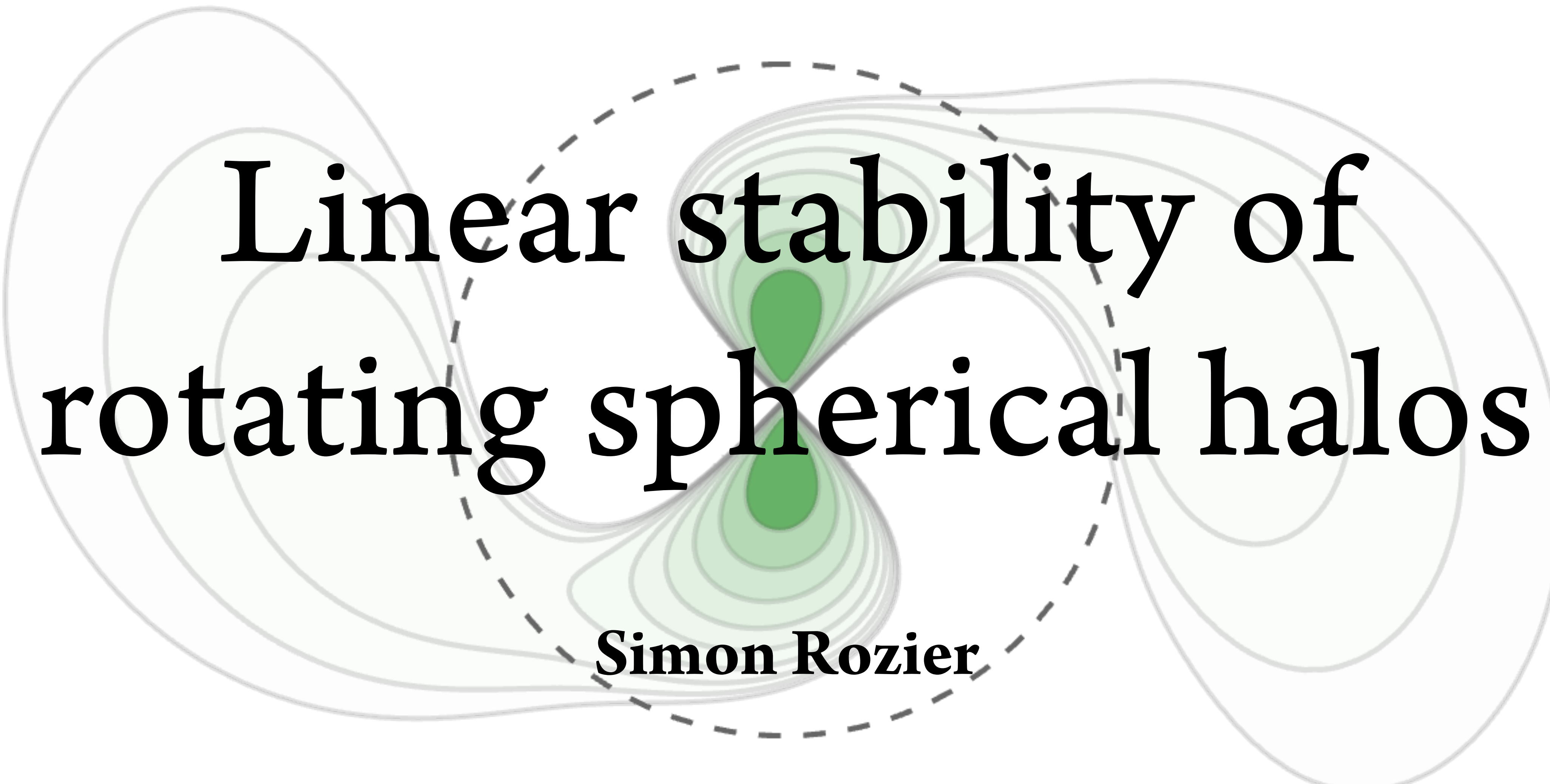


Linear stability of rotating spherical halos

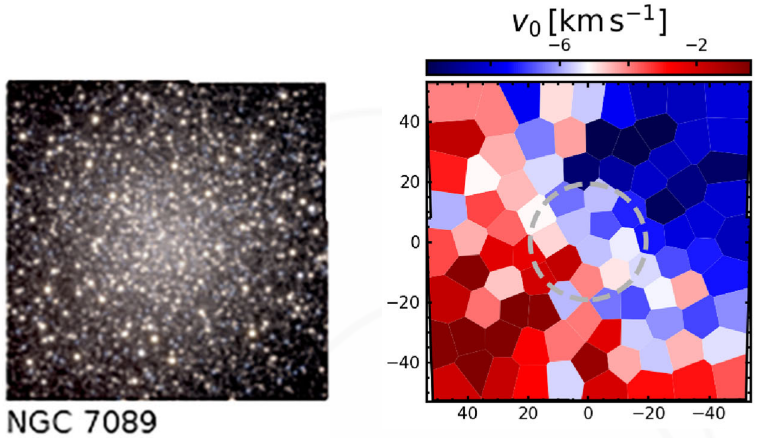
Simon Rozier

J.-B. Fouvry, P. G. Breen, A. L. Varri, C. Pichon, D. C. Heggie

News from the Dark - LUPM - 20-22/03/2019

Spheriodal stellar systems are rotating

Globular Clusters

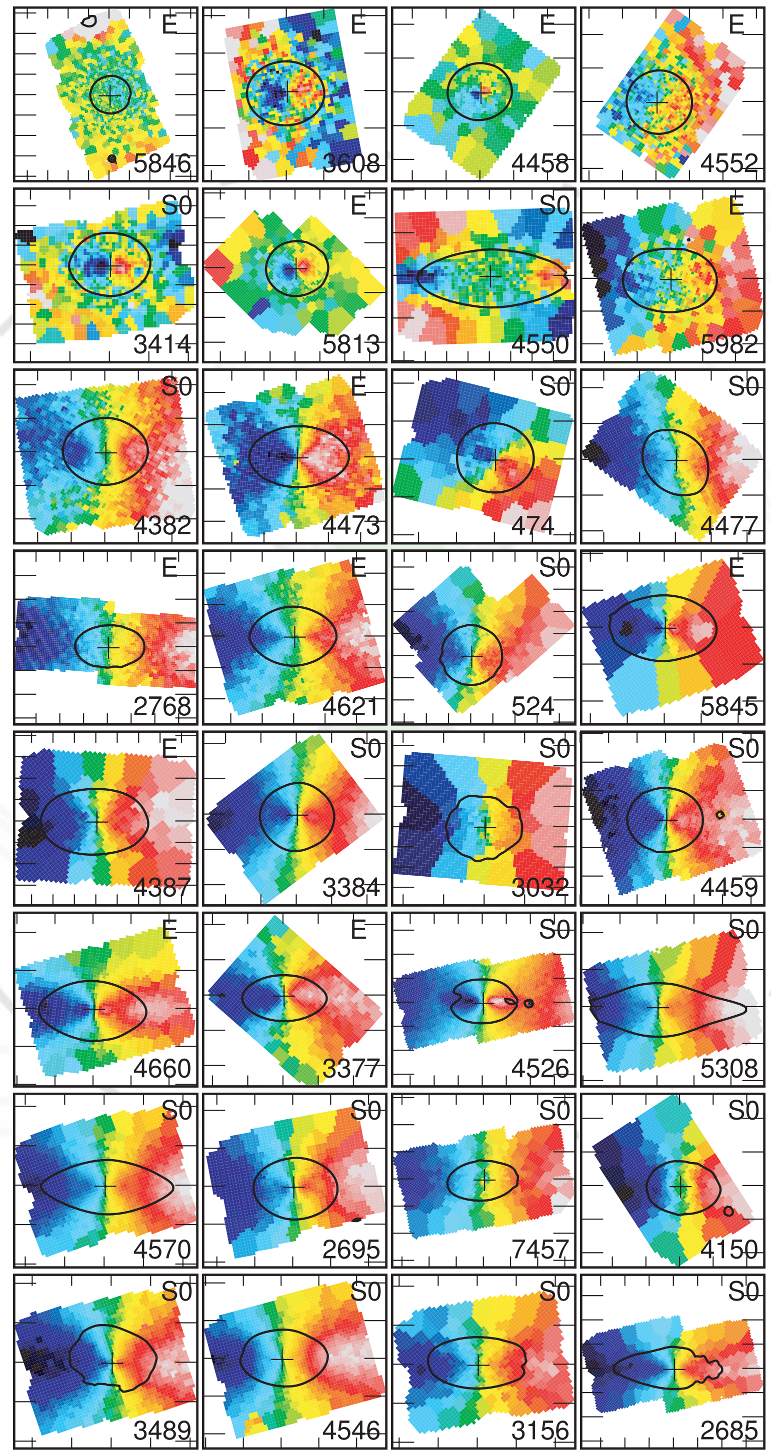


Kamann et al. 2017

Name	L10	B12	F14	L15	K15	K18	F18	G18	B18	V18	this work
NGC 104	✓	✓			✓	✓		✓	✓	✓	✓
NGC 288	X	X			X		✓		X	X	X
NGC 362					X	∼	✓		X	X	X
NGC 1261							∼				X
NGC 1851		✓		✓		✓			∼	X	X
NGC 1904		X					✓		X		(✓)
NGC 2808		✓		✓	X	✓			X	X	✓
NGC 3201		✓				✓	✓		∼	X	∼
NGC 4372				X					✓	∼	∼
NGC 4590	X	✓			X				X	X	X
NGC 5024	X	X	✓		X						∼
NGC 5139		✓				✓		✓	✓	✓	✓
NGC 5272			✓		X		✓	✓	✓	∼	(✓)
NGC 5286									∼	X	X
NGC 5466					✓						X
NGC 5824											(✓)
NGC 5904		✓	✓		✓	✓		✓	✓	✓	✓
NGC 5927				✓			∼		X	X	X
NGC 5986									X	X	∼
NGC 6093			✓			✓			∼	X	(✓)
NGC 6121	✓	✓			∼	∼			∼	X	X
NGC 6171		∼					✓		X	X	X
NGC 6205			✓						∼	X	✓
NGC 6218	X	X	✓		X				X	X	(✓)
NGC 6254		X	✓			✓	∼		∼	X	X
NGC 6266						✓			∼	✓	✓
NGC 6273									✓	✓	✓
NGC 6293						✓				X	
NGC 6341			✓		∼				X	X	(✓)
NGC 6388		✓				✓			X	X	X
NGC 6397		X							∼	X	✓
NGC 6402					X				∼	X	∼
NGC 6441		✓			X	∼				X	X
NGC 6496							∼				X
NGC 6522						∼			X		X
NGC 6539									∼	X	∼
NGC 6541						✓			∼	X	✓
NGC 6553										X	✓
NGC 6626			✓						X	X	✓
NGC 6656	✓	✓			X	✓		✓	✓	✓	✓
NGC 6681						∼			X	X	
NGC 6715		✓			X					X	X
NGC 6723							∼			X	X
NGC 6752	X	X		X	X	∼		✓	✓	∼	(✓)
NGC 6779			✓						∼	X	X
NGC 6809	∼	∼			∼			✓	✓	∼	(✓)
NGC 6838		∼			X				X	X	∼
NGC 6934			✓		X						
NGC 7078		✓		✓	✓	✓		✓	✓	✓	✓
NGC 7089					✓	✓			✓	✓	✓
NGC 7099	X	X			X	∼			X	X	(✓)
Terzan 5											✓

