Alesha Zamolodchikov's meeting Saclay'09

From Teichmüller spaces to Yangian algebras

Leonid Chekhov and Marta Mazzocco

- Combinatorial description of Teichmüller spaces $\mathcal{T}_{g,s,n}$ of Riemann surfaces $\Sigma_{g,s,n}$ of genus g with s holes and n orbifold points.

- geodesic algebras A_n ($\Sigma_{0,1,n}$) and D_n ($\Sigma_{0,2,n}$); braid-group action.

- associated Fuchsian systems $\rightarrow$ (...possibly next time...)

- geodesic algebras $\mathfrak{D}_n$ as semiclassical limit of twisted Yangian algebras

- Finite-dimensional reductions: p-level reduction for $\mathfrak{D}_n$ and the representation for D_n.

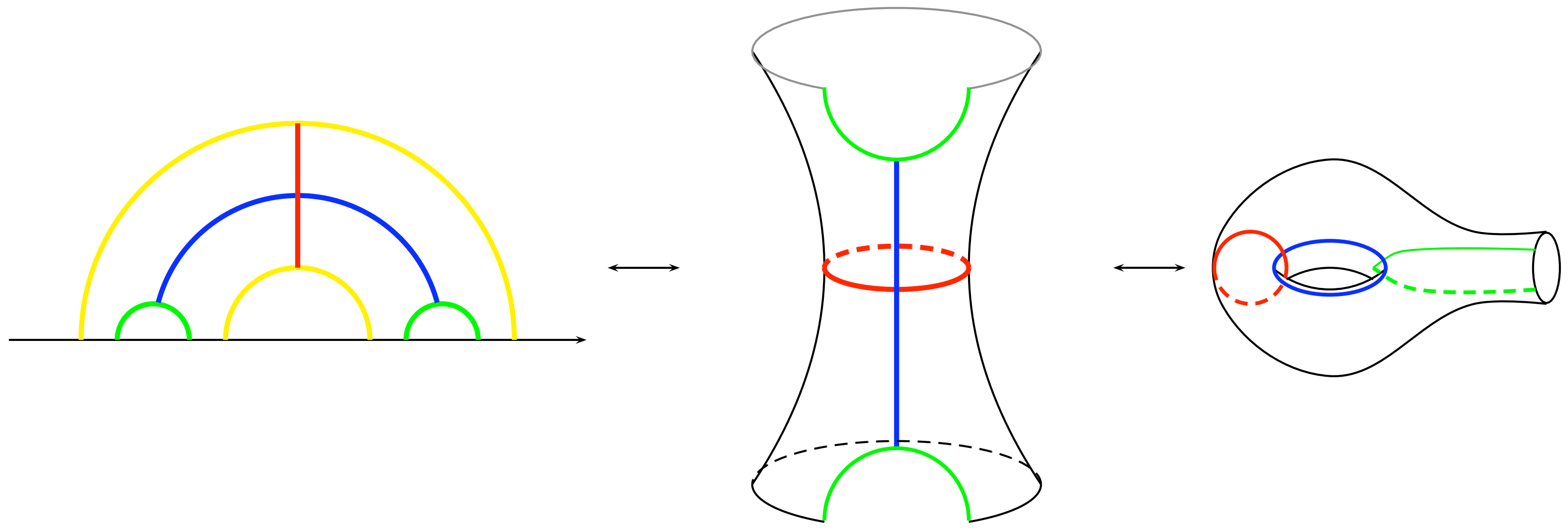
Combinatorial description of Teichmüller spaces of Riemann surfaces $\Sigma_{g,s,n}$ of genus g, with $s > 0$ holes and $n \geq 0$ orbifold points of order 2.

Poincaré uniformization: $\Sigma_{g,s,n} = \mathbb{H}/\Delta_{g,s,n}$,

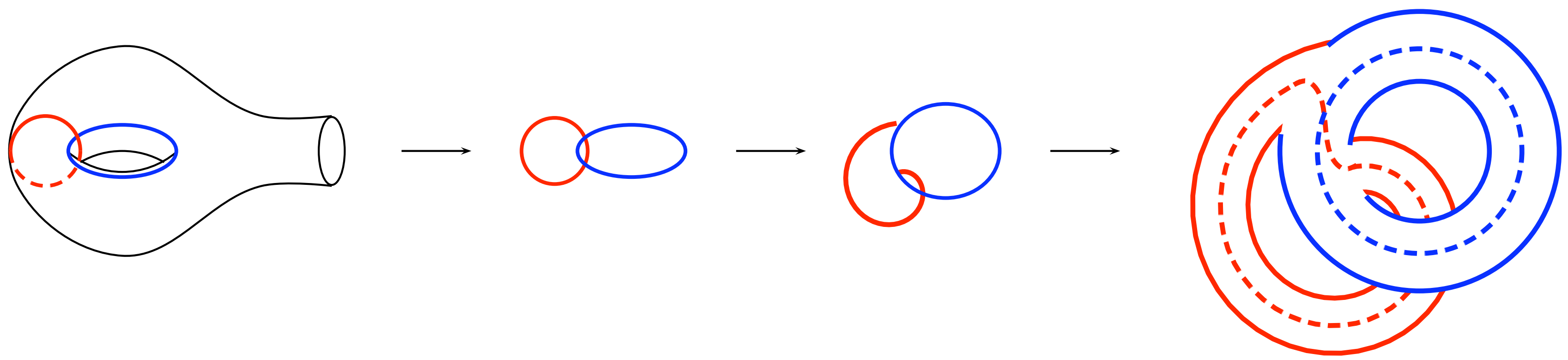
$\Delta_{g,s,n}$ – a *Fuchsian group*—discrete finitely generated subgroup of $PSL(2,\mathbb{R})$, its generators $\gamma_1, \ldots, \gamma_{2g+s-1} \in \mathbb{P}SL(2,\mathbb{R})$ are *hyperbolic elements* and the remaining n generators F_i are *elliptic elements* of rotations through the angle π.

