A walk throught symmetries
Revisit some notions with a new regard

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1 Some preliminaries

2 General principles
- Basic definitions
- Lie algebras
- Representations of Lie algebras

3 Examples with $\mathfrak{su}(3)$

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Symmetries during physicist's study

Symmetries in physics

Symmetry is a *leitmotiv*

1. Computation of electric/magnetic field using Gauss/Ampère theorem
   $\rightarrow$ needs symmetries
2. In mechanics choosing a preferred frame (as the rest frame)
   $\rightarrow$ needs symmetries
3. Classification of Mendeleev periodical table
   $\rightarrow$ needs symmetries
4. *etc.*

What is a symmetry?

Symmetries at IPHC

Symmetries at IPHC

1. Symmetries in subatomic physics
   Symmetries of spacetime
   $\to$ mass and spin of particles
2. Symmetries in nuclear physics
   Symmetry of space
   $\to$ shell model
   Symmetry proton-neutron
   $\to$ isospin
3. Symmetries in particles physics
   The Standard Model
   $\to$ classifies particles
   $\to$ dictates their interactions
   Concept of symmetry breaking
   $\to$ gives a mass to particles
   Concept of anomalies
   $\to$ restrict the quantum numbers of particles

Some strange points

Complex or real

1. In standard Quantum Mechanics lectures (L3)

   Angular momentum: (L_1, L_2, L_3) operators of rotations
   To introduce the spin, *i.e.*, the states $|\ell, m\rangle$ we define

   $$L_\pm = L_1 \pm i\ L_2$$

   Since the angles of rotation are real: $L = i\alpha^a L_a, \alpha^i \in \mathbb{R}$.
   $\rightarrow$ why can we make complex linear combination ?????

2. When studying the spin of the electron: Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\rightarrow$ Pauli spinor $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$

$\rightarrow$ The Pauli matrices are complex $\Rightarrow \psi^1, \psi^2 \in \mathbb{C}$

$\rightarrow$ A Pauli spinor has $\mathbf{4} = 2 \times 2$ degrees of freedom

$\rightarrow$ An electron has two degrees of freedom: spin $s = \pm\frac{1}{2}$?????

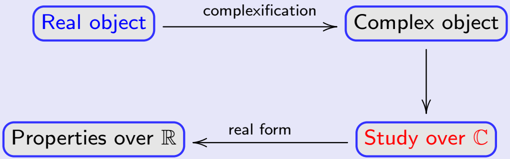
Confusion between complex/real numbers

Complex is more simple

1. On the complex number life is more easy
   1. $X^2 + 1 = 0$ two solutions on $\mathbb{C}$ no solution on $\mathbb{R}$
   2. The matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \xrightarrow[\text{Diagonalisable}]{\text{over } \mathbb{C}} \Delta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

2. Sometimes a back and forth between $\mathbb{R}$ and $\mathbb{C}$ is possible $\cdots$



3. Back and forth between $\mathbb{R}$ and $\mathbb{C}$ not always possible

Confusion between complex/real numbers

Complexification/real form

3. One example of back and forth

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \xrightarrow[z=x+iy]{\text{complexification}} R = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Acts on $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ ⟵ real form ($x = 1/2(z+\bar z)$, $y = -i/2(z-\bar z)$) ⟵ Acts on $\begin{pmatrix} z \\ \bar z \end{pmatrix} \in \mathbb{C} \oplus \overline{\mathbb{C}}$

E.g. electromagnetic waves

$$A^\mu = \begin{pmatrix} A^0 = 0 \\ A^1 \\ A^2 \\ A^3 = 0 \end{pmatrix}$$

⟵ E.g two polarisation of the light

$$A_\pm = \frac{\pm A^1 + iA^2}{\sqrt{2}}$$

arXiv:0711.0770v1 [hep-th] 6 Nov 2007

An Exceptionally Simple Theory of Everything

WRONG

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ABSTRACT: All fields of the standard model and gravity are unified as an E8 principal bundle connection. A non-compact real form of the E8 Lie algebra has G2 and F4 subalgebras which break down to strong su(3), electroweak su(2) x u(1), gravitational so(3,1), the frame-Higgs, and three generations of fermions related by triality. The interactions and dynamics of these 1-form and Grassmann valued parts of an E8 superconnection are described by the curvature and action over a four dimensional base manifold.

KEYWORDS: ToE.

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PLANÈTE

Anthony Garrett Lisi : " La théorie est mathématiquement et esthétiquement superbe "

Physicien amateur, Anthony Garrett Lisi, un Américain de 39 ans, a posté, début novembre, sur un serveur, un papier de 31 pages stipulant que toutes les lois de l'univers seraient décrites par une seule et même théorie. Son travail est, depuis début novembre, au centre de vives discussions dans la communauté scientifique.