Sollima et al. 2019

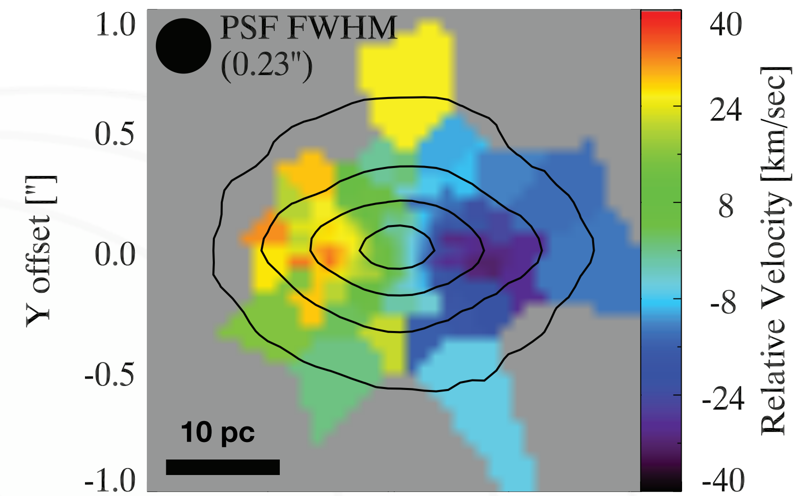
Elliptical & Spheroidal Galaxies



Emsellem et al. 2007

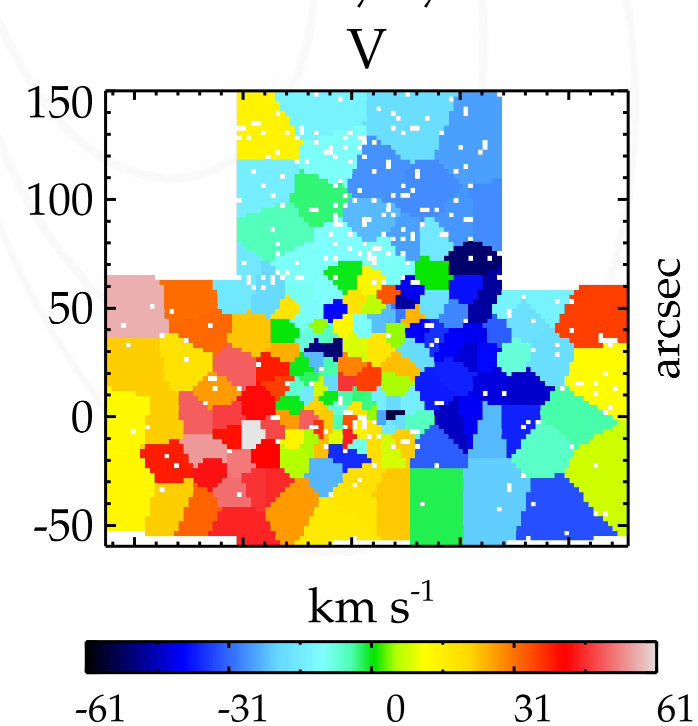
Nuclear Star Clusters

NGC 4244
(nearby, spiral, edge on)



Seth et al. 2008

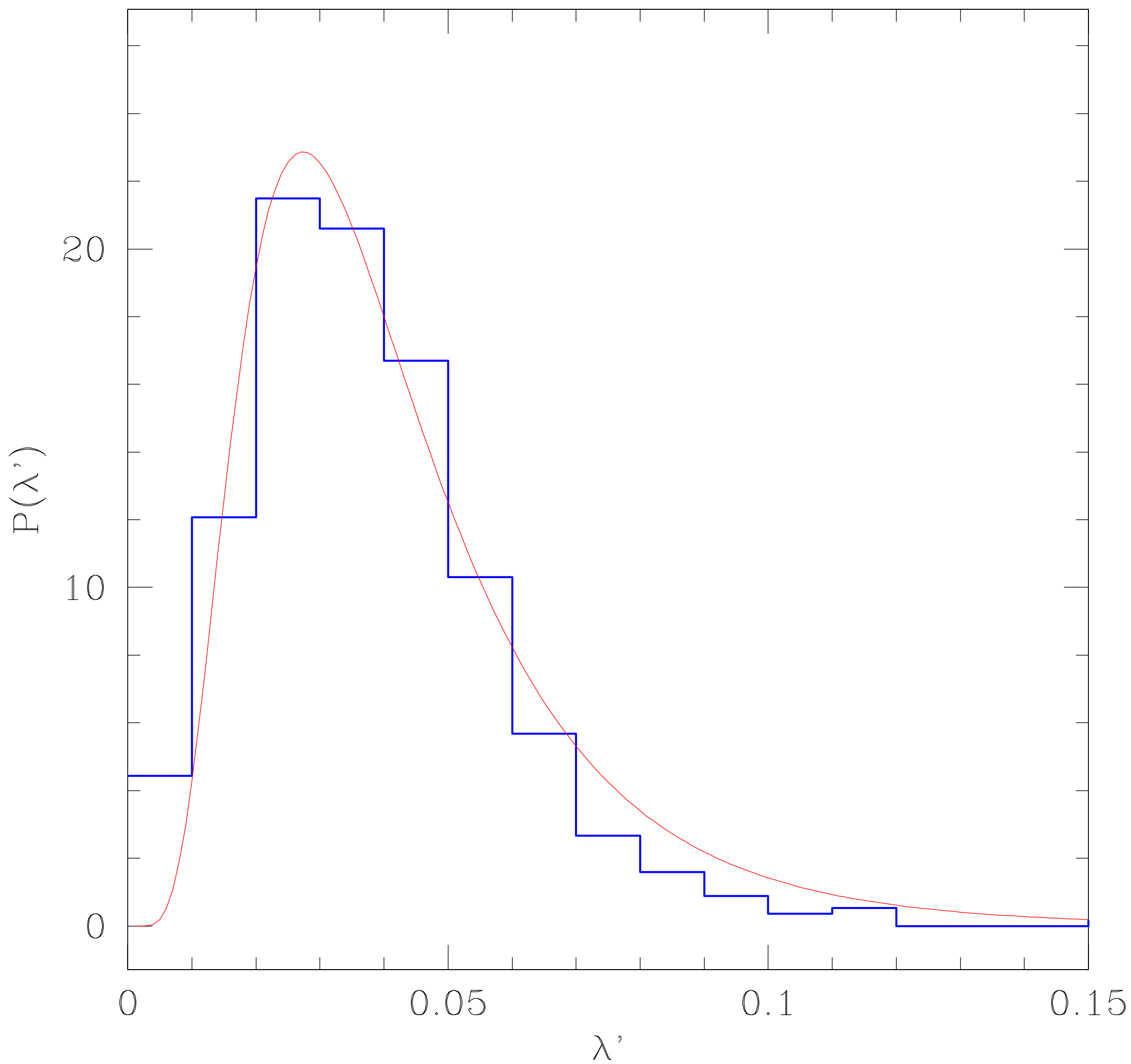
Milky Way



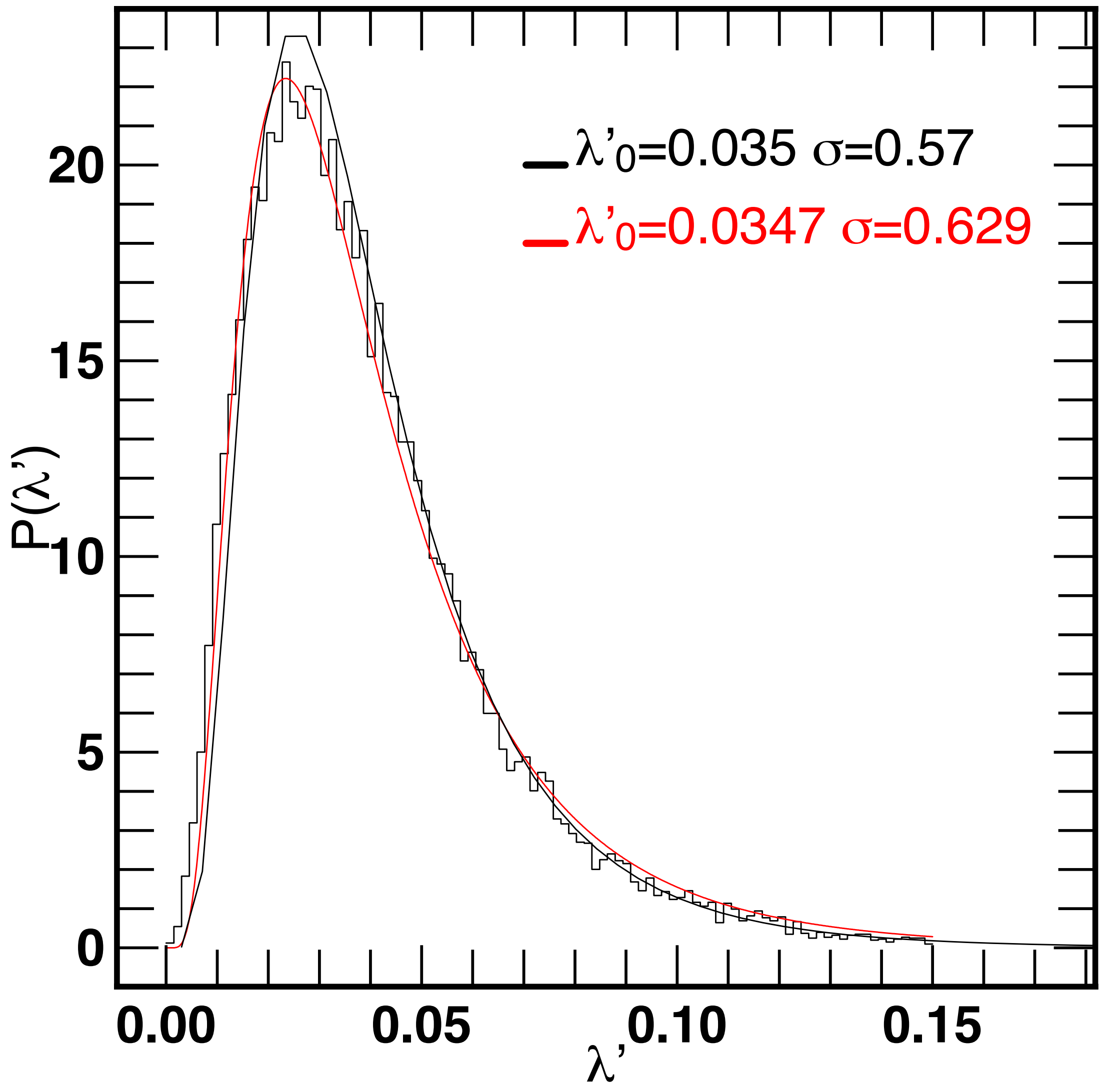
Feldmeier-Krause et al. 2017

DM halos are rotating

$$\lambda' = \frac{J}{\sqrt{2}MVR}$$



Bullock et al. 2001



Aubert et al. 2004

Rotation in DM halos

- Theory:

Rotation emerging from tidal torques at the time of structure formation. *Peebles 1969*
Accretion of satellites (assembly). *Vitvitska et al. 2002*

- Simulations: Tidal torquing and assembly do not seem to be independent. *Lopez et al. 2019*

- Observations: Open question. Imprint of the rotating halo on the baryon dynamics?

- Problem: Rotation has an influence on the dynamical evolution of the halo, and its shape. Can transform a spherical halo into a triaxial one, e.g through instabilities.

Linear Response Theory: the response matrix

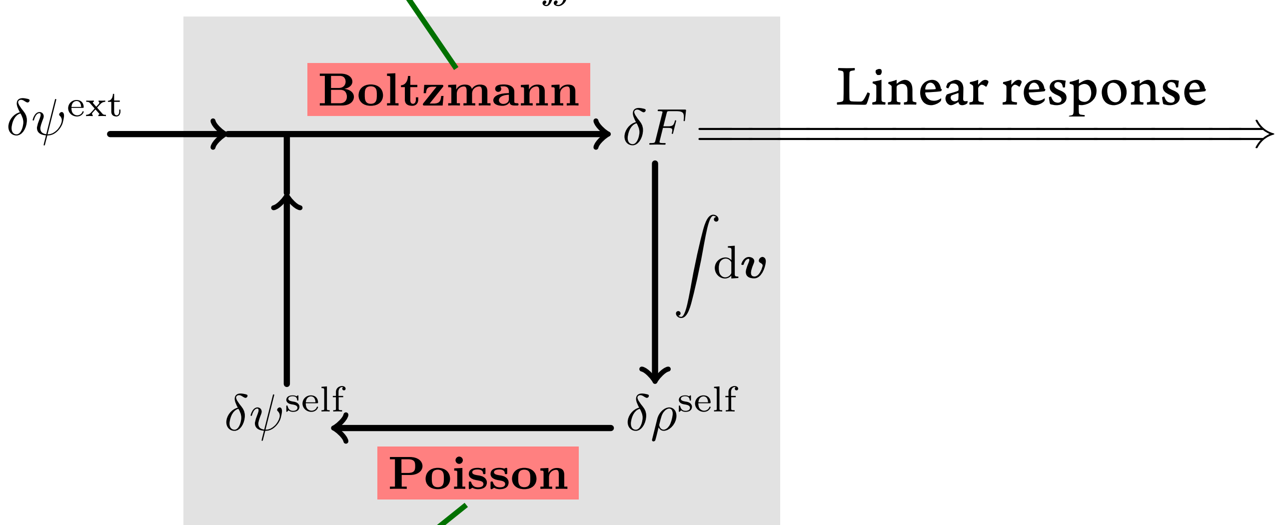
How does a stellar system respond to an exterior perturbation?