- fat graph technique of R.Penner for punctured RS'89. Introducing coordinates in $\mathcal{T}_{g,s}$; generalized to surfaces with holes (V.V.Fock)'93 with the coordinates in the decorated Teichmüller spaces $\mathcal{T}^H_{g,s} = \mathcal{T}_{g,s} \otimes \mathbb{R}^s$
- quantization of coordinates of $\mathcal{T}^H_{g,s}$ (L.Ch., V.V.Fock)'97.
- Poisson and quantum algebras of geodesic functions [L.Ch. Fock]'99
- quantum Thurston theory [L.Ch., R.Penner]'04

Hyperbolic elements in $\mathbb{P}SL(2,\mathbb{R})$ are in one-to-one correspondence with **closed geodesics** on the Riemann surface and with **closed paths** in the fat graph.



Origin of graphs:



• We associate a **fat-graph** to the topology of the Riemann surface:

1. All the inner vertices are trivalent

2. The vertices terminating at the orbifold points are one-valent.

3. Each face must contain exactly one hole.

4. We associate a real number Z_α to every nonoriented edge. These numbers are the coordinates of the decorated Teichmüller space $\mathcal{T}^H_{g,s,n} = \mathcal{T}_{g,s,n} \otimes \mathbb{R}^s$.

Define R/L matrices for right/left turns, F for reflection at an orbifold point, X_{Z_α} for passage through the αth edge:

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_{z_\alpha} := \begin{pmatrix} 0 & -\exp\left(\frac{z_\alpha}{2}\right) \\ \exp\left(-\frac{z_\alpha}{2}\right) & 0 \end{pmatrix}.$$

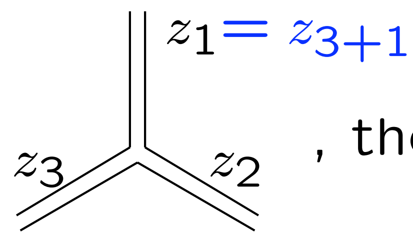
We obtain the matrix product representation $\gamma_I = RX_{z_{ip}} \dots X_{z_{i_k}} F X_{z_{i_k}} \dots LX_{z_{i_1}}$ for Fuchsian group elements following its path in the graph.

• The main algebraic object is the geodesic function G_{γ_I}:

$$G_{\gamma_I} := tr\,\gamma_I = tr(RX_{z_{ip}} \dots X_{z_{i_k}} F X_{z_{i_k}} \dots LX_{z_{i_1}}) = e^{\ell_{\gamma_I}/2} + e^{-\ell_{\gamma_I}/2}$$

for a hyperbolic element ($\left|\ell_{\gamma_I}\right|$ is the length of the corresponding closed geodesic), and $tr\,F_i = 0$.

Poisson bracket

Label all edges entering one vertex clockwise 

, then $\{z_i, z_{i+1}\} = 1$.

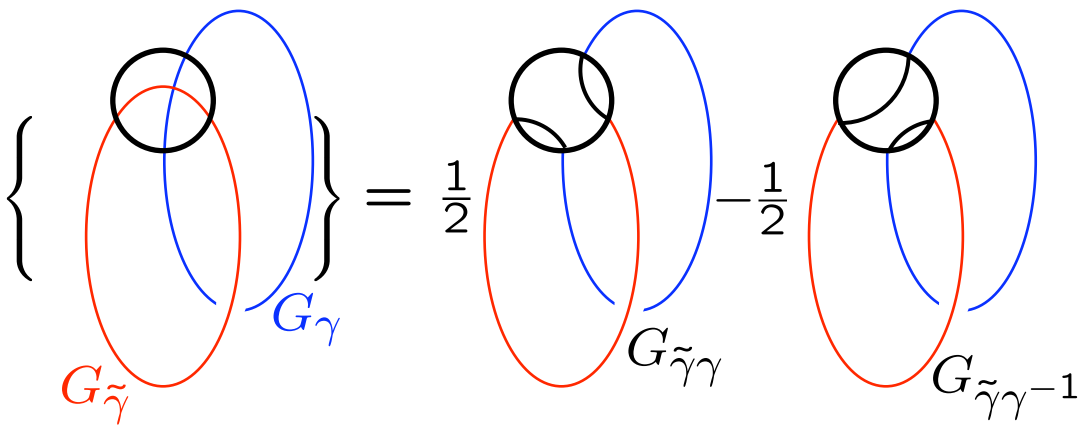
These brackets induce the *Goldman bracket* between geodesic functions [L.Ch., V.Fock] (B.Goldman had obtained this bracket using the Chern–Simons action for 2+1 gravity).