Par Propos recueillis par courriel par Stéphane Foucart · Publié le 19 novembre 2007 à 14h00 - Mis à jour le 19 novembre 2007 à 14h00

Some properties over $\mathbb{C}$ do not pass to $\mathbb{R}$ ☹

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September 2010, Volume 298, Issue 2, pp 419–436 | Cite as

There is No "Theory of Everything" Inside E_8

Authors | Authors and affiliations
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Article
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Abstract

We analyze certain subgroups of real and complex forms of the Lie group E_8, and deduce that any "Theory of Everything" obtained by embedding the gauge groups of gravity and the Standard Model into a real or complex form of E_8 lacks certain representation-theoretic properties required by physical reality. The arguments themselves amount to representation theory of Lie algebras in the spirit of Dynkin's classic papers and are written for mathematicians.

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   - Basic definitions
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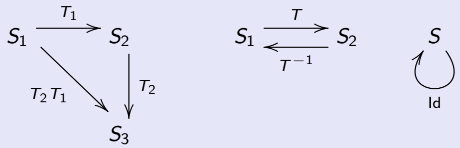
Mathematical structure associated to symmetries

- symmetry = transformation which leaves a system invariant

Set of Symmetries = group



1. the principle of symmetry is extremely powerful in physics

   implies the fundamental laws

2. In Quantum mechanics the principle of symmetry takes a **stronger** dimension

Symmetries in Quantum Mechanics

Symmetries in Hilbert space

- States $\big|\psi\big\rangle$ lives in **Hilbert** space
- A transformation $G : \big|\Psi\big\rangle \longrightarrow \big|\Psi_G\big\rangle = G\big|\Psi\big\rangle$ is a **symmetry**

  if it preserves the **transition amplitude**
  $\big\langle\Psi_G\big|\Phi_G\big\rangle = \big\langle\Psi|G^\dagger G\big|Phi\big\rangle = \big\langle\Psi\big|\Phi\big\rangle.$
- If G is unitary

$$GG^\dagger = \mathrm{Id} \ .$$

G preserves the **transition amplitude**

The Wigner Theorem

Theorem (Wigner, 1959)

Let a quantum system be invariant under a symmetry group G. To any element $g \in G$ one can associate an operator $\mathcal{U}(g)$ acting on the state $\big|\Psi\big\rangle \in H$

$$\big|\Psi\big\rangle \to \big|\Psi'\big\rangle = \big|\mathcal{U}(g)\Psi\big\rangle = \mathcal{U}(g)\big|\Psi\big\rangle \ ,$$

which is either unitary and linear

$$\big\langle\mathcal{U}(g)\Psi_1\big|\mathcal{U}(g)\Psi_2\big\rangle = \big\langle\Psi_1\big|\Psi_2\big\rangle \ ,$$

$$\mathcal{U}(g)\Big[\lambda_1\big|\Psi_1\big\rangle + \lambda_2\big|\Psi_2\big\rangle)\Big] = \lambda_1\mathcal{U}(g)\big|\Psi_1\big\rangle + \lambda_2\mathcal{U}(g)\big|\Psi_2\big\rangle \ ,$$

or anti-unitary and anti-linear

$$\big\langle\mathcal{U}(g)\Psi_1\big|\mathcal{U}(g)\Psi_2\big\rangle = \big\langle\Psi_1\big|\Psi_2\big\rangle^* \ ,$$

$$\mathcal{U}(g)\Big[\lambda_1\big|\Psi_1\big\rangle + \lambda_2\big|\Psi_2\big\rangle)\Big] = \lambda_1^*\mathcal{U}(g)\big|\Psi_1\big\rangle + \lambda_2^*\mathcal{U}(g)\big|\Psi_2\big\rangle \ .$$

Continuous and discrete symmetries

There are two types of symmetries

◇ Discrete symmetries

Example (The parity transformation in $\mathbb{R}^3$)

$$\begin{aligned} \mathrm{Id} : \vec{x} &\to \vec{x} \,, \\ P : \vec{x} &\to -\vec{x} \,. \end{aligned}$$

$\longrightarrow$ **finite group or countable group** $G = \left\{g_1, \cdots, g_n\right\}, G = \left\{g_i \; i \in \mathbb{N}\right\}$

◇ Continuous symmetries

Example (The rotations in $\mathbb{R}^3$)

$$R(\vec{\alpha}) : \vec{x} \to R(\vec{\alpha})\vec{x} \,,$$

$\vec{\alpha} \in \mathbb{R}^3$ is the angle of rotation

$$\lim_{\vec{\alpha} \to \vec{0}} R(\vec{\alpha}) = \mathrm{Id} \,.$$

$\longrightarrow$ **unitary and linear operators**

$\longrightarrow$ **continuously** connected to the **identity operator** Id.