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0}$$

$\longrightarrow$ Angle-Action variables

$(\mathbf{J}, \boldsymbol{\theta})$

Collective effects



Mode?

$$\widehat{\boldsymbol{M}}(\omega)\underline{A} = \underline{A}$$

$$\det[\boldsymbol{I} - \widehat{\boldsymbol{M}}(\omega)] = 0$$

Complex frequency $\omega = \omega_0 + i\eta$

ω_0 = precession rate

η = growth rate

$\eta < 0 \rightarrow$ stable; $\eta > 0 \rightarrow$ unstable

$$\boxed{\Delta\psi = 4\pi G\rho}$$

$\longrightarrow$ Projection

$$\delta\psi(\boldsymbol{x}) = \sum a_p \psi^{(p)}(\boldsymbol{x})$$

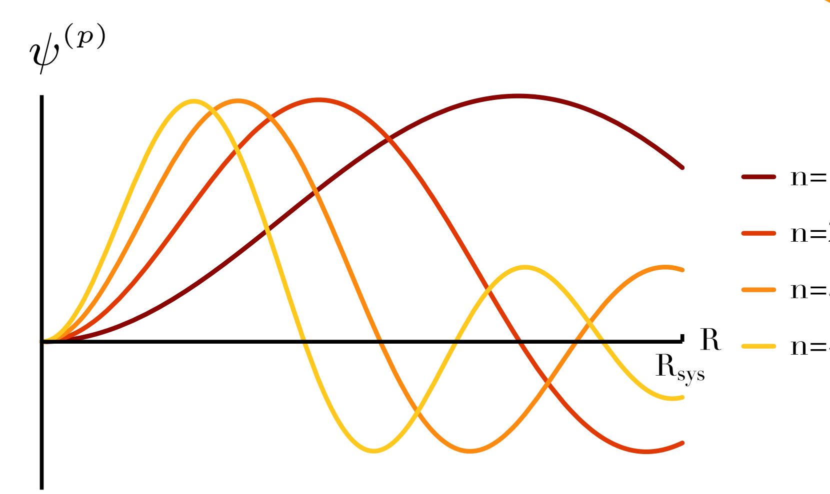
$$\delta\rho(\boldsymbol{x}) = \sum a_p \rho^{(p)}(\boldsymbol{x})$$

Kalnajs 1976

Basis

$$\Delta\psi^{(p)} = 4\pi G\rho^{(p)}$$

$$\int \mathrm{d}\boldsymbol{x}\, \psi^{(p)*}(\boldsymbol{x})\rho^{(q)}(\boldsymbol{x}) = -\delta_p^q$$



Key points of the method

Angle-action variables:
Guarantee a simple (proportional) relation between the response and the perturbation

$$\boxed{\tilde{f}^1_{\mathbf{m}}(\mathbf{J},\omega) = \mathbf{m}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\frac{\tilde{\psi}^1_{\mathbf{m}}(\mathbf{J},\omega)}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}}$$

Projection onto Bi-Orthogonal Basis:
Transforms the differential equations into a vectorial problem

$$\tilde{\boldsymbol{\psi}}^{resp}(\omega) = \widehat{\mathbf{M}}(\omega)(\hat{\mathbf{I}} - \widehat{\mathbf{M}}(\omega))^{-1}\cdot\tilde{\boldsymbol{\psi}}^{ext}(\omega)$$

Response matrix:

$$\boxed{\widehat{M}_{pq}(\omega) = -(2\pi)^3\sum_{\mathbf{m}}\int \mathrm{d}\mathbf{J}\,\frac{\mathbf{m}\cdot\frac{\partial f_0}{\partial \mathbf{J}}}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}\,\psi^{(p)*}_{\mathbf{m}}(\mathbf{J})\psi^{(q)}_{\mathbf{m}}(\mathbf{J})}$$

Unstable mode:

$$\boxed{\det[\hat{\mathbf{I}} - \widehat{\mathbf{M}}(\omega_m)] = 0,\quad \mathrm{Im}(\omega_m) > 0}$$

$$\boldsymbol{\psi}^{resp}(t) \propto e^{\mathrm{i}\omega_m t}$$

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \boxed{\sum_{\mathbf{n}}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Sum over **resonance vectors**

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial\mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Sum over **resonance vectors**

Integral over **action space**

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Sum over **resonance vectors**

Integral over **action space**

Gradient of the distribution function

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Sum over **resonance vectors**

Integral over **action space**

Gradient of the distribution function

Resonant denominator at the **intrinsic frequencies**

$\widehat{M}(\omega)$ computation

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Sum over **resonance vectors**

Integral over **action space**

Gradient of the distribution function

Resonant denominator at the **intrinsic frequencies**

Potential **basis functions**

$\widehat{M}(\omega)$ computation
Assumptions and Technical challenges

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial\mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Computing the frequencies:

Requires to sample the integral in peri- and apo-centre space.

Computing the integral:

Computation on a static grid. 4 parameters allow some flexibility in the choice of the grid in peri- and apo-centres.

The regions where the denominator resonates have to be particularly well sampled.

$\widehat{M}(\omega)$ computation
Assumptions and Technical challenges

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Computing the Bi-Orthogonal Basis:

Basis functions defined through nested integrals.

Runge-Kutta scheme implemented, with a significant gain in performance as compared to previous response matrix codes.

$\widehat{M}(\omega)$ computation

Assumptions and Technical challenges

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \boxed{\sum_{\mathbf{n}}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Truncation of the sum:

Infinite sum transformed into finite by assuming that **only the first several resonance vectors contribute**. Motivated by the literature (Lynden-Bell, Polyachenko): significant resonances should have $|n_1|, |n_2|, |n_3| \leq 2$

1 additional parameter

$\widehat{M}(\omega)$ computation

Assumptions and Technical challenges

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot \boxed{\partial f/\partial \mathbf{J}}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Adding rotation to the theory:

No rotation: $f(J_r, L)$

With rotation: $f(J_r, L, L_z)$

System:

Spherical cluster $\left(\psi_0(r)\text{, Plummer potential}\right)$, steady state with tunable anisotropy $\left(f_q(E,L)\right)$ and rotation $\left(f_{q,\alpha}(E,L,L_z)\right)$:

Dejonghe 1987 *Lynden-Bell 1962*

$$\boxed{f_{q,\alpha}(E,L,L_z) = f_q(E,L)(1+\alpha\ \mathrm{Sign}(L_z/L))}$$

α rotation parameter

$\alpha = 0$: no rotation

$\alpha = 1$: maximal rotation

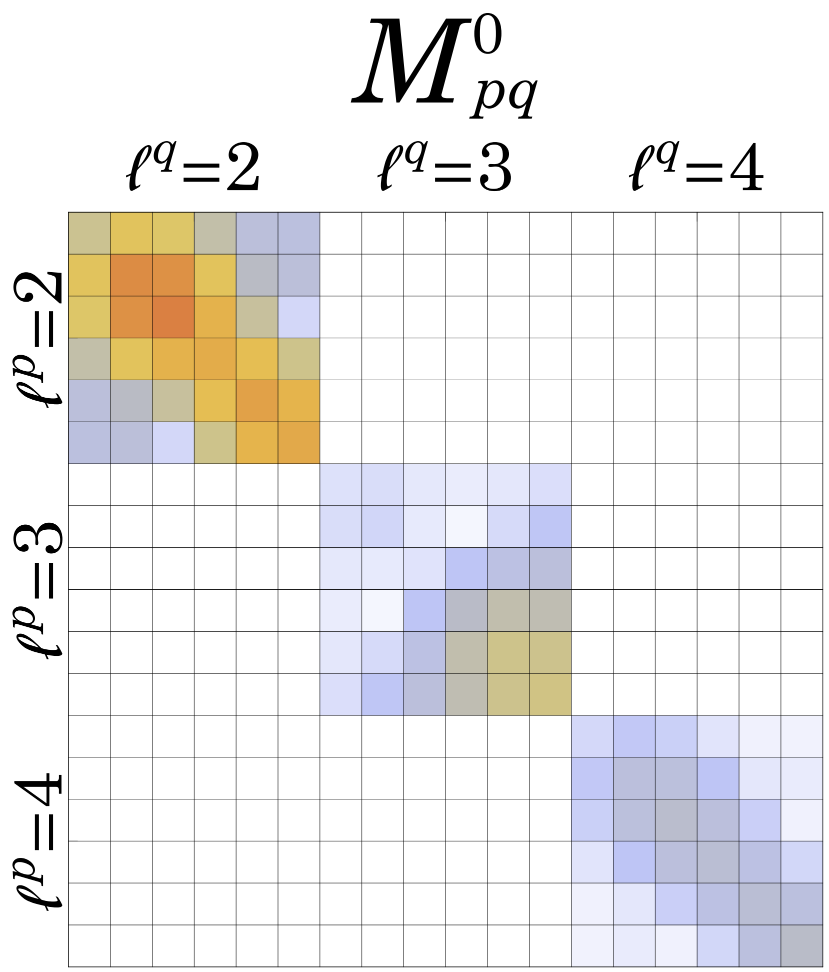
Response matrix for rotating systems

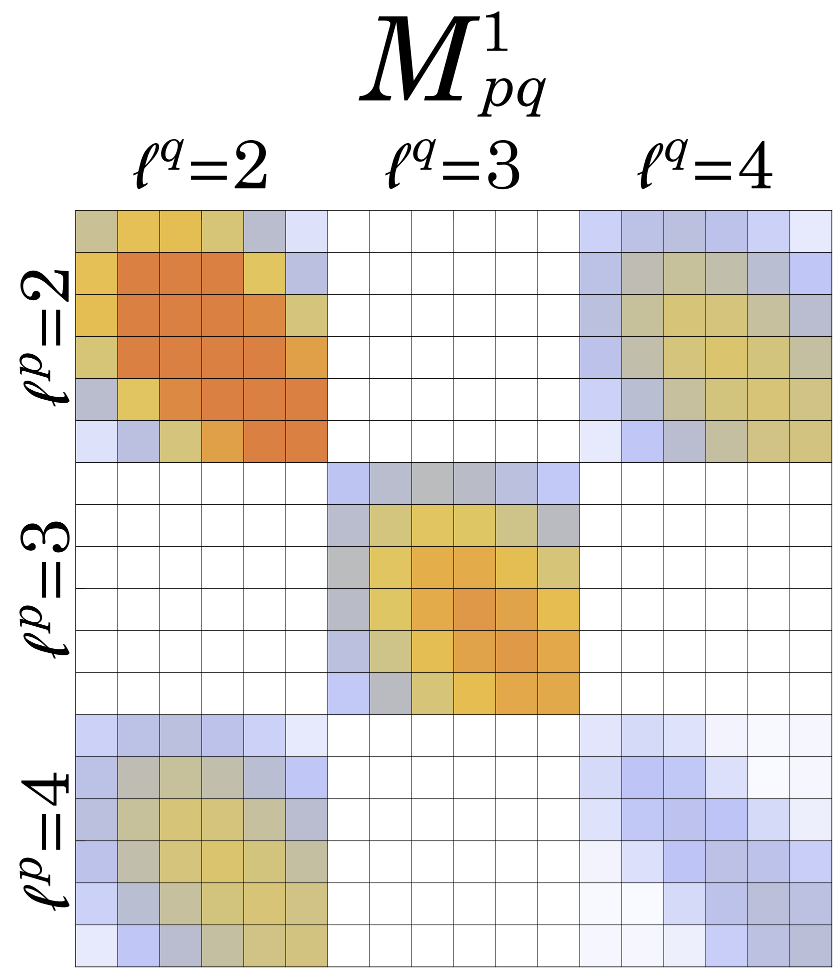
$$\boxed{\widehat{M}_{pq}(\omega) = \widehat{M}^{0}_{pq}(\omega) + \alpha\, \widehat{M}^{1}_{pq}(\omega)}$$

α rotation parameter
$\alpha = 0$: no rotation
$\alpha = 1$: maximal rotation

p and q described by their harmonic numbers $m^p, \ell^p, n^p, m^q, \ell^q, n^q$

Good properties of M⁰: independent of m^p, m^q ; proportional to $\delta_{\ell^p}^{\ell^q}$
M¹ does not have these properties anymore: the matrix is much larger





$\widehat{M}(\omega)$ computation

Assumptions and Technical challenges

$$\widehat{M}_{pq}(\omega) = (2\pi)^3 \sum_{\mathbf{n}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{n}\cdot\partial f/\partial \mathbf{J}}{\omega - \mathbf{n}\cdot\mathbf{\Omega}} \, [\psi_{\mathbf{n}}^{(p)}(\mathbf{J})]^* \, \psi_{\mathbf{n}}^{(q)}(\mathbf{J})$$

Truncation of the matrix:

Infinite matrix truncated in $m^p, \ell^p, n^p, m^q, \ell^q, n^q$.

- $m^p = m^q = 2$: searching for bars and two-armed spirals
- Cut-off in ℓ^p, ℓ^q: assuming the first resonances only matter + the background and the instabilities are well-described by low-order ℓ terms.
- Cut-off in n^p, n^q: assuming the first radial basis functions are sufficient to describe both the background and the instabilities.