If two closed geodesics do not intersect, their geodesic functions Poisson commute; hence

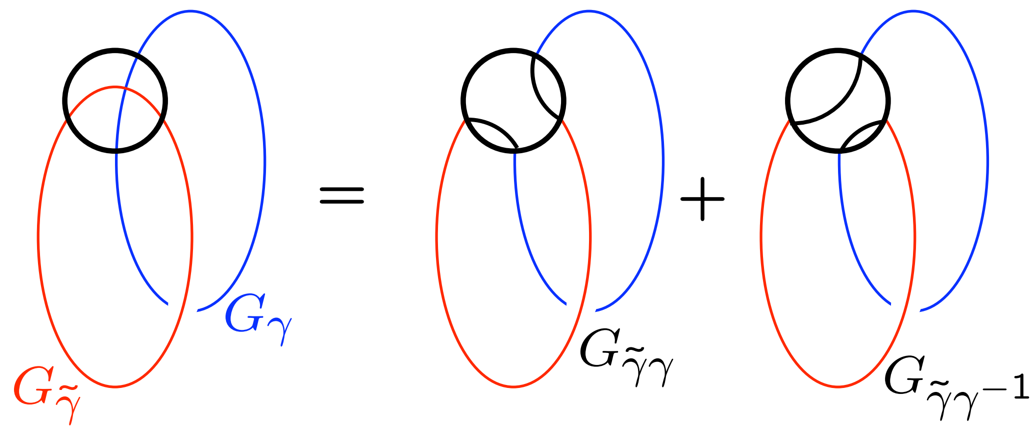
The lengths of the geodesics going around the holes are **central elements**

Poisson algebra of G_γ is usually infinite (and has an exponential growth). Interesting particular cases are those when we can **close** this algebra on the level of finitely many geodesic functions, or may introduce a "regular" structure on an infinite set.

Poisson relation (for a single intersection)



Classical skein relation (for a single intersection)



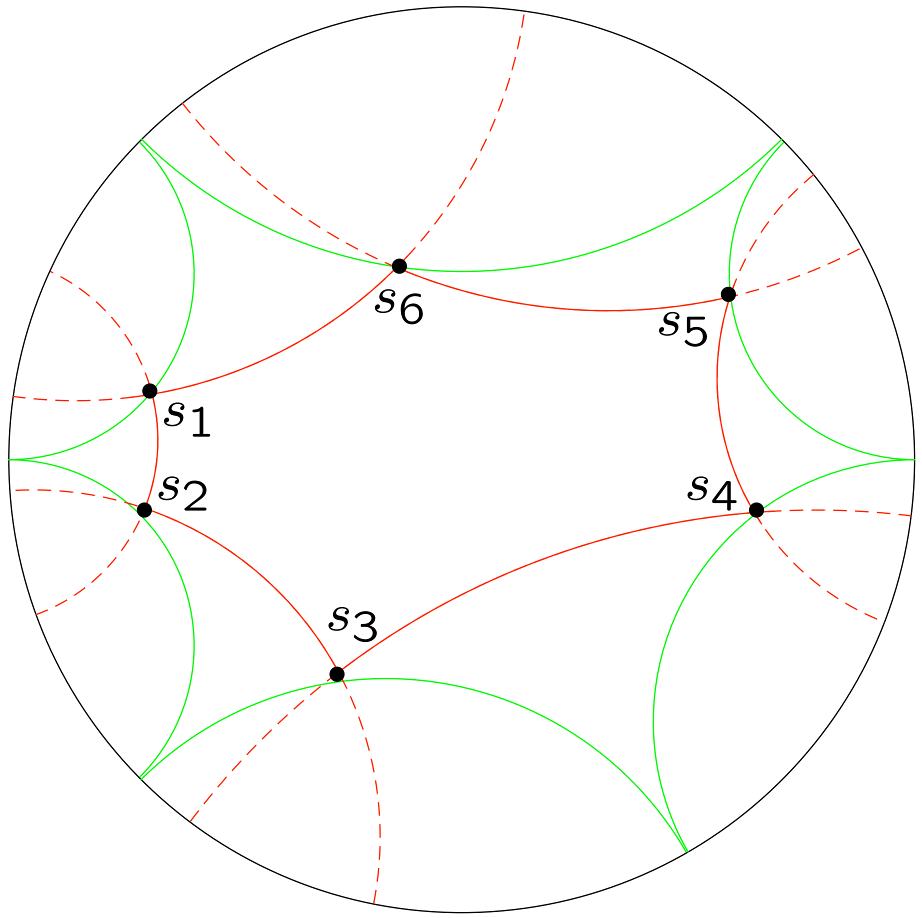
Using these two relations, we construct all the Poisson algebras of geodesic functions: $\{G_{\tilde{\gamma}}, G_\gamma\} = \frac{1}{2}G_{\tilde{\gamma}\gamma} - \frac{1}{2}G_{\tilde{\gamma}\gamma^{-1}} = \frac{1}{2}G_{\tilde{\gamma}}G_\gamma - G_{\tilde{\gamma}\gamma^{-1}}$.