Continuous symmetries

- A continuous symmetry depends on parameters

Example

1. The rotation in $\mathbb{R}^3$ has three parameters
2. The Galilean group has ten parameters
3. The Lorentz group has six parameters
4. The Poincaré group has ten parameters
5. The gauge group of electromagnetism has one parameter
6. Many continuous groups in physics

Infinitesimal transformations

Symmetries = group

- Consider a group of symmetry with n parameters
  - To any $g \in G$ is associated $n-$parameters: $g(\theta^1, \cdots, \theta^n) \equiv g(\theta)$.
  - If the group is real the parameters are real
  - If the group is complex the parameters are complex
- Infinitesimally $\mathcal{U}(g(\theta)) = 1 + iA(\theta) = 1 + i\theta^a T_a$,
- Assume G to be **real**
  - T_a have **very restricted properties**.
  - Since $\mathcal{U}(g(\theta))^\dagger = \mathcal{U}(g(\theta))^{-1}$ we have $T_a{}^\dagger = T_a$
- The product of two symmetries is a symmetry

$$\mathcal{U}(g(\theta))\mathcal{U}(g(\theta')) = \mathcal{U}(g(\theta'')) \Rightarrow [T_a, T_b] = if_{ab}{}^c T_b \ .$$

- If G acts on a state $|\psi\rangle$. Infinitesimally:

$$\begin{aligned} |\psi'\rangle &= \mathcal{U}(g(\theta))\,|\psi\rangle = (1 + iA(\theta))\,|\psi\rangle \\ &\Rightarrow \\ \delta\,|\psi\rangle &= |\psi'\rangle - |\psi\rangle = iA(\theta)\,|\psi\rangle = i\theta^a T_a\,|\psi\rangle \ . \end{aligned}$$

A natural structure emerges: Lie algebras and Lie groups

Definition (Lie algebra $\mathfrak{g}$ associated to a Lie group G)

If $\mathfrak{g}$ is finite dimensional choosing a basis $\mathfrak{g} = \text{Span}\{T_1, \cdots, T_n\}$ we have

$$x = i\theta^a T_a \ , \quad [T_a, T_b] = if_{ab}{}^c T_c \ , [T_a, T_b] = -[T_b, T_a] \quad .$$
$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0$$

The (real) coefficients $f_{ab}{}^c$ are called the structure constants of $\mathfrak{g}$.
The identity $[T_a, [T_b, T_c]]$+perm $= 0$ is called the Jacobi identity.

Remark

Lie algebras were introduced and classified by mathematicians (Cartan, Dynkin, *etc*) and subsequently applied in physics.

Relationship between Lie algebras and Lie groups

- From Lie **algebra** to Lie **group** $\mathfrak{g} \xrightarrow{\exp} G$
  - To any $1 + i\theta^a T_a$ one can associate an element in the Lie group G

$$1 + i\theta^a T_a \longrightarrow \lim_{n\to\infty} \left(1 + i\frac{\theta^a}{n} T_a\right)^n = e^{i\theta^a T_a} \ .$$

  - Composition of an **infinite** number of **infinitesimal** transformations
- From Lie **groups** to Lie **algebras** $G \xrightarrow{\partial_{\theta^a}} \mathfrak{g}$
  - To any element $e^{i\theta^a T_a}$ one can associate n independent elements in the **Lie** algebra $\mathfrak{g}$

$$\mathcal{U}(g(\theta)) = e^{i\theta^a T_a} \longrightarrow -i \frac{\partial \mathcal{U}(g(\theta))}{\partial \theta^a}\Big|_{\theta^a=0} = T_a \ .$$

  - We have a geometrical interpretation
    - When θ^a varies $\rightarrow$ curve Γ_a.
    - T_a is the **vector tangent** to Γ_a at the identity.