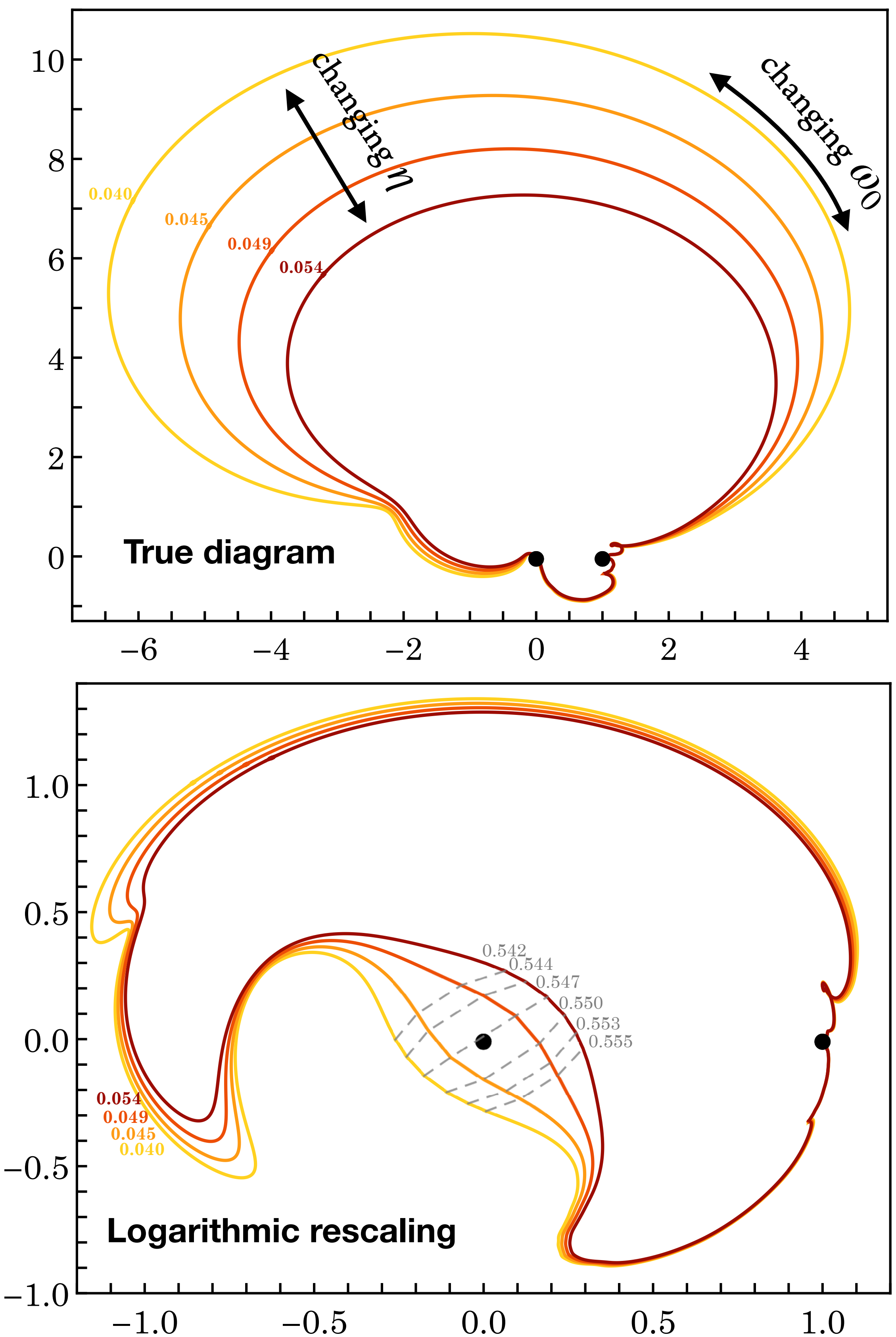
2 more parameters + 1 parameter controlling the characteristic radius of the basis.

Summary:

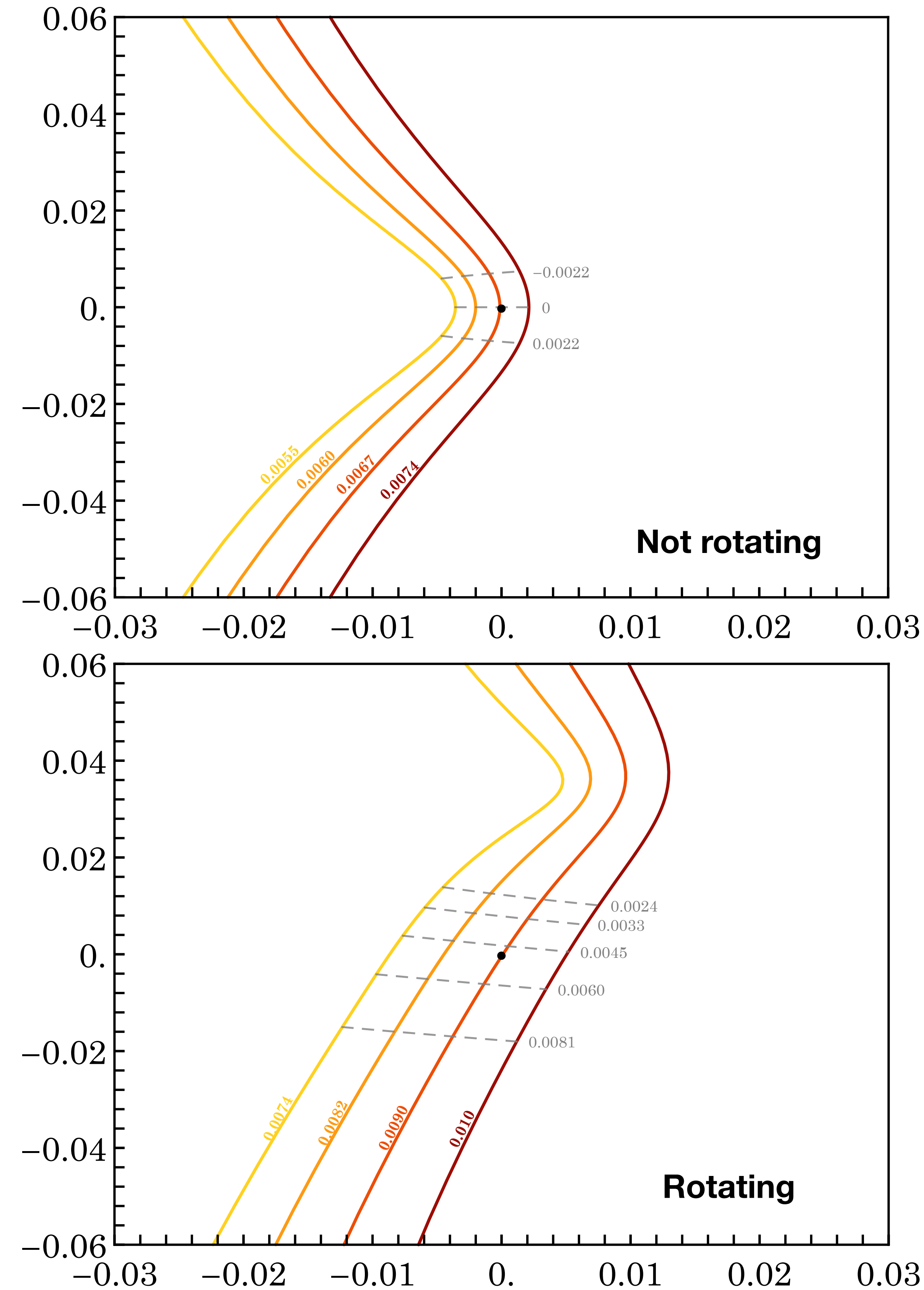
8 parameters controlling 1 computation of $\widehat{M}^0(\omega)$ and $\widehat{M}^1(\omega)$
Convergence study needed on each of these parameters.

Identifying unstable modes in rotating systems

Nyquist diagrams: $\omega_0 \to \det[\boldsymbol{I} - \widehat{\boldsymbol{M}}(\omega_0 + i\eta)]$