The group $\mathfrak{G}_n = \Delta_{0,1,n}$

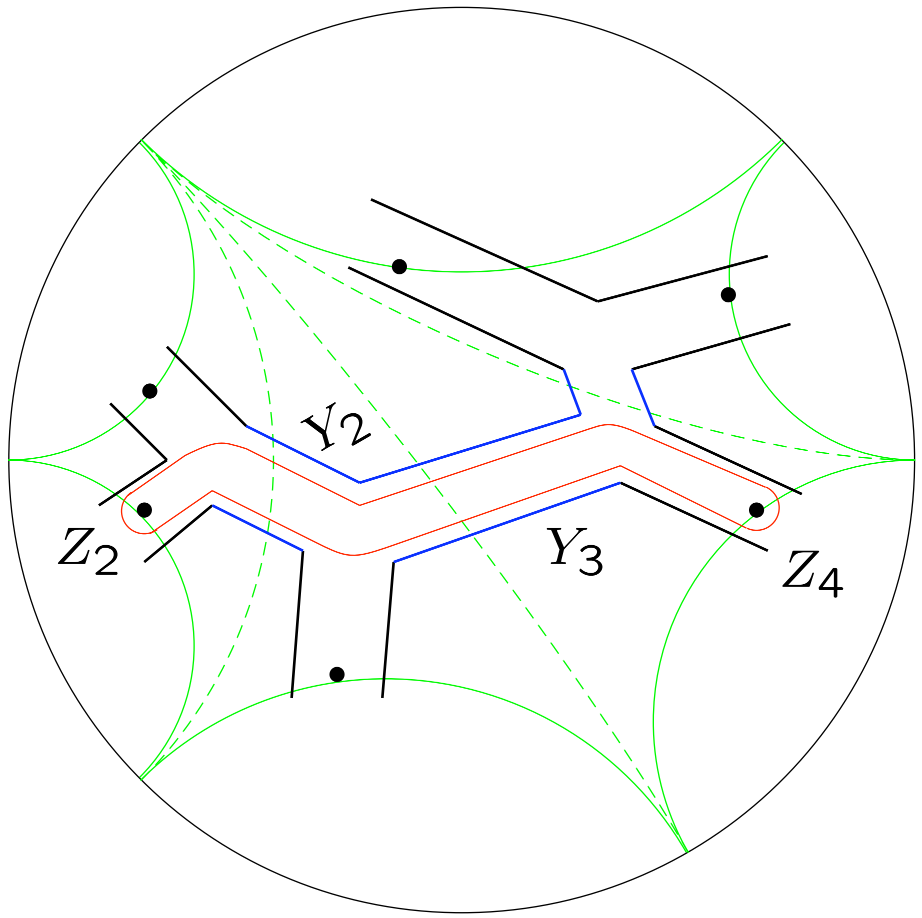
The Poincaré disc (constant curvature -1) with n marked points s_i, $i = 1, \ldots, n$ in the interior. At each point s_i, we introduce the element F_i of the rotation through π; each $F_i = U_i F U_i^{-1}$ is a conjugate of the matrix $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The group $\mathfrak{G}_n$ is generated by all the F_i. The element $\gamma_{ij} = F_i F_j$ is always a hyperbolic element whose *invariant axis* is a unique geodesic that passes through the points s_i and s_j with the *length* being the double geodesic distance between s_i and s_j (red geodesic lines in the figure).

$$G_{ij} = tr\,\gamma_{ij}.$$

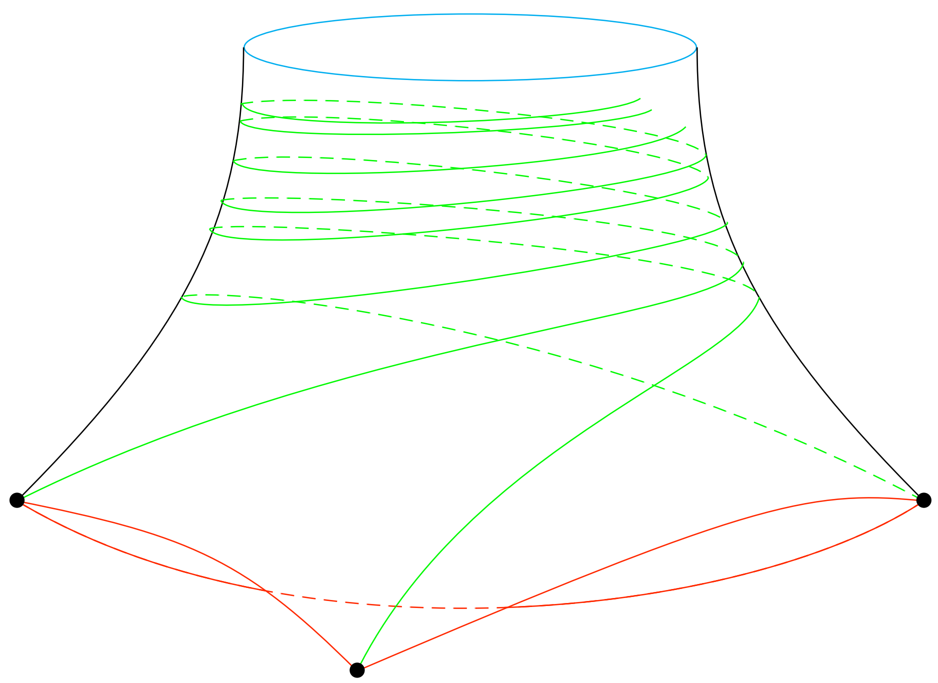


The Poincaré disc with $n = 6$ orbifold points s_i. The group generated by F_i is hyperbolic (modulo exactly the conjugates of the elements F_i) iff there exists a pattern of green geodesic lines, each passing through exactly one point s_i, that are pairwise parallel at infinity, as shown. Red geodesic lines are the invariant axes of elements $\gamma_{i,i+1} = F_i F_{i+1}$, the part of an axis that lies in the fundamental domain is drawn as the solid line.