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Three-dimensional simple Lie algebras and Lie groups – Matrix Lie groups

$$SL(2,\mathbb{C}) = \left\{ U \in \mathcal{M}_2(\mathbb{C}), \det(U) = 1 \right\}$$

$$U = 1 + u, u \in \mathfrak{sl}(2,\mathbb{C})$$

$$u = \alpha^0 X_0 + \alpha^+ X_+ + \alpha^- X_-, \alpha^i \in \mathbb{C}$$

Lie Algebra ⟷ Lie group

$$X \in \mathfrak{sl}(2,\mathbb{C}) \Leftrightarrow \mathrm{Tr}(X) = 0$$

$$X_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$SU(2) = \left\{ U \in \mathcal{M}_2(\mathbb{C}), \det(U) = 1, UU^\dagger = 1 \right\}$$

$$U = 1 + iu, u \in \mathfrak{su}(2), u = \alpha^i J_i, \alpha^i \in \mathbb{R}$$

Lie Algebra ⟷ Lie group

$$J \in \mathfrak{su}(2) \Leftrightarrow \mathrm{Tr}(J) = 0, J^\dagger = J$$

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$SL(2,\mathbb{R}) = \left\{ U \in \mathcal{M}_2(\mathbb{R}), \det(U) = 1 \right\}$$

$$U = 1 + iu, u \in \mathfrak{sl}(2,\mathbb{R}), u = \alpha^i K_i, \alpha^i \in \mathbb{R}$$

Lie Algebra ⟷ Lie group

$$K \in \mathfrak{sl}(2,\mathbb{R}) \Leftrightarrow \mathrm{Tr}(K) = 0$$

$$K_0 = -\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$K_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad K_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

A back and forth for $\mathfrak{su}(2)$

$\mathfrak{sl}(2,\mathbb{C})$

$$[X_0, X_\pm] = \pm X_\pm \ ,$$
$$[X_+, X_-] = 2X_0 \ .$$

$$\begin{cases} J_3 = X_0 \\ J_1 = \frac{1}{2}(X_+ + X_-) \\ J_2 = -\frac{i}{2}(X_+ - X_-) \end{cases}$$

$$\begin{cases} K_0 = -iX_0 \\ K_1 = \frac{i}{2}(X_+ + X_-) \\ K_2 = \frac{1}{2}(X_+ - X_1) \end{cases}$$

$J_a^\dagger = J_a$

$K_a^\dagger = -K_a$

$\mathfrak{su}(2)$

$$[J_i, J_j] = iJ_k \ , i, j, k, \text{ perm.}$$

$\mathfrak{su}(2) \otimes \mathbb{C}$

$$X_+ = J_1 + iJ_2$$
$$X_- = J_1 - iJ_2$$

$\mathfrak{sl}(2,\mathbb{R})$

$$[K_0, K_1] = iK_2$$
$$[K_1, K_2] = -iK_0$$
$$[K_2, K_0] = iK_1$$

Solve the first puzzle ☺

The Lie group $SU(3)$ and its Lie algebra $\mathfrak{su}(3)$

Lie group

$$SU(3) = \left\{ U \in \mathcal{M}_3(\mathbb{C}) \ , \det U = 1 \ , U^\dagger U = 1 \right\}$$

$U = 1 + iu$

Lie algebra

$$\mathfrak{su}(3) = \left\{ u \in \mathcal{M}_3(\mathbb{C}) \ , \mathrm{Tr} u = 0 \ , u - u^\dagger = 0 \right\}$$

generators

$$u = i\theta^a \frac{1}{2}\lambda_a$$

exp

group elements

$$U = e^{i\theta^a \frac{1}{2}\lambda_a}$$

The Gell-Mann matrices

The Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

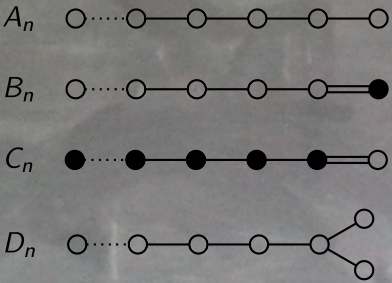
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The **Gell-Mann matrices** $\rightarrow$ **generators** $T_a = 1/2\lambda_a$

$$\mathrm{Tr}(T_a T_b) = 1/2\delta_{ab} \;, [T_a, T_b] = i f_{ab}{}^c T_c \;.$$

Classical Lie algebras



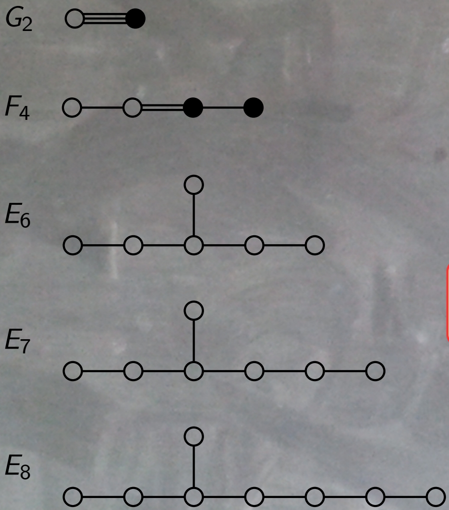
All these algebras have a matrix definition