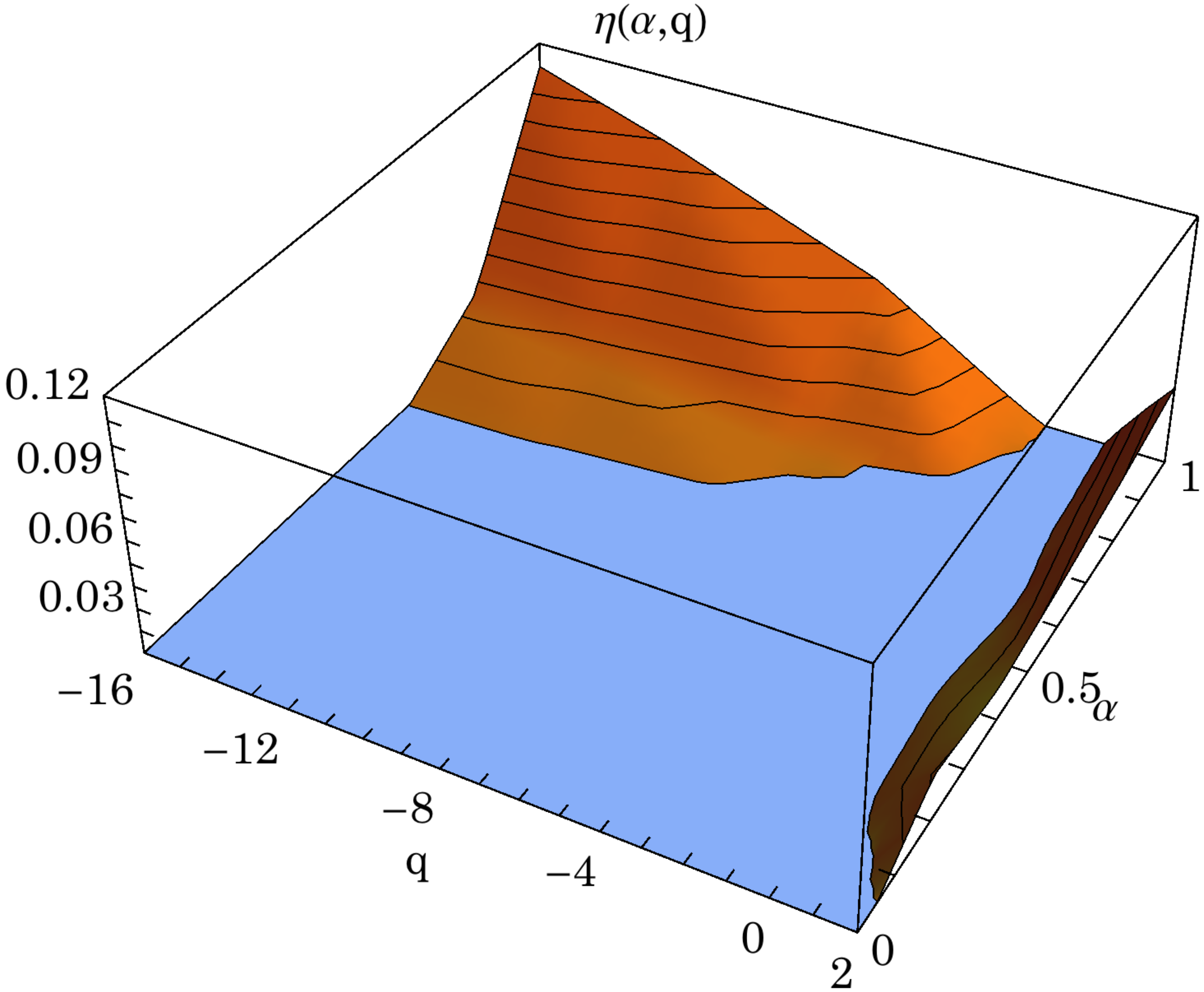


Influence of rotation

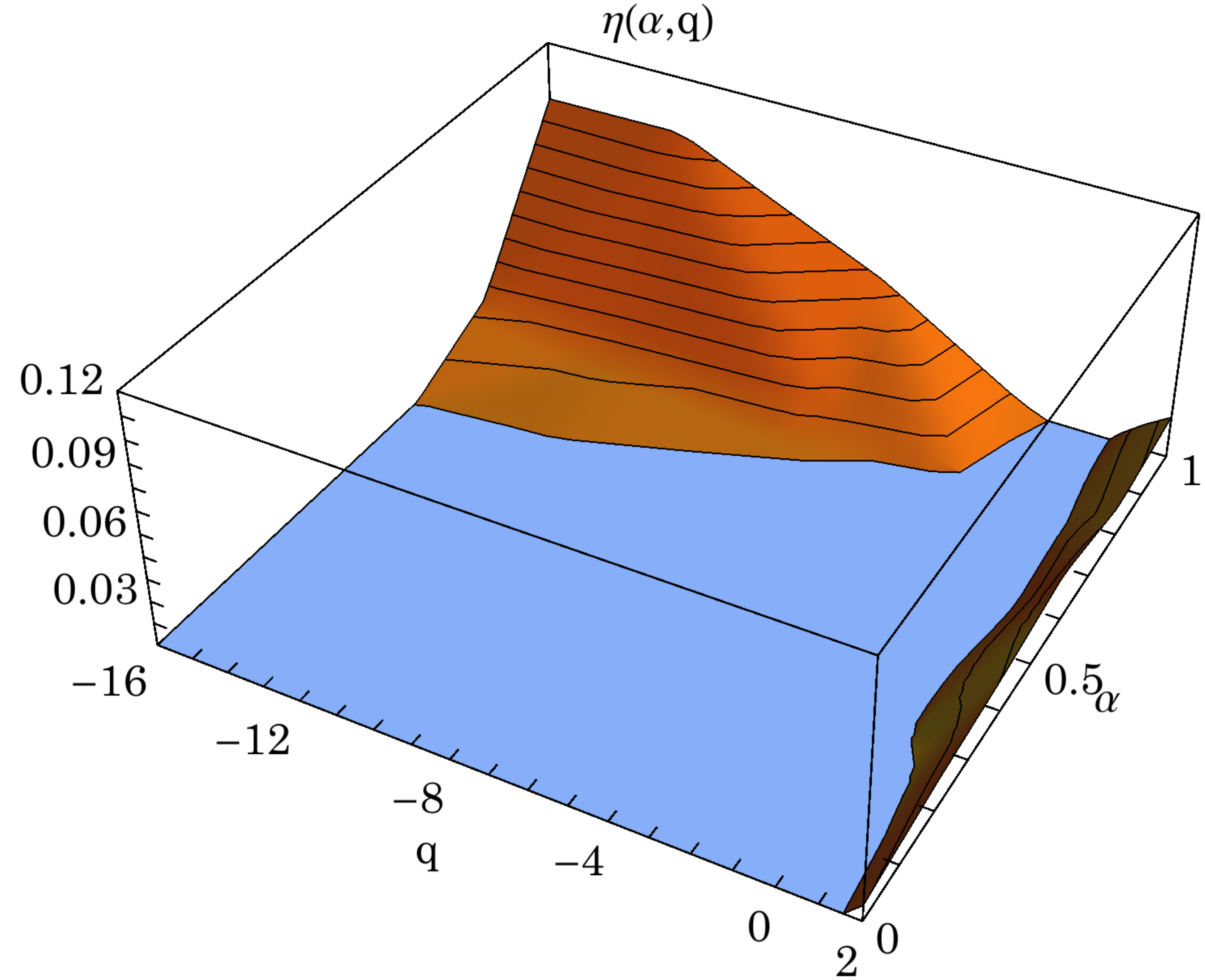


Results: instability mapped in anisotropy-rotation space

$\eta(\alpha,q)$ = growth rate



Linear Response Theory

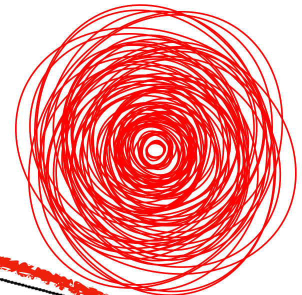


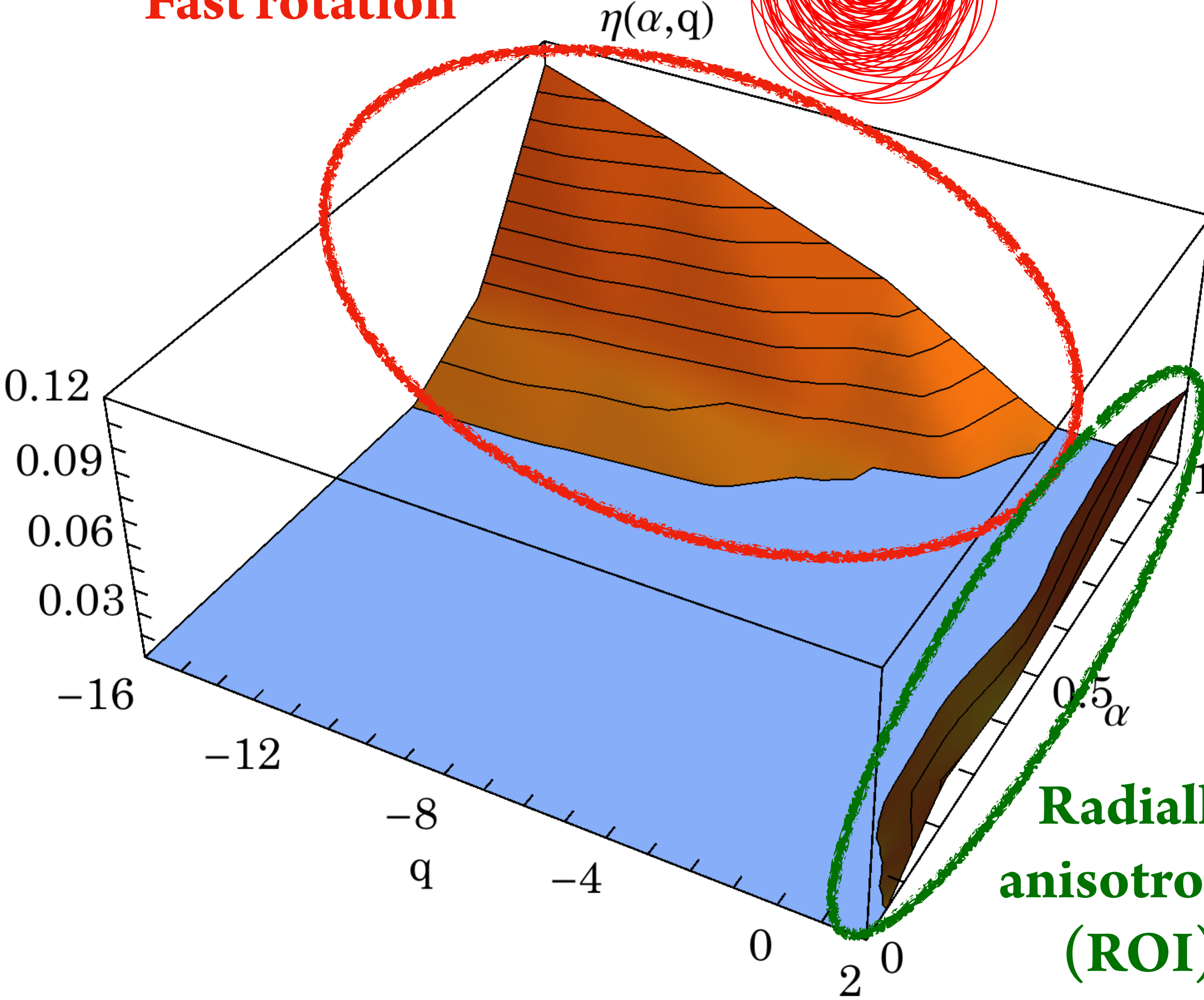
N-body simulations

q: anisotropy; α: rotation

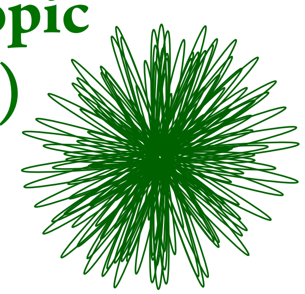
Results: instability mapped in anisotropy-rotation space

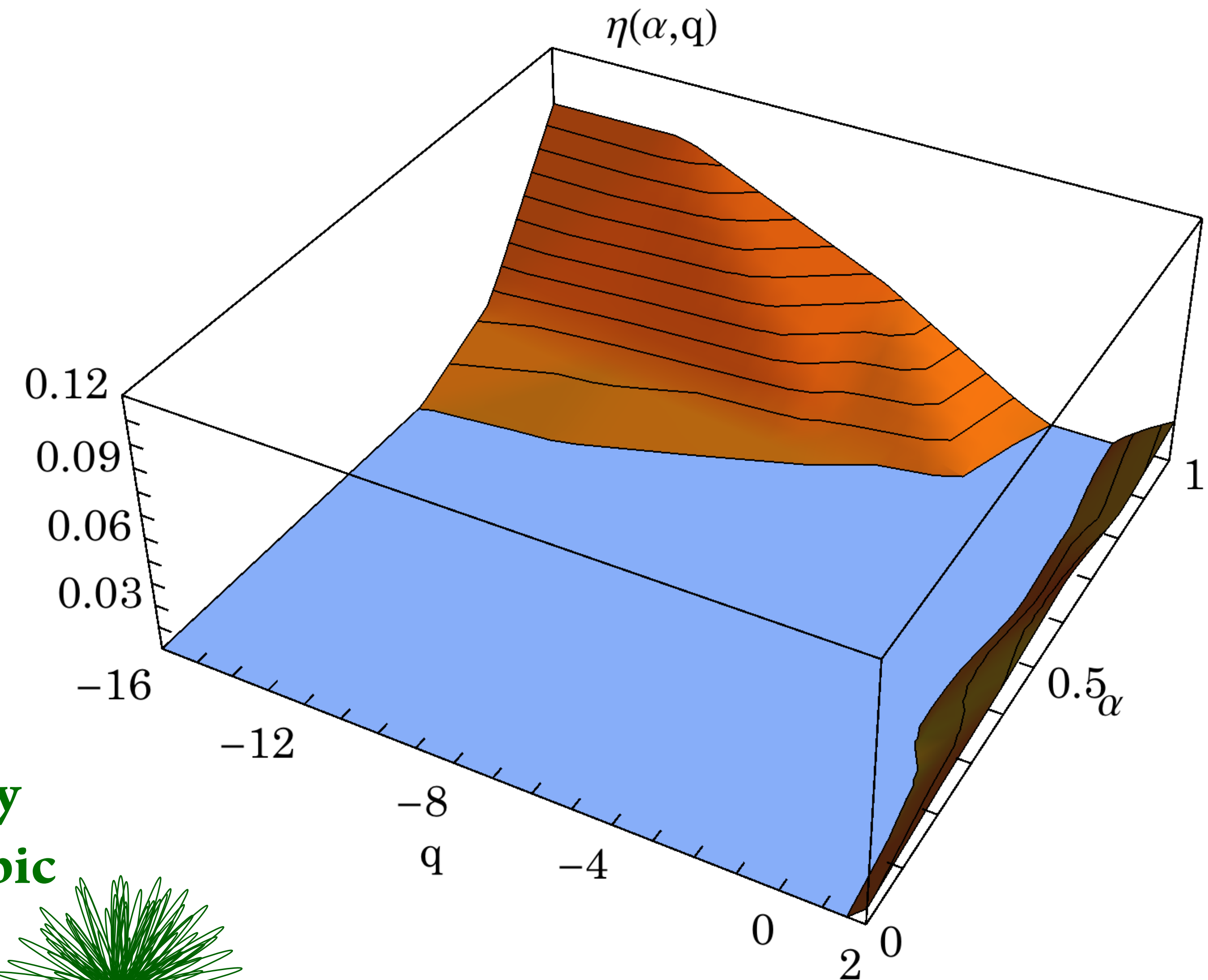
Tangentially anisotropic
Fast rotation





Radially
anisotropic
(ROI)