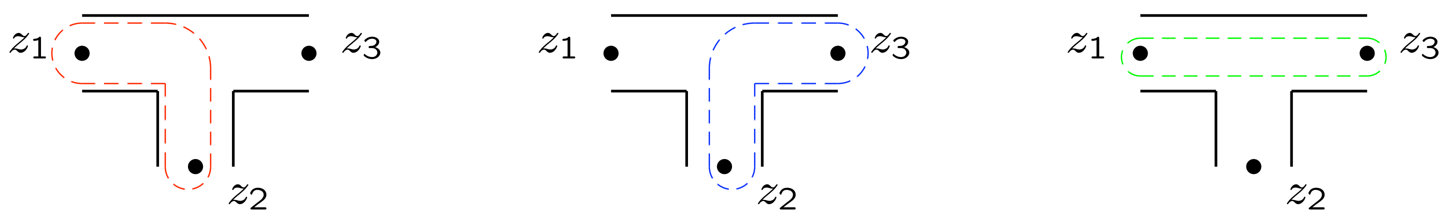
The associated fat graph dual to the ideal triangle decomposition of the fundamental domain. Real numbers Z_i, $i = 1, \ldots, 6$ and Y_2, Y_3, and Y_4 associated to all the edges; we indicate those relevant for constructing the geodesic function $G_{2,4}$.

$$G_{2,4} = tr\, LX_{Y_2}RX_{Y_3}LX_{Z_4}FX_{Z_4}RX_{Y_3}LX_{Y_2}RX_{Z_2}FX_{Z_2}$$



The orbifold Riemann surface $\Sigma_{0,1,3}$. Lines decomposing into ideal triangles (green) start at the obrifold points and spiral asymptotically to the geodesic boundary of the hole whose perimeter is $\ell_P = |Z_1 + Z_2 + Z_3|$,

Example: Surface with one hole and three orbifold points (A_3)

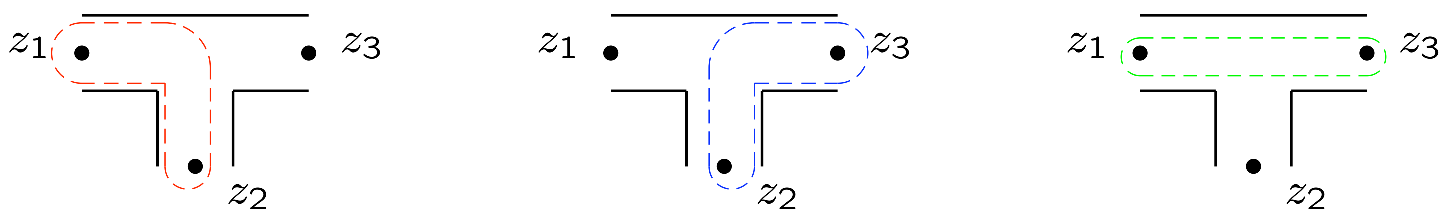


$$\mathrm{Tr}(RX_{z_1}FX_{z_1}LX_{z_2}FX_{z_2}) = e^{z_1+z_2} + e^{-z_1-z_2} + e^{-z_1+z_2}$$
$$\mathrm{Tr}(RX_{z_2}FX_{z_2}LX_{z_3}FX_{z_3}) = e^{z_2+z_3} + e^{-z_2-z_3} + e^{-z_2+z_3}$$
$$\mathrm{Tr}(RX_{z_3}FX_{z_3}LX_{z_1}FX_{z_1}) = e^{z_3+z_1} + e^{-z_3-z_1} + e^{-z_3+z_1}$$

Central element:

$2-e^{2z_1+2z_2+2z_3}-e^{-2z_1-2z_2-2z_3}$

Example: Surface with one hole and three orbifold points (A_3)



$$G_{12} = \mathrm{Tr}(RX_{z_1}FX_{z_1}LX_{z_2}FX_{z_2}) = e^{z_1+z_2} + e^{-z_1-z_2} + e^{-z_1+z_2}$$
$$G_{23} = \mathrm{Tr}(RX_{z_2}FX_{z_2}LX_{z_3}FX_{z_3}) = e^{z_2+z_3} + e^{-z_2-z_3} + e^{-z_2+z_3}$$
$$G_{13} = \mathrm{Tr}(RX_{z_3}FX_{z_3}LX_{z_1}FX_{z_1}) = e^{z_3+z_1} + e^{-z_3-z_1} + e^{-z_3+z_1}$$

Central element:

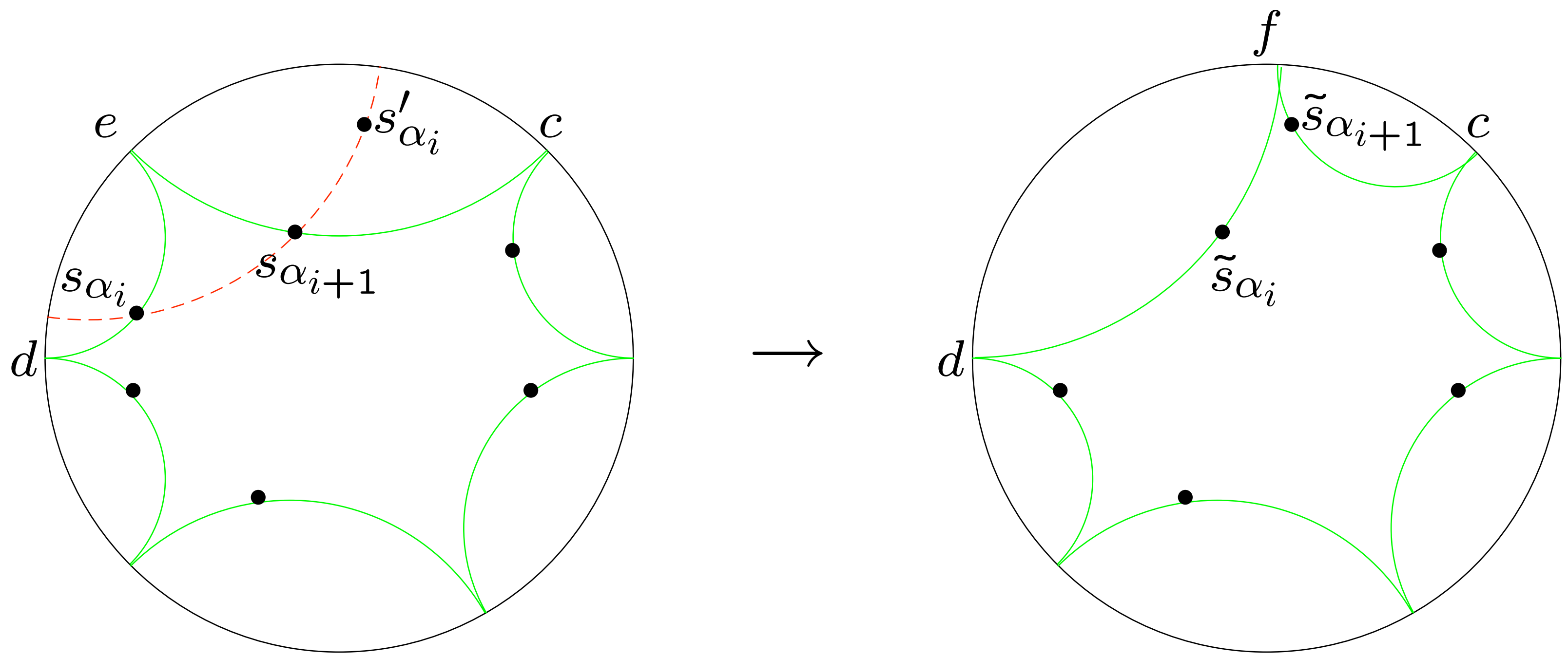
$$2-e^{2z_1+2z_2+2z_3}-e^{-2z_1-2z_2-2z_3} = G_{12}^2 + G_{13}^2 + G_{23}^2 - G_{12}G_{13}G_{23},$$ the Markov element $\mathcal{M}$

Goldman bracket=semiclassical Nelson–Regge (NR) brackets:

$$\{G_{12}, G_{13}\} = 2G_{23} - G_{12}G_{13}, \quad \{G_{12}, G_{23}\} = G_{12}G_{23} - 2G_{13} \quad \{G_{13}, G_{23}\} = 2G_{12} - G_{13}G_{23}.$$

Action of the braid group for A_n

The braid-group transformation is interchanging of order of orbifold points by rotating the ith point about the $i+1$th point:



Globally: natural braid group action: $B_n = \{\beta_{1,2}, \ldots, \beta_{n-1,n}\}$.

subject to the standard braid-group relations: $\beta_{i-1,i}\beta_{i,i+1}\beta_{i-1,i}(\mathcal{A}) = \beta_{i,i+1}\beta_{i-1,i}\beta_{i,i+1}(\mathcal{A})$.

Braid-group invariants = modular invariants = Poisson invariants

For instance, for $n=3$ we have

$$\beta_{1,2}(G_{12},G_{13},G_{23}) = (G_{12},G_{23},G_{13}-G_{12}G_{23}) \qquad \beta_{2,3}(G_{12},G_{13},G_{23}) = (G_{13},G_{12}-G_{13}G_{23},G_{23})$$

We can present this action in the **matrix form**:

$$\beta_{i,i+1}(\mathcal{A}) = B_{i,i+1}(G_{i,i+1})\mathcal{A}\left[B_{i,i+1}(G_{i,i+1})\right]^T.$$

where (in the case $n=3$)

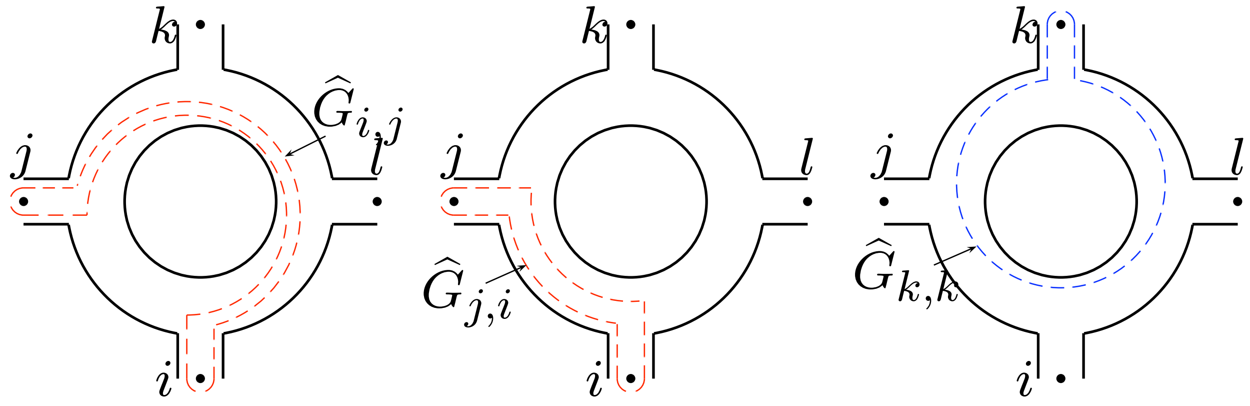
$$\mathcal{A} = \begin{pmatrix} 1 & G_{12} & G_{13} \\ 0 & 1 & G_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{1,2}(G) = \begin{pmatrix} G_{12} & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{2,3}(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & G_{23} & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Meanwhile $\beta_{i,i+1}(\mathcal{A}^T) = B_{i,i+1}(G)\mathcal{A}^T\left[B_{i,i+1}(G)\right]^T$, therefore, **any** linear combination $\mathcal{A}+\lambda^{-1}\mathcal{A}^T$ transforms in the same way, so the generating function for braid-group invariants is

$$\det\left(\mathcal{A}+\lambda^{-1}\mathcal{A}^T\right);$$

For $n=3$, the invariant is again the Markov element $\det\left(\mathcal{A}+\mathcal{A}^T\right) = 8-2\mathcal{M}$.