1. We have

complex algebras		compact real form	Definition: preserves a scalar product
$A_n \cong \mathfrak{su}(n+1,\mathbb{C})$	$\to$	$\mathfrak{su}(n+1)$	$z_i \in \mathbb{C}: \ \|z_1\|^2+\cdots+\|z_{n+1}\|^2$
$B_n \cong \mathfrak{so}(2n+1,\mathbb{C})$	$\to$	$\mathfrak{so}(2n+1)$	$x_i \in \mathbb{R}: \ x_1^2+\cdots+x_{2n+1}^2$
$C_n \cong \mathfrak{sp}(2n,\mathbb{C})$	$\to$	$\mathfrak{usp}(2n)$	$q_i \in \mathbb{H}: \ \|q_1\|^2+\cdots+\|q_n\|^2$
$D_n \cong \mathfrak{so}(2n,\mathbb{C})$	$\to$	$\mathfrak{so}(2n)$	$x_i \in \mathbb{R}: \ x_1^2+\cdots+x_{2n}^2$

2. Real forms are classified. For $\mathfrak{so}(2n,\mathbb{C}) \to \mathfrak{so}(p,q)$ with $p+q=2n$ and $\mathfrak{so}^*(2n)$.

Exceptional Lie algebras



All these algebras have not a matrix definition. Related to octonions

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Practical realisation of Lie algebras

Representations of Lie algebras

1. A Lie algebra
   - $\rightarrow$ formal definition $\mathfrak{g} = \{T_1, \cdots, T_n\}$
   - $\rightarrow$ commutation relations $[T_a, T_b] = if_{ab}{}^c T_c$
   - $\rightarrow$ Jacobi identity $[T_a, [T_b, T_c]] +$ perm $= 0$
2. In physics a Lie algebra is a symmetry of a certain system
   - $\rightarrow$ acts on physical states
     - if the state is a vector with $d-$components
       - $\diamond T_a \rightarrow M_a$ $d \times d$ matrices
     - if the state is a function
       - $\diamond T_a \rightarrow M_a$ differential operator
   - $\rightarrow \{M_1, \cdots, M_n\}$ is called a representation of $\mathfrak{g}$
3. Problem find all unitary representations.

Again Real or complex

Real, complex pseudo-real

Assume $\mathfrak{g}$ to be a real Lie algebra with representation $T_a \to M_a$

1. To a rep. specified by M_a: three other reps.

$$\big[M_a, M_b\big] = i f_{ab}{}^c M_c \Rightarrow \begin{cases} \big[-\bar{M}_a, -\bar{M}_b\big] & = & i f_{ab}{}^c(-\bar{M}_c) \\ \big[M_a^\dagger, M_b^\dagger\big] & = & i f_{ab}{}^c M_c^\dagger \\ \big[-M_a^t, -M_b^t\big] & = & i f_{ab}{}^c(-M_c^t) \end{cases}$$

2. Unitarity: we always have $M_a^\dagger = M_a$ and $\bar{M}_a = M_a^t$
3. Different types of representations
   - a Real representation: the matrices are purely imaginary
     $-\bar{M}_a = M_a$ and the four rep. are the **same**
     For example rotations in $\mathbb{R}^3$
   - b Pseudo real representation: the matrices are complex but
     $-\bar{M}_a = P M_a P^{-1}$
     For example Spinor rep. of $SU(2)$
   - c Complex matrices −the two rep. not equivalent
     E.g $\mathfrak{su}(3)$ Gell-Mann matrices: quarks and antiquarks

Representation of $\mathfrak{su}(2)$

1. **Unitary** representation of $\mathfrak{su}(2)$ are **finite dimensional**.
2. $Q = \vec{J} \cdot \vec{J} = J_1^2 + J_2^2 + J_3^2$ is a **Casimir operator**.
3. To **any** $\ell \in \frac{1}{2}\mathbb{N}$ corresponds a $(2\ell+1)-$dimensional representation.

$$\mathcal{D}_\ell = \Big\{ \big|\ell, m\big\rangle \ , -\ell \le m \le \ell \Big\} \ .$$

4. Introducing $L_\pm = L_1 \pm i\, L_2$ **using the back and forth** $\mathbb{R} \leftrightarrow \mathbb{C}$

$$\begin{aligned} Q\big|\ell, m\big\rangle &= \ell(\ell+1)\big|\ell, m\big\rangle \\ J_0\big|\ell, m\big\rangle &= m\big|\ell, m\big\rangle \\ J_+\big|\ell, m\big\rangle &= \sqrt{(\ell - m)(\ell + m + 1)}\big|\ell, m+1\big\rangle \\ J_-\big|\ell, m\big\rangle &= \sqrt{(\ell + m)(\ell - m + 1)}\big|\ell, m-1\big\rangle \end{aligned}$$

The representation $\mathcal{D}_\ell$ is uniquely defined by the vector $\big|\ell, \ell\big\rangle$

$$J_-^{2\ell+1}\big|\ell, \ell\big\rangle = 0 \ .$$

The vector $\big|\ell, \ell\big\rangle$ is uniquely defined by

$$\begin{aligned} J_0\big|\ell, \ell\big\rangle &= \ell\big|\ell, \ell\big\rangle \\ J_+\big|\ell, \ell\big\rangle &= 0 \ . \end{aligned}$$

Real, Complex or pseudo-real?