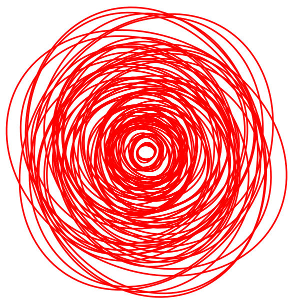


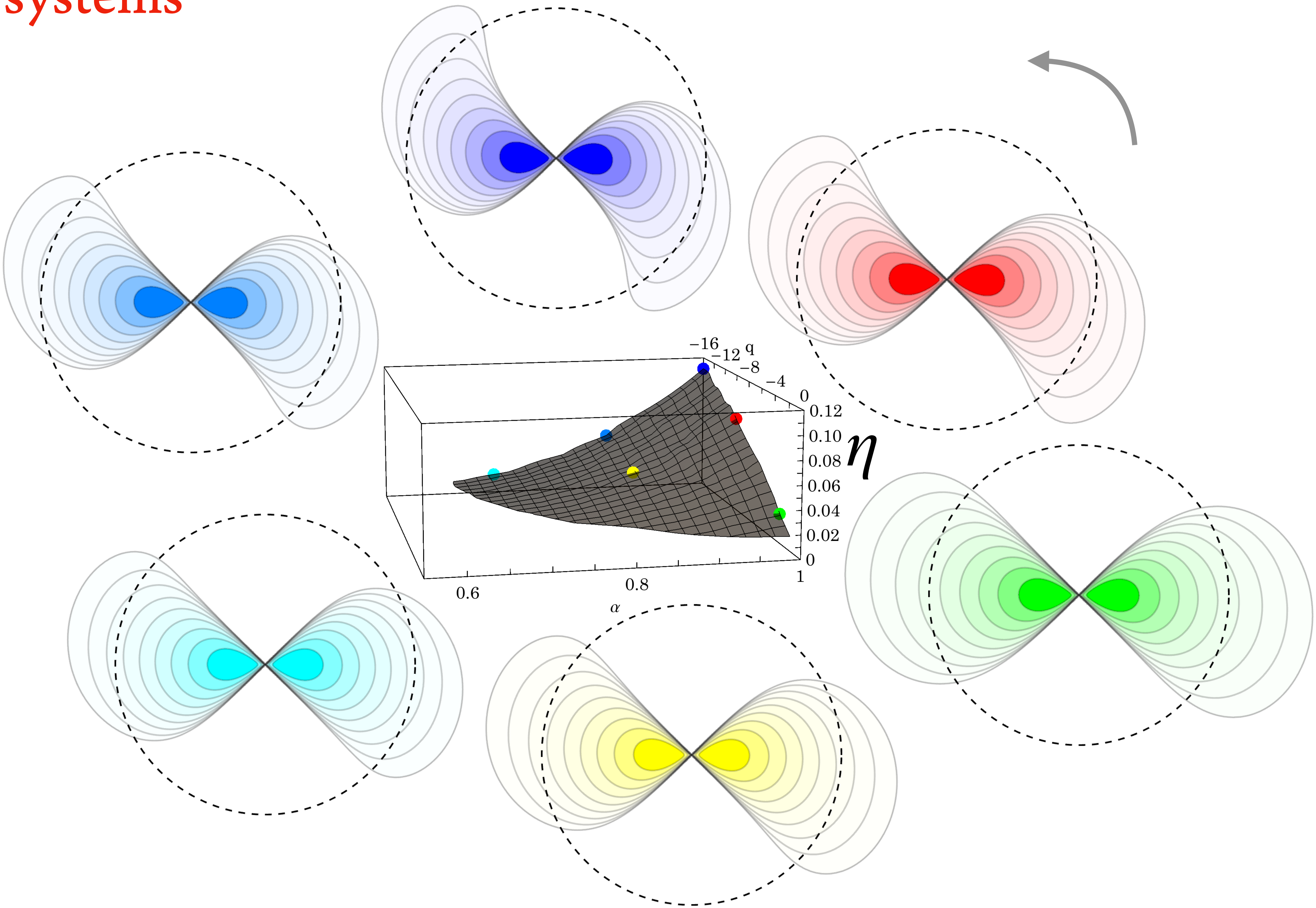
Linear Response Theory

N-body simulations

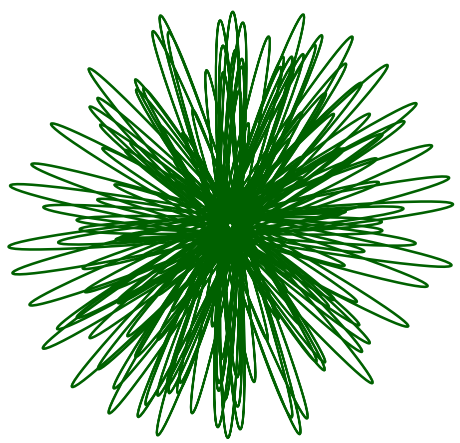
q: anisotropy; α: rotation

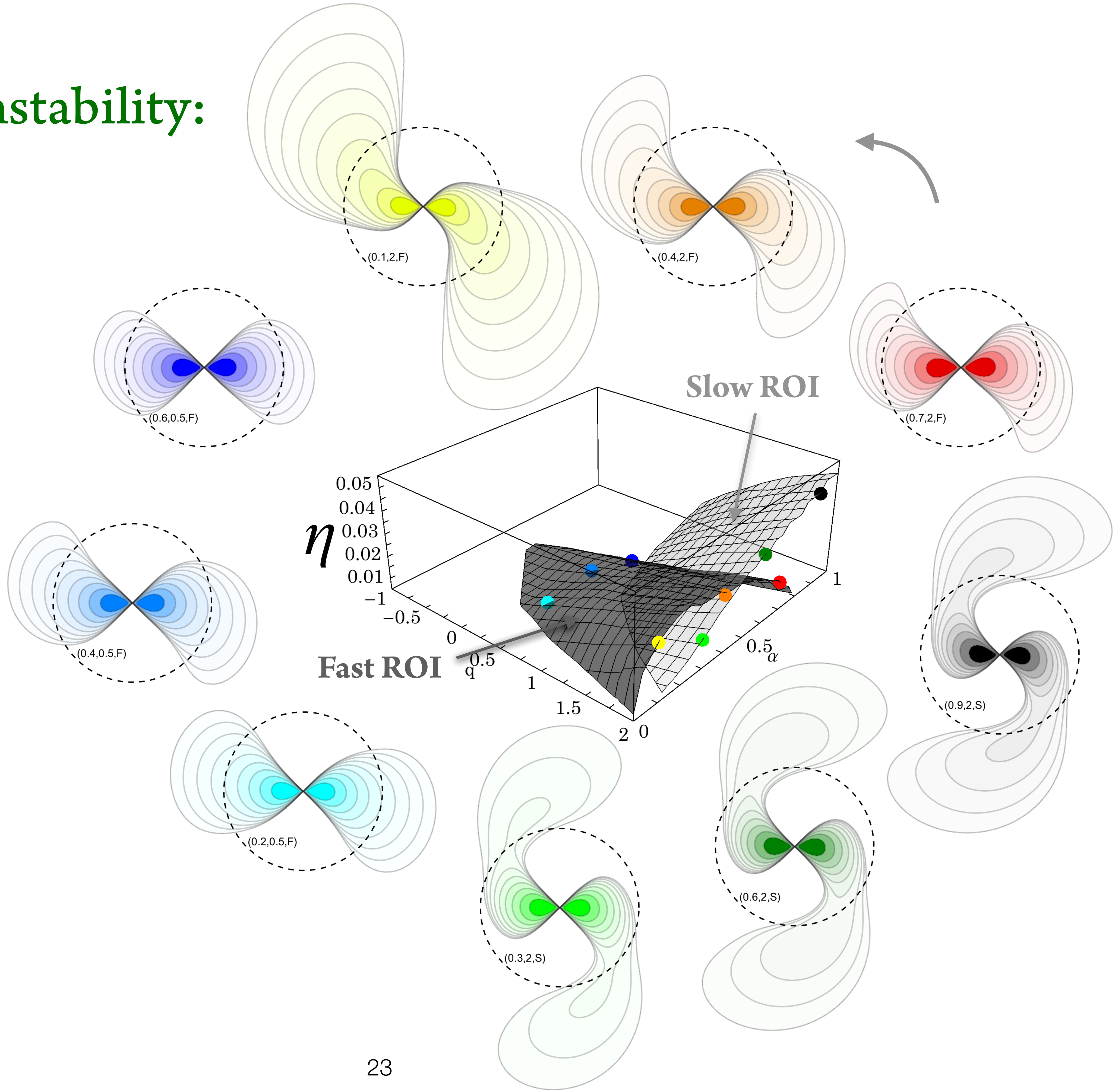
Instability in tangentially-biased systems