<u>Surface with two holes and n orbifold points (D_n)</u>



D_n algebra has therefore n^2 generators $\widehat{G}_{i,j}$, $i,j=1,\dots,n$.

Braid group $\langle\beta_{1,2},\dots,\beta_{n-1,n},\beta_{n,1}\rangle$:

$$\beta_{i,i+1}\widehat{G}_{k,l}=\widetilde{\widehat{G}}_{k,l},: \begin{cases} \widetilde{\widehat{G}}_{i+1,k}=\widehat{G}_{i,k} & k\neq i,i+1, \\ \widetilde{\widehat{G}}_{i,k}=\widehat{G}_{i,k}\widehat{G}_{i,i+1}-\widehat{G}_{i+1,k} & k\neq i,i+1, \\ \widetilde{\widehat{G}}_{k,i+1}=\widehat{G}_{k,i} & k\neq i,i+1, \\ \widetilde{\widehat{G}}_{k,i}=\widehat{G}_{k,i}\widehat{G}_{i,i+1}-\widehat{G}_{k,i+1} & k\neq i,i+1, \\ \widetilde{\widehat{G}}_{i,i+1}=\widehat{G}_{i,i+1} & \\ \widetilde{\widehat{G}}_{i+1,i+1}=\widehat{G}_{i,i} & \\ \widetilde{\widehat{G}}_{i,i}=\widehat{G}_{i,i}\widehat{G}_{i,i+1}-\widehat{G}_{i+1,i+1} & \\ \widetilde{\widehat{G}}_{i+1,i}=\widehat{G}_{i+1,i}+\widehat{G}_{i,i+1}\widehat{G}^2_{i,i}-2\widehat{G}_{i,i}\widehat{G}_{i+1,i+1} & \end{cases} \qquad i=1,\dots,n-1,$$

$$
\beta_{n,1}\widehat{G}_{k,l} = \widetilde{\widehat{G}}_{k,l}: \quad \begin{cases}
\widetilde{\widehat{G}}_{1,k} = \widehat{G}_{n,k} & k \neq n, 1, \\
\widetilde{\widehat{G}}_{n,k} = \widehat{G}_{n,k}\widehat{G}_{n,1} - \widehat{G}_{1,k} & k \neq n, 1, \\
\widetilde{\widehat{G}}_{k,1} = \widehat{G}_{k,n} & k \neq n, 1, \\
\widetilde{\widehat{G}}_{k,n} = \widehat{G}_{k,n}\widehat{G}_{n,1} - \widehat{G}_{k,1} & k \neq n, 1, \\
\widetilde{\widehat{G}}_{n,1} = \widehat{G}_{n,1} & \\
\widetilde{\widehat{G}}_{1,1} = \widehat{G}_{n,n} & \\
\widetilde{\widehat{G}}_{n,n} = \widehat{G}_{n,n}\widehat{G}_{n,1} - \widehat{G}_{1,1} & \\
\widetilde{\widehat{G}}_{1,n} = \widehat{G}_{1,n} + \widehat{G}_{n,1}\widehat{G}^2_{n,n} - 2\widehat{G}_{n,n}\widehat{G}_{1,1} &
\end{cases} .
$$

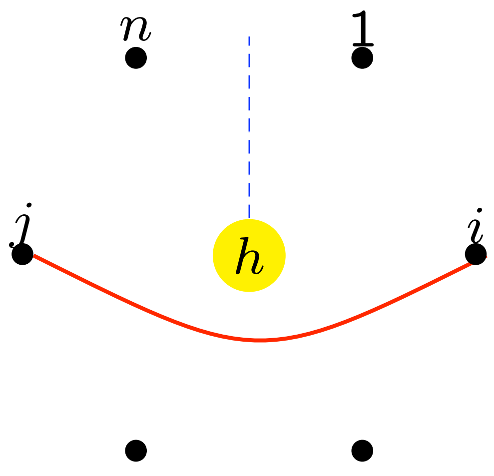
The D_n Poisson algebra (with terms up to the third order in $\widehat{G}_{i,j}$ in the r.h.s.) is an abstract algebra (satisfies the Jacobi relations).

Remark. The quantum version of these relations below was presented in [L.Ch. J.Phys.A'09]

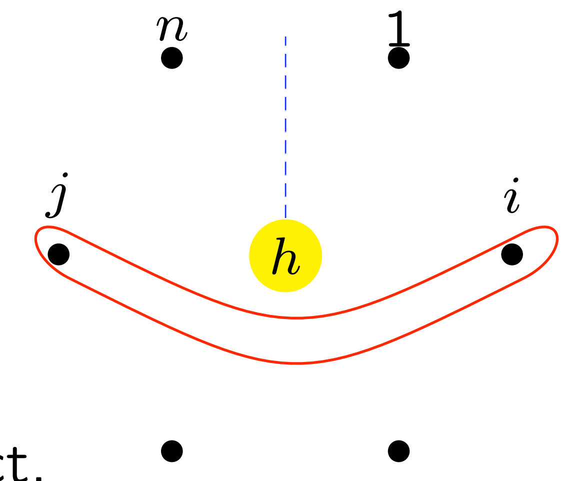
Infinite-dimensional algebras $\mathfrak{D}_n$

We introduce a new hole with the perimeter $|P_h|$ generated by the element M_h, $\mathrm{Tr}M_h = e^{P_h/2} + e^{-P_h/2} := \Pi$, and consider the algebraic elements $G_{ij}^{(k)}$

$G_{i,j}^{(0)}$

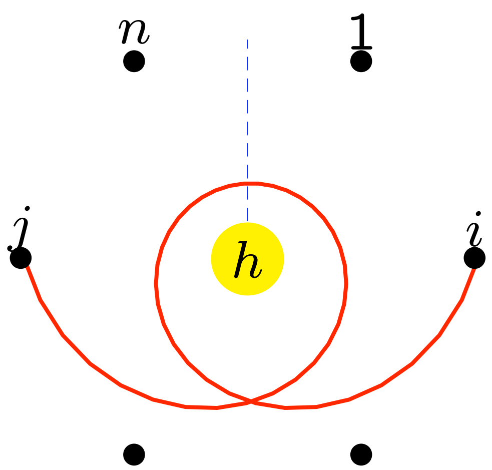


in fact,

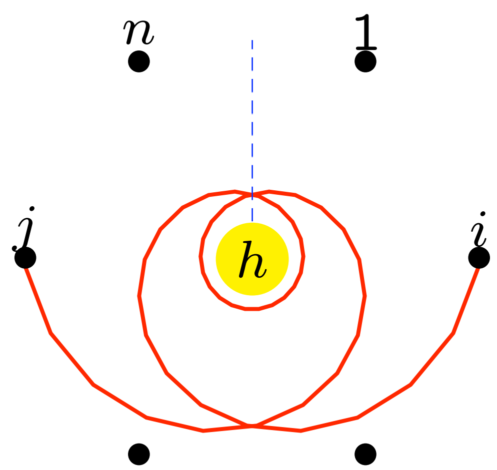


all the lines are double lines (reflected back at orbifold points)

$G_{i,j}^{(1)}$



$G_{i,j}^{(2)}$



We introduce the generating function

$$\mathcal{G}_{i,j}(\lambda) := \mathcal{A}^{(0)}_{i,j} + \sum_{k=1}^{\infty} G^{(k)}_{i,j}\lambda^{-k},$$

where $\mathcal{A}^{(0)}$ is an upper-triangular matrix with the entries $G^{(0)}_{i,j}$ above the diagonal and unities on the diagonal.

For $\mathcal{G}_{i,j}(\lambda)$, using the Goldman brackets and skein relations, we obtain the algebra

$$\begin{aligned}&\left\{\mathcal{G}_{j,i}(\lambda), \mathcal{G}_{p,l}(\mu)\right\} = \\ &\quad \left(\epsilon(j-p) - \frac{\lambda+\mu}{\lambda-\mu}\right)\mathcal{G}_{p,i}(\lambda)\mathcal{G}_{j,l}(\mu) + \left(\epsilon(i-l) + \frac{\lambda+\mu}{\lambda-\mu}\right)\mathcal{G}_{p,i}(\mu)\mathcal{G}_{j,l}(\lambda) + \\ &\quad + \left(\epsilon(i-p) - \frac{1+\lambda\mu}{1-\lambda\mu}\right)\mathcal{G}_{j,p}(\lambda)\mathcal{G}_{i,l}(\mu) + \left(\epsilon(j-l) + \frac{1+\lambda\mu}{1-\lambda\mu}\right)\mathcal{G}_{l,i}(\lambda)\mathcal{G}_{p,j}(\mu)\end{aligned}$$

This is an abstract infinite dimensional Poisson algebra $\mathfrak{D}_n$.

Relation to the reflection equation

Twisted Yangians by Molev, Ragoucy, and Sorba: for trigonometric R-matrix

$$\begin{aligned} R(u,v) &= (u-v)\sum_{i\neq j} E_{ii}\otimes E_{jj} + (q^{-1}u - qv)\sum_{i} E_{ii}\otimes E_{ii} \\ &+ (q^{-1}-q)u\sum_{i>j} E_{ij}\otimes E_{ji} + (q^{-1}-q)v\sum_{i<j} E_{ij}\otimes E_{ji}, \qquad q = e^{-i\pi\hbar}, \end{aligned}$$

acting in the tensor product of spaces 1 and 2 and for the (quantum) quantities $s_{i,j}(u) = \sum_{k=0}^{\infty} s_{i,j}^{(k)} u^{-k}$ such that $s_{ii}^{(0)} = 1$, $i = 1,\dots,n$ and $s_{ij}^{(0)} = 0$ for $1 \le j < i \le n$, we have the (matrix) reflection equation

$$R(u,v)S_1(u)R^t(u^{-1},v)S_2(v) = S_2(v)R^t(u^{-1},v)S_1(u)R(u,v)$$

(the symbol t denotes the transposition only in the first space).

Lemma Poisson algebra $\mathfrak{D}_n$ of $G_{i,j}^{(k)}$ is the semiclassical limit $\hbar \to 0$ of the quantum twisted Yangian algebra for $S_{i,j}(u) = \mathcal{G}_{i,j}(u)$.

Braid-group action for $\mathfrak{D}_n$.

For $\mathcal{G}(\lambda)$, the matrix representation for the braid-group action reads

$$\beta_{i,i+1}\mathcal{G}(\lambda) = B_{i,i+1}\mathcal{G}(\lambda)\Big(B_{i,i+1}\Big)^T, \qquad \beta_{n,1}\mathcal{G}(\lambda) = B_{n,1}(\lambda)\mathcal