$\mathfrak{su}(2)$ Vectors

1. Vector representation: $\mathcal{D}_1 = \{|1,-1\rangle, |1,0\rangle, |1,1\rangle\}$
   - $\overline{|1,0\rangle} = |1,0\rangle$, *i.e.*, real
   - $\overline{|1,-1\rangle} = -|1,1\rangle$
2. The matrices are after change of basis

$$J_1 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & i\sqrt{2} & 0 \\ -i\sqrt{2} & 0 & i\sqrt{2} \\ 0 & -i\sqrt{2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Purely imaginary matrices

   Real representation: a vector has three degrees of freedom

Real, Complex or pseudo-real?

$\mathfrak{su}(2)$ Spinors

1. Spinor representation: $\mathcal{D}_{\frac{1}{2}} = \{\left|\frac{1}{2},-\frac{1}{2}\right\rangle, \left|\frac{1}{2},\frac{1}{2}\right\rangle\}$
   - $\left|\frac{1}{2},-\frac{1}{2}\right\rangle, \left|\frac{1}{2},\frac{1}{2}\right\rangle$ complex
   - $\overline{\left|\frac{1}{2},-\frac{1}{2}\right\rangle} \neq \left|\frac{1}{2},\frac{1}{2}\right\rangle$
2. Matrices acting on spinor Pauli matrices
3. For $\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we have for the Pauli matrices

$$-\bar{\sigma}_i = \epsilon^{-1}\sigma_i\epsilon \ .$$

4. The representation is pseudo-real

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \Rightarrow \bar{\psi} = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} \sim \epsilon\psi = \begin{pmatrix} \psi^2 \\ -\psi^1 \end{pmatrix}$$

5. We thus have two degrees of freedom.

Solve the second puzzle ☺

1 Some preliminaries

2 General principles
- Basic definitions
- Lie algebras
- Representations of Lie algebras

3 Examples with $\mathfrak{su}(3)$

Representations of Lie algebras

Any Lie algebra

Extends to any Lie algebras

1. Any Lie algebra has
   - a simultaneously commuting generators *i.e.*, of type J_0
     their eigenvalues characterise the representation
   - b creation operators, *i.e.*, of type J_+
     all states obtained from a vacuum
   - c annihilation operators, *i.e.*, of type J_-
     annihilated the vacuum
   - d Enough to consider an equal number of generator each type : (h_i, E_i^+, E_i^-)

$$[h_i, E_i^\pm] = \pm 2 E_i^\pm \ , \quad [E_i^+, E_i^-] = h_i$$

   Satisfying an $\mathfrak{sl}(2,\mathbb{C})$ algebra : so all is known !

2. For a compact Lie algebra all representations
   1. unitary are finite dimensional
   2. can be obtained in a way similar to $\mathfrak{su}(2)$

$\longrightarrow$ H. Weyl

Differential realisation of $\mathfrak{su}(3)$ ▸ Gell-Mann

Fundamental and anti-fundamental representations

- Fundamental representation : $1/2\lambda_a \to z^1, z^2, z^3 \in \mathbb{C}^3$.
- Anti-fundamental representation : $-1/2\bar{\lambda}_a \to \bar{z}_1, \bar{z}_2, \bar{z}_3$.
- From Gell-Mann matrices we deduce

$$\lambda_3 \to h_1 = (z^1\partial_1 - z^2\partial_2) - (\bar{z}_1\bar{\partial}^1 - \bar{z}_2\bar{\partial}^2)$$

$$-\frac{1}{4}\lambda_3 + \frac{\sqrt{3}}{4}\lambda_8 \to h_2 = (z^2\partial_2 - z^3\partial_3) - (\bar{z}_2\bar{\partial}^2 - \bar{z}_3\bar{\partial}^3)$$

$$\frac{1}{2}(\lambda_1 + i\lambda_2) \to E_1^+ = z^1\partial_2 - \bar{z}_2\bar{\partial}^1 \quad \frac{1}{2}(\lambda_1 - i\lambda_2) \to E_1^- = z^2\partial_1 - \bar{z}_1\bar{\partial}^2$$