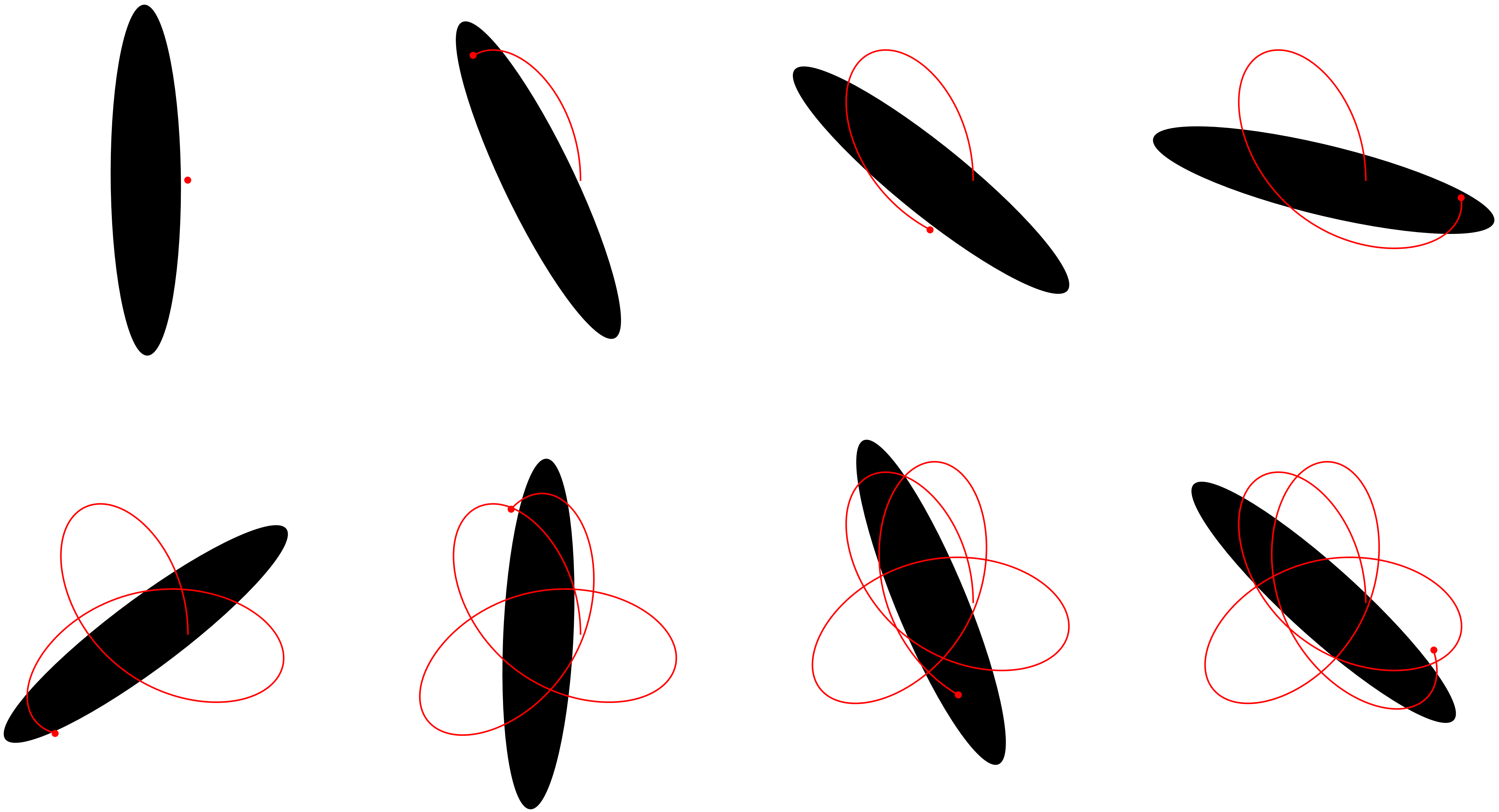


Radial Orbit Instability:
2 regimes

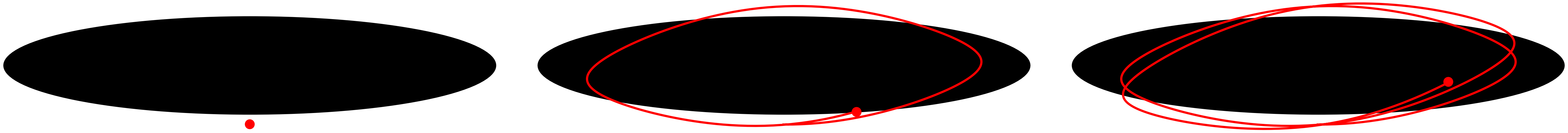




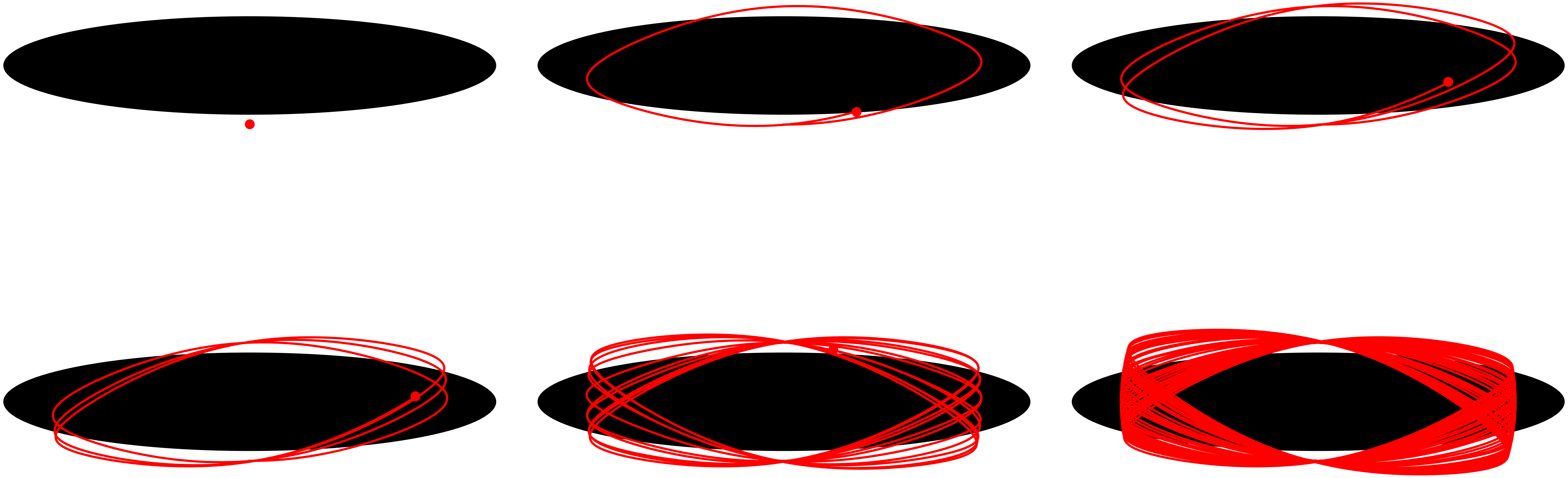
ROI in 2 sketches



ROI in 2 sketches

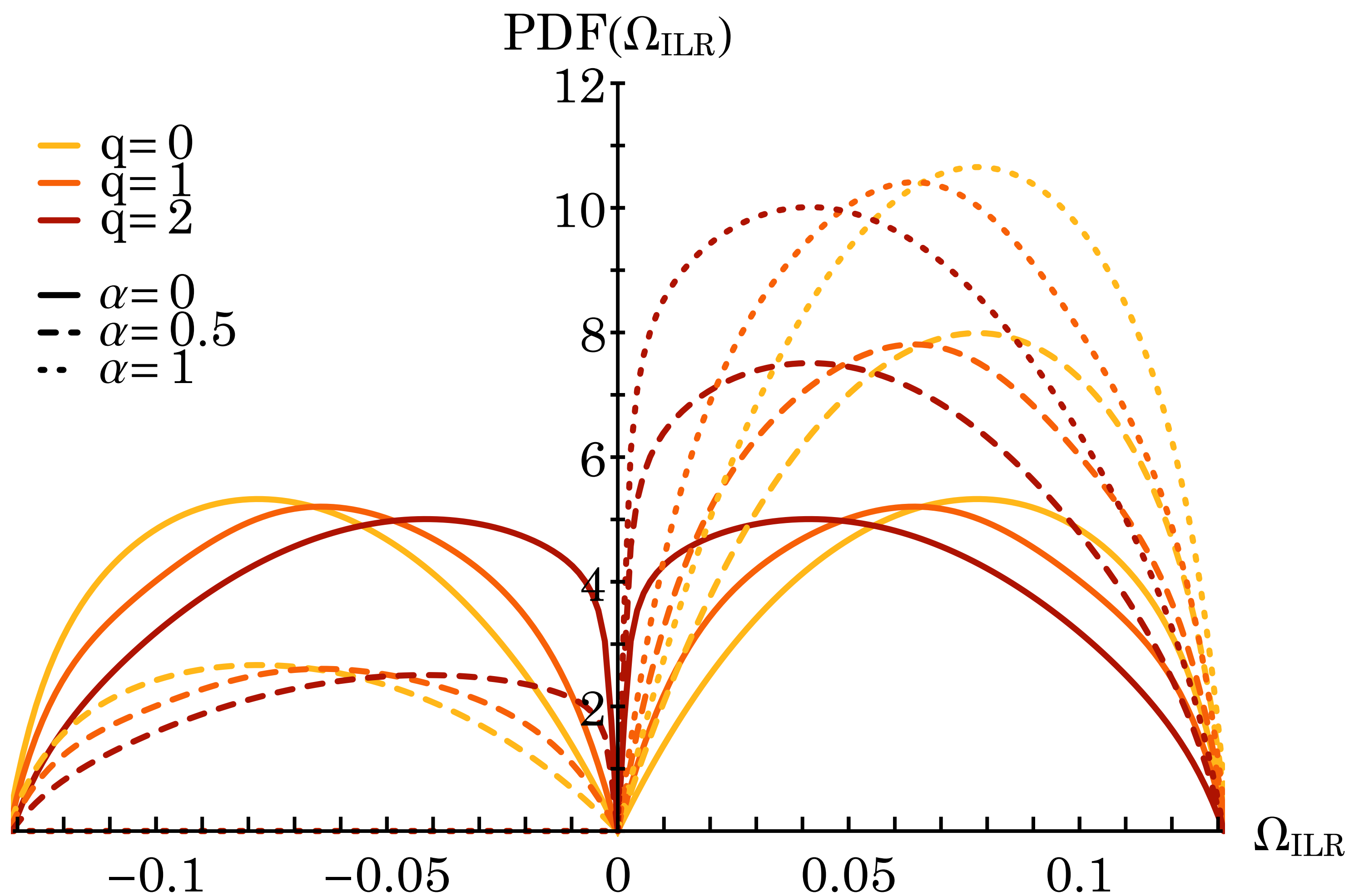


ROI in 2 sketches



Competition between attracting torque (destabilizing) and kinetic pressure in tumbling rates (stabilizing)

ROI in rotating spheres: qualitative argument



When q or **α** increases, the dispersion in tumbling rates decreases

Conclusion - Prospects

- Stability analysis of Plummer spheres with various rotation & anisotropy. Checked against N-body.

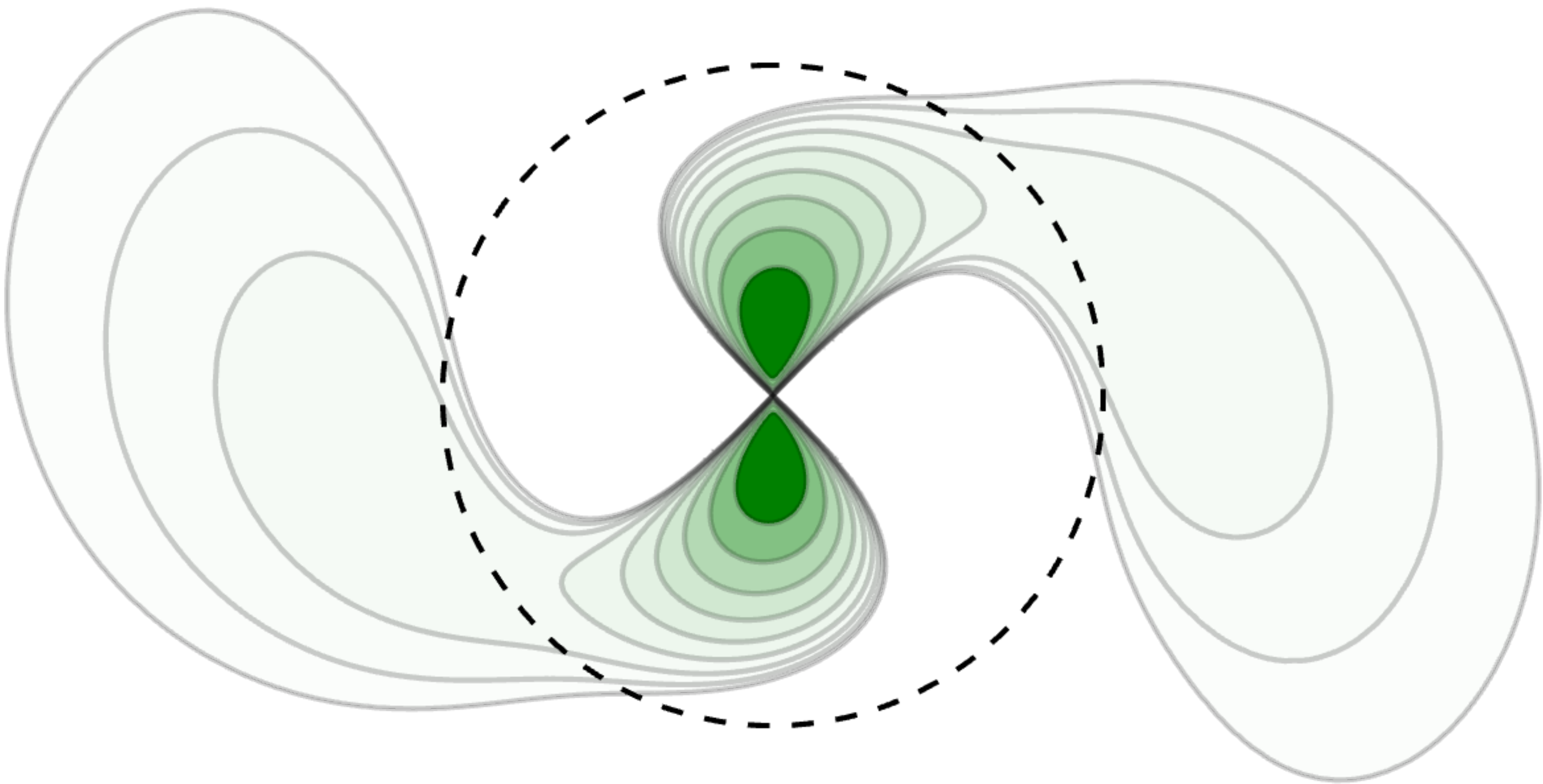
  ⟶ Generalized the radial orbit instability to rotating systems.

  ⟶ Discovery of a new regime of instability in tangentially anisotropic fast rotators.

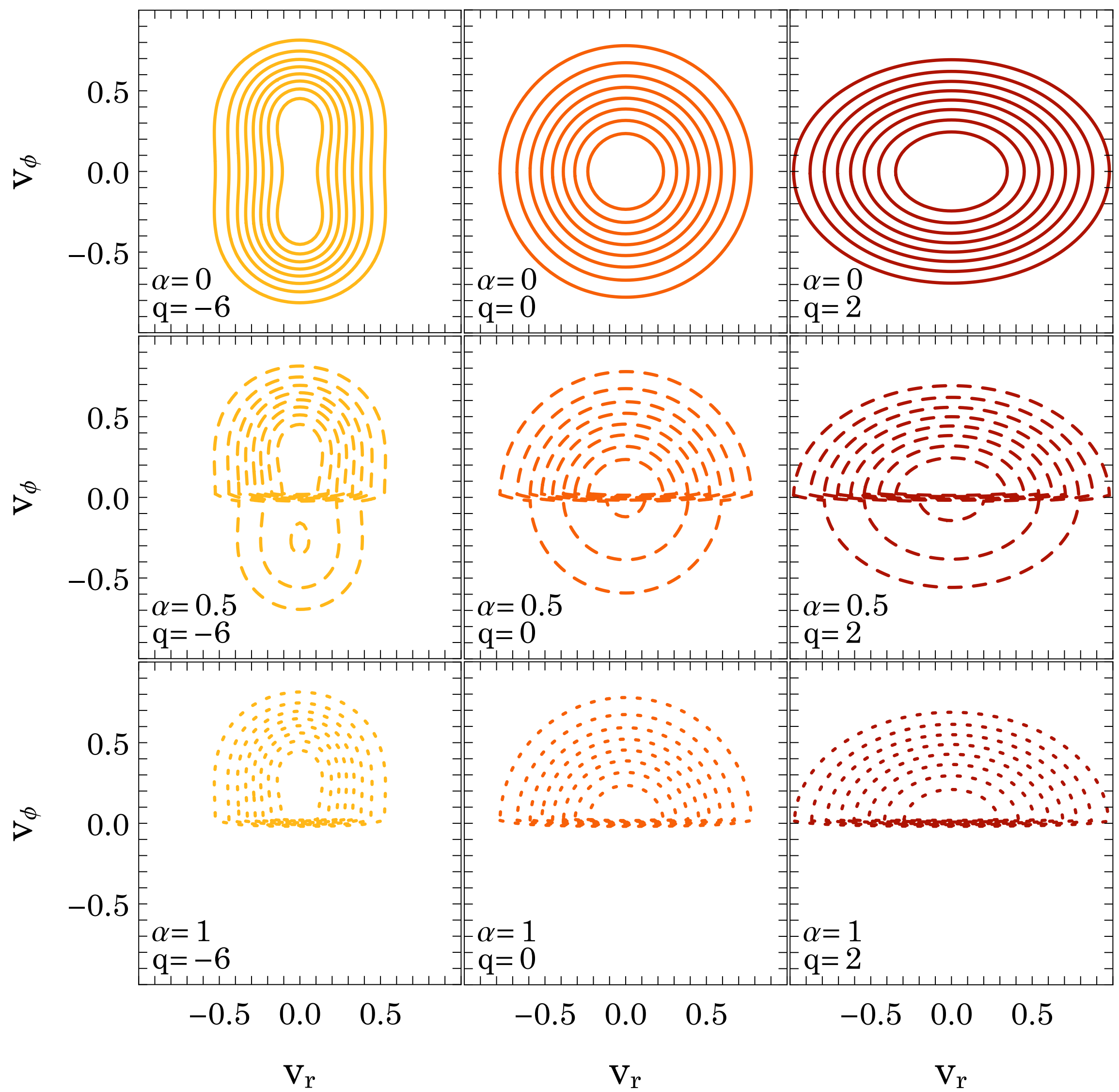
- Next step: Explain the mechanisms behind the instabilities (resonances?)

