + \sin^2(\theta-\theta')\sin\theta\} + 4p^4\sin^4\theta\sin\theta'$$

- all the wave-function dependance is in W

$$W^{(|\lambda|)}_{\mathbf{k},\mathbf{p}}(t) = u^{(|\lambda|)}_{\mathbf{p},+}(t)u^{(|\lambda|)}_{\mathbf{p}',+}(t) - u^{(|\lambda|)}_{\mathbf{p},-}(t)u^{(|\lambda|)}_{\mathbf{p}',-}(t)$$

helicity $\pm 1/2$

- presence of ϵ_0 and $\epsilon_{\pm 1}$
- In the relativistic regime we have

$$\epsilon^{\mu}_{\mathbf{p},0} = \frac{1}{m_{3/2}}(p, \sqrt{p^2 + m_{3/2}^2}\hat{\mathbf{p}}) \propto \frac{p^{\mu}}{m_{3/2}} + \dots$$

- The main contribution is with two ϵ_0

$$\Pi^2_{\frac{1}{2}}(k,t,t') \simeq$$
$$\frac{1}{2\pi^2}\int_{p,p' \gg m_{3/2}} dp\, d\theta K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) \;\; W^{(\frac{1}{2})}_{\mathbf{k},\mathbf{p}}(t) W^{(\frac{1}{2})*}_{\mathbf{k},\mathbf{p}}(t')$$

$$K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) = \frac{1}{36 m_{3/2}^2} p^4 p'^2 \sin\theta \{(\cos\theta - \cos\theta')^2$$
$$+ 4\sin^4(\frac{\theta-\theta'}{2})(1+\sin\theta\sin\theta')\} + \dots$$

In the relativistic regime, helicity $\pm 1/2$ production is enhanced compare to $\pm\frac{3}{2}$.

$$\frac{d\rho_{GW}}{d\log k}(k,t) \simeq$$
$$\frac{Gk^3}{\pi^3 a^4(t)} \int dp\, d\theta\, K^{(\frac{1}{2})}(p,k,\theta,m_{3/2})\ \{|I_c(k,p,\theta,t)|^2 + |I_s(k,p,\theta,t)|^2\},$$

$$I_c(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')} \cos(kt')\, W_{\mathbf{k},\mathbf{p}}^{(\frac{1}{2})}(t'),$$
$$I_s(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')} \sin(kt')\, W_{\mathbf{k},\mathbf{p}}^{(\frac{1}{2})}(t')$$

- In order to compute the GW spectrum, we need the evolution of the wave-function $u_{\mathbf{p},\pm}(t)$.
- the equation of motion is given by

$$[i\gamma^0\partial_0 - a\, m_{3/2} + (A + iB\gamma^0)\mathbf{p}\cdot\gamma]\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0,$$

- initial condition : $u_{\mathbf{p},\pm}$ satisfies the vanishing occupation number condition .
- we take the momentum $\mathbf{p}$ to lie along the z direction.

$$A+iB = \exp\left(2i\int\Theta(t)dt\right) \qquad f(t)_\pm = \exp\left(\mp i\int\Theta(t)dt\right)u_\pm$$

- the equation of motion becomes

$$\ddot{f}_\pm + [p^2 + (\Theta + m_{3/2}a)^2 \pm i(\dot{\Theta} + \dot{m_{3/2}a})]f_\pm = 0.$$

- Polonyi model of inflation, z Polonyi field

$$\mathcal{K} = |z|^2 - \frac{|z|^4}{\Lambda^2},$$
$$\mathcal{W} = \mu^2 z + \mathcal{W}_0,$$

- An estimate of the mass order near the minimum is

$$m_{3/2} \simeq \frac{\mu^2}{\sqrt{3}M_{Pl}} \simeq \frac{\mathcal{W}_0}{M_{Pl}^2}, \qquad m_z \simeq 2\sqrt{3}\frac{m_{3/2}M_{Pl}}{\Lambda}.$$

- Requiring $\Lambda < M_{Pl}$ leads to $m_z > m_{3/2}$.
- We assumed that the F term of z does not contribute to the Hubble expansion but is large enough to lead to a gravitino mass that satisfies

$$\mathcal{H} \ll m_{3/2} < m_z,$$

- We are in the sub-horizon limit : we can apply all the previous calculation.
- Polonyi model contains a nontrivial source term $\Theta(t)$ to produce helicity-1/2 gravitino,

$$\Theta(t) = -\frac{am_z^2\delta z}{2\sqrt{3}m_{3/2}M_{Pl}} = -\frac{am_z^2\delta z}{2F},$$

- $\delta z = z - z_0$ is the displacement of z from its value z_0 at the minimum of the the scalar potential and $F = \sqrt{3}m_{3/2}M_{Pl}$ is the supersymmetry breaking scale.
- The coupling between this source term and the gravitino is given by

$$\ddot{f}_\pm + [k^2 + (a\, m_{3/2} - \frac{am_z^2\delta z}{2F})^2 \mp i\frac{am_z^2\delta\dot{z}}{2F}]f_\pm = 0.$$

- This equation is similar to the production of the spin $1/2$ from a quadratic scalar with a Yukawa coupling.
- The effective Yukawa coupling is

$$\tilde{y} = \frac{m_z^2}{2F}.$$

- The fermion production in this case fill up a fermi sphere with comoving radius

$$k_F \sim (a/a_I)^{1/4} q^{1/4} m_z, \qquad q \equiv \frac{\tilde{y}^2 z_I^2}{m_z^2},$$

- The peak of the GW is reached at this radius, and has a frequency given by

$$f_p \simeq 6 \cdot 10^{10} \tilde{y}^{\frac{1}{2}} \mathrm{Hz}.$$