$$\frac{1}{2}(\lambda_6 + i\lambda_7) \to E_2^+ = z^2\partial_3 - \bar{z}_3\bar{\partial}^2 \quad \frac{1}{2}(\lambda_6 - i\lambda_7) \to E_2^- = z^3\partial_2 - \bar{z}_2\bar{\partial}^3$$

$$\frac{1}{2}(\lambda_4 + i\lambda_5) \to E_3^+ = z^1\partial_3 - \bar{z}_3\bar{\partial}^1 \quad \frac{1}{2}(\lambda_4 - i\lambda_5) \to E_3^- = z^3\partial_1 - \bar{z}_1\bar{\partial}^3$$

Satisfying

$$i = 1, 2, \quad \left[h_i, E_i^{\pm}\right] = \pm 2E_i^{\pm}\ , \quad \left[E_i^+, E_i^-\right] = h_i\ ,$$
$$\left[E_1^+, E_2^+\right] = E_3^+\ , \quad \left[E_1^-, E_2^-\right] = -E_3^-\ .$$

- All representations are constructed from $(h_i, E_i^+, E_i^-), i = 1, 2$.

Polynomial realisation of $\mathfrak{su}(3)$ representations

Polynomial representations

- The vacuum of the representation $\mathcal{D}_{m_1,m_2}$ is
  - annihilated by annihilation operators

$$\left.\begin{array}{l} E_1^+\Phi_{m_1,m_2} = 0 \\ E_2^+\Phi_{m_1,m_2} = 0 \end{array}\right\} \Longrightarrow \Phi_{m_1,m_2}(z,\bar{z}) = \Phi_{m_1,m_2}(z^1,\bar{z}_3)$$

  - Specified by the eigenvalues of h_1, h_2

$$\left.\begin{array}{l} h_1\Phi_{m_1,m_2} = m_1\Phi_{m_1,m_2} \\ h_2\Phi_{m_1,m_2} = m_2\Phi_{m_1,m_2} \end{array}\right\} \Longrightarrow \Phi_{m_1,m_2}(z,\bar{z}) = (z^1)^{m_1}(\bar{z}_3)^{m_2}$$

- Action of the annihilation operators: representation.
- Scalar product

$$(f,g) = \frac{i}{8\pi^3}\int \mathrm{d}^3z\mathrm{d}^3\bar{z}\ \bar{f}(z,\bar{z})g(z,\bar{z})e^{-|z^1|^2-|z^2|^2-|z^3|^2}\ .$$

The representation is unitary if $m_1, m_2 \in \mathbb{N}$.

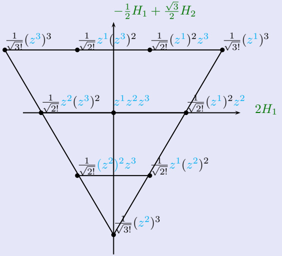
The ten-dimensional representation

Polynomial representations

- The representations $\mathcal{D}_{m,0}$ vacuum: $\Phi_{m,0} = (z^1)^m$

$$\mathcal{D}_{m,0} = \Big\{\psi_{a_1,a_2,a_3}(z) = \frac{1}{\sqrt{a_1!a_2!a_3!}}(z^1)^{a_1}(z^2)^{a_2}(z^3)^{a_3} \ , \quad 0 \le a_1, a_2, a_3 \le m, a_1 + a_2 + a_3 = m\Big\} \ .$$

- Othonormal basis: $(\psi_{a'_1,a'_2,a'_3}, \psi_{a_1,a_2,a_3}) = \delta_{a'_1 a_1}\delta_{a'_2 a_2}\delta_{a'_3 a_3}$.
- $\psi_{a_1,a_2,a_3} = \Phi_{a_1-a_2,a_2-a_3}$, $h_i\Phi_{m_1,m_2} = m_i\Phi_{m_1,m_2}$
- The ten-dimensional representation $\mathcal{D}_{3,0}$.