Switching off some resonance vectors gives good results in the case of the ROI.

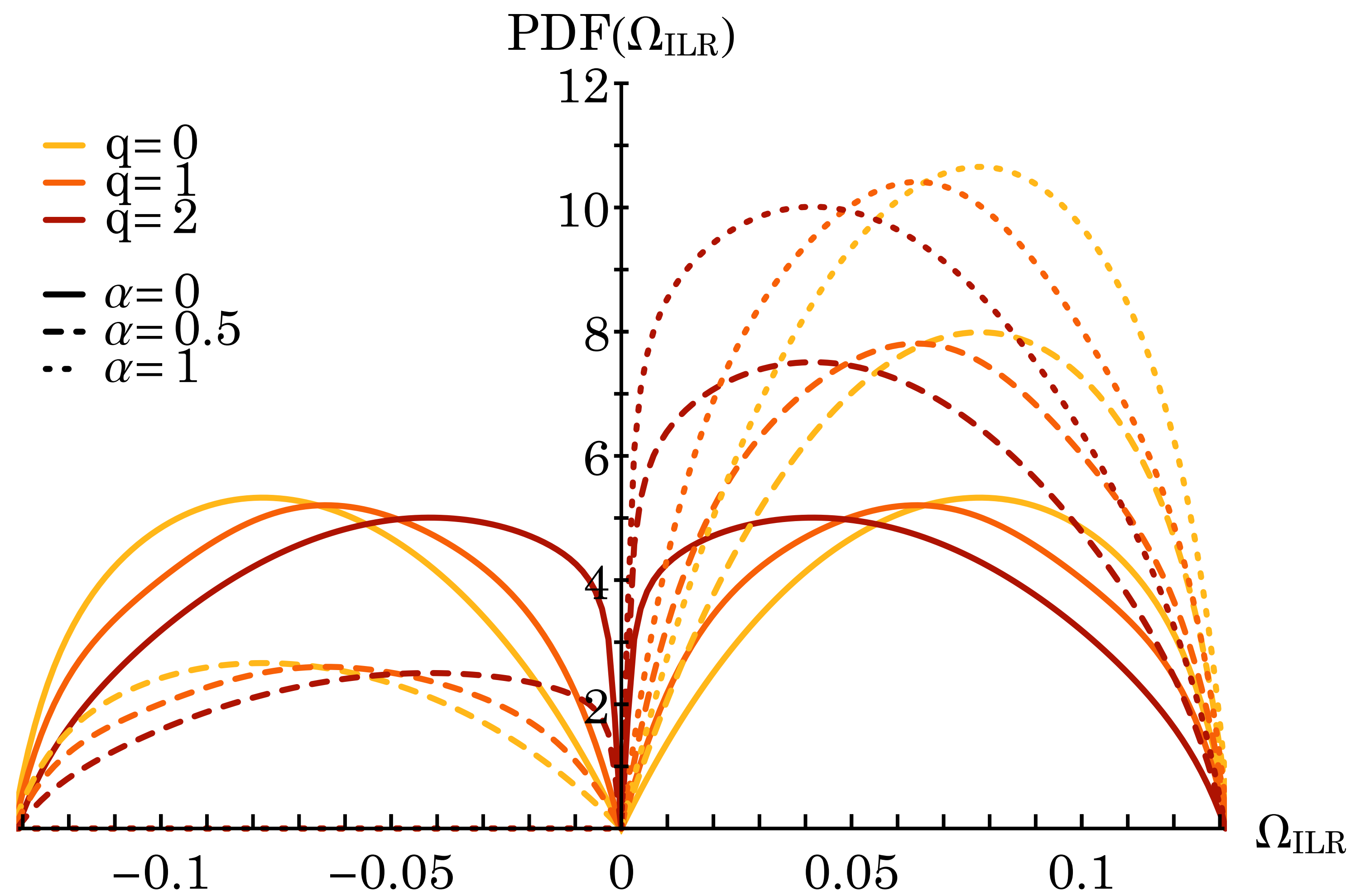
Thanks for your attention



Backup



v_ϕ
0.5
0.0
−0.5
$\alpha=0$
q= −6
q= 0
q= 2
$\alpha=0.5$
$\alpha=1$
−0.5 0.0 0.5
v_r



PDF(Ω_{ILR})
12
10
8
6
4
2
q= 0
q= 1
q= 2
$\alpha=0$
$\alpha=0.5$
$\alpha=1$
−0.1
−0.05
0
0.05
0.1
Ω_{ILR}

The matrix formalism

Projection of the potential and density onto a **bi-orthogonal basis** solving the Poisson equation

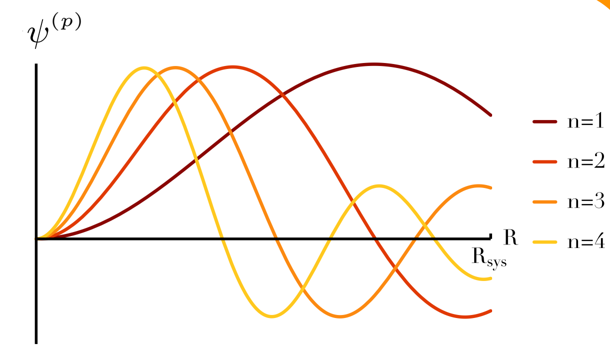
$$\boxed{\Delta\psi = 4\pi G\rho} \rightarrow \text{Projection} \quad \begin{aligned} \delta\psi(\boldsymbol{x}) &= \sum a_p \psi^{(p)}(\boldsymbol{x}) \\ \delta\rho(\boldsymbol{x}) &= \sum a_p \rho^{(p)}(\boldsymbol{x}) \end{aligned}$$

$$\Delta\psi^{(p)} = 4\pi G\rho^{(p)}$$

Kalnajs 1976

Basis

$$\int \mathrm{d}\boldsymbol{x}\, \psi^{(p)*}(\boldsymbol{x})\rho^{(q)}(\boldsymbol{x}) = -\delta_p^q$$



Self-induced perturbation	External perturbation	Total perturbation	
$\psi^{s1}(\mathbf{x},t) = \sum_p a_p(t)\psi^{(p)}(\mathbf{x})$	$\psi^{e}(\mathbf{x},t) = \sum_p b_p(t)\psi^{(p)}(\mathbf{x})$	$\psi^{1}(\mathbf{x},t) = \sum_p c_p(t)\psi^{(p)}(\mathbf{x})$	$c_p = a_p + b_p$

$$a_p(t) = -\int \mathrm{d}\mathbf{x}\, \rho^{s1}(\mathbf{x},t)\,\psi^{(p)*}(\mathbf{x}) = -\sum_{\mathbf{m}} \int \mathrm{d}\mathbf{x}\, \rho^{s1}_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}\,\psi^{(p)*}(\mathbf{x}) = -\sum_{\mathbf{m}} \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\, f^{1}_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}\,\psi^{(p)*}(\mathbf{x})$$

Bi-orthogonal basis $\quad X(\boldsymbol{\theta},\mathbf{J},t) = \sum_{\mathbf{m}} X_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}} \quad \rho = \int \mathrm{d}\mathbf{v}\, f$

$$a_p(t) = -\sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\mathrm{d}\boldsymbol{\theta}\, f^{1}_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}\,\psi^{(p)*}(\mathbf{x}) = -(2\pi)^3\sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, f^{1}_{\mathbf{m}}(\mathbf{J},t)\,\psi^{(p)*}_{\mathbf{m}}(\mathbf{J})$$

Change of canonical variables $\quad X(\boldsymbol{\theta},\mathbf{J},t) = \sum_{\mathbf{m}} X_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}$

The response matrix

Laplace transform $\tilde{a}_p(\omega) = -(2\pi)^3 \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, \tilde{f}^1_{\mathbf{m}}(\mathbf{J},\omega)\, \psi^{(p)*}_{\mathbf{m}}(\mathbf{J}) = -(2\pi)^3 \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, \mathbf{m}\cdot \frac{\partial f_0}{\partial \mathbf{J}} \frac{\tilde{\psi}^1_{\mathbf{m}}(\mathbf{J},\omega)}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}\, \psi^{(p)*}_{\mathbf{m}}(\mathbf{J})$

$\tilde{X}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, X(t)\, e^{\mathrm{i}\omega t}$

Linearized Vlasov: $\tilde{f}^1_{\mathbf{m}}(\mathbf{J},\omega) = \mathbf{m}\cdot \frac{\partial f_0}{\partial \mathbf{J}} \frac{\tilde{\psi}^1_{\mathbf{m}}(\mathbf{J},\omega)}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}$

$$\tilde{a}_p(\omega) = -(2\pi)^3 \sum_q \tilde{c}_q(\omega) \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, \mathbf{m}\cdot \frac{\partial f_0}{\partial \mathbf{J}} \frac{1}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}\, \psi^{(p)*}_{\mathbf{m}}(\mathbf{J}) \psi^{(q)}_{\mathbf{m}}(\mathbf{J})$$

$\tilde{\psi}^1_{\mathbf{m}}(\mathbf{J},\omega) = \sum_q \tilde{c}_q(\omega) \psi^{(q)}_{\mathbf{m}}(\mathbf{J})$

Matrix equation $\tilde{\mathbf{a}}(\omega) = \widehat{\mathbf{M}}(\omega)\cdot\tilde{\mathbf{c}}(\omega)$ or $\tilde{\mathbf{a}}(\omega) = \widehat{\mathbf{M}}(\omega)(\hat{\mathbf{I}} - \widehat{\mathbf{M}}(\omega))^{-1}\cdot\tilde{\mathbf{b}}(\omega)$

Response matrix

$$\boxed{\widehat{M}_{pq}(\omega) = -(2\pi)^3 \sum_{\mathbf{m}} \int \mathrm{d}\mathbf{J}\, \frac{\mathbf{m}\cdot \frac{\partial f_0}{\partial \mathbf{J}}}{\mathbf{m}\cdot\mathbf{\Omega}-\omega}\, \psi^{(p)*}_{\mathbf{m}}(\mathbf{J}) \psi^{(q)}_{\mathbf{m}}(\mathbf{J})}$$

Condition for an unstable mode: $\boxed{\det[\hat{\mathbf{I}} - \widehat{\mathbf{M}}(\omega_m)] = 0, \quad \mathrm{Im}(\omega_m) > 0}$ $\quad \mathbf{a}(t) \propto e^{\mathrm{i}\omega_m t}$

Linearizing the Vlasov equation

System:
Spherical cluster ($\psi_0(r)$, Plummer potential), steady state with tunable anisotropy ($f_q(E,L)$)
Dejonghe 1987
and rotation ($f_{q,\alpha}(E,L,L_z)$):
Lynden-Bell 1962

$$f_{q,\alpha}(E,L,L_z) = f_q(E,L)(1+\alpha \ \mathrm{Sign}(L_z/L))$$

Angle-action variables:
The actions $\mathbf{J}$ are constants of the motion, so the angles do not appear in the Hamiltonian. $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$
Spherically-symmetric system $\longrightarrow$ 3 actions available Binney & Tremaine 2008
Then the corresponding angles are straightforward (linear in time).

Usual choice: $$J_1 = J_r = \frac{1}{\pi}\int_{r_p}^{r_a} v_r \, \mathrm{d}r \qquad J_2 = J_\phi = L = r\, v_T \qquad J_3 = J_z = \mathbf{L}\cdot\mathbf{e_z}$$

Vlasov: $$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{\theta}}\cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$$

$\boldsymbol{\Omega}(\mathbf{J}) = \frac{\partial H}{\partial \mathbf{J}}$

Linearized Vlasov: $$\frac{\partial f^1}{\partial t} + \frac{\partial f^1}{\partial \boldsymbol{\theta}}\cdot \boldsymbol{\Omega}(\mathbf{J}) - \frac{\partial f_0}{\partial \mathbf{J}}\frac{\partial \psi^1}{\partial \boldsymbol{\theta}} = 0$$

Fourier transform in angles: 2π-periodicity implies all functions can be written $X(\boldsymbol{\theta},\mathbf{J},t) = \sum_{\mathbf{m}} X_{\mathbf{m}}(\mathbf{J},t)e^{\mathrm{i}\mathbf{m}\cdot\boldsymbol{\theta}}$

Laplace transform in time: $$\tilde{X}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, X(t)\, e^{\mathrm{i}\omega t}$$

Complex frequency $\omega = \omega_0 + i\eta$
ω_0 = precession rate
η = growth rate
$\eta < 0 \rightarrow$ stable; $\eta > 0 \rightarrow$ unstable

$$\tilde{f}^1_{\mathbf{m}}(\mathbf{J},\omega) = \mathbf{m}\cdot\frac{\partial f_0}{\partial \mathbf{J}}\frac{\tilde{\psi}^1_{\mathbf{m}}(\mathbf{J},\omega)}{\mathbf{m}\cdot\boldsymbol{\Omega} - \omega}$$