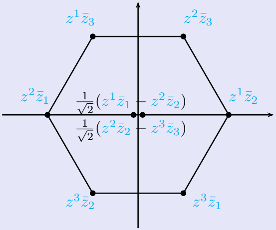
The adjoint representation

Polynomial representations

- Adjoint representation: $\mathcal{D}_{1,1}$ vacuum $\Phi_{1,1} = z^1 \bar{z}_3$

$$\mathcal{D}_{1,1} = \Big\{\Phi_{1,1}(z) = z^1 \bar{z}_3 \ , \ \Phi_{0,-1}(z) = z^1 \bar{z}_2 \ , \ \Phi_{-1,0}(z) = z^2 \bar{z}_3 \ ,$$
$$\Phi_{-1,-1}(z) = z^3 \bar{z}_1 \ , \Phi_{0,1}(z) = z^2 \bar{z}_1 \ , \ \Phi_{1,0}(z) = z^2 \bar{z}_2 \ ,$$
$$\Phi_{0,0}(z) = \frac{1}{2}(|z^1|^2 - |z^2|^2) \ , \ \Phi'_{0,0}(z) = \frac{1}{2}(|z^2|^2 - |z^3|^2)\Big\}$$

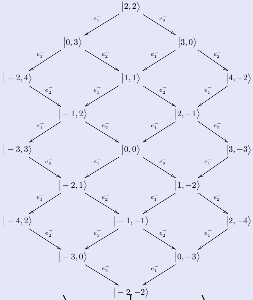
- Orthonormal basis



More complicated case

Polynomial representations

- Representation : $\mathcal{D}_{2,2} = \underset{\sim}{\mathbf{27}}$: vacuum $\Phi_{2,2} = (z^1)^2(\bar{z}_3)^2$



- We have $h_i\big|m_1, m_2\big\rangle = m_i\big|m_1, m_2\big\rangle$
- The algorithm gives precisely the dimension of each space. *E.g*:

$$\big|2,-1\big\rangle = \begin{cases} \Phi_{2,-1} = \frac{1}{2}\Big(z^1 z^2(\bar{z}_2)^2 - (z^1)^2\bar{z}_1\bar{z}_2\Big) \\ \Phi'_{2,-1} = \frac{1}{\sqrt{14}}\Big(-(z^1)^2\bar{z}_1\bar{z}_2 + 2z^1 z^2(\bar{z}_2)^2 - 2z^1 z^3\bar{z}_2\bar{z}_3\Big) \end{cases}$$

Some extensions

Some generalisations

1. There exists analogous differential realisation
   for **the Lie algebras** A_n, B_n, C_n, D_n
   enables to have "polynomial" reps, (except spinors of $SO(n)$)
2. The eigenvalues of h_1, h_2 do not completely characterise states
   needs new operator commuting with h_1, h_2:
   Casimir of the first $\mathfrak{su}(2)$ $J = h_1^2 + 2E_1^- E_1^+ + 2E_1^+ E_1^-$

$$\left.\begin{array}{l}\Phi_{2,-1}\\ \Phi'_{2,-1}\end{array}\right\} \to \left\{\begin{array}{l}\rho_{24,2,-1} = \frac{1}{2} z^1 z^2 (\bar{z}_2)^2 - \frac{1}{2}(z^1)^2 \bar{z}_1 \bar{z}_2 \\ \rho_{\ 8,2,-1} = \frac{1}{6}(z^1)^2 \bar{z}_1 \bar{z}_2 - \frac{1}{6} z^1 z^2 (\bar{z}_2)^2 + \frac{2}{3} z^1 z^3 \bar{z}_2 \bar{z}_3\end{array}\right.$$

   Missing label problem: extends to any Lie algebra
3. Clebsch-Gordan coefficients: tensor product of reps.

$$\begin{array}{ccc} \mathcal{D} \otimes \mathcal{D}' & = & \oplus_k \mathcal{D}_k \\ \uparrow \quad \uparrow & & \uparrow \\ \begin{pmatrix} z \\ \bar{z}\end{pmatrix}\begin{pmatrix} w \\ \bar{w}\end{pmatrix} & & \begin{pmatrix} z, w \\ \bar{z}, \bar{w}\end{pmatrix}\end{array}$$

   doubling the variables leads to Clebsch-Gordan coefficients
   Extends to **the Lie algebras** A_n, B_n, C_n, D_n

Thank you for your attention !

$\mathfrak{su}(2)$

$$\left[J_i, J_j\right] = iJ_k \ , i, j, k, \ \text{perm.}$$

$$J_\pm = J_1 \pm iJ_2$$

$$\begin{cases} \left[J_0, J_\pm\right] = \pm J_\pm \\ \left[J_+, J_-\right] = 2J_0 \end{cases}$$

$\mathfrak{sl}(2, \mathbb{R})$

$$\begin{aligned} \left[K_0, K_1\right] &= iK_2 \\ \left[K_1, K_2\right] &= -iK_0 \\ \left[K_2, K_0\right] &= iK_1 \end{aligned}$$

$$K_\pm = K_1 \pm iK_1$$

$$\begin{cases} \left[K_0, K_\pm\right] = \pm K_\pm \\ \left[K_+, K_-\right] = -2K_0 \end{cases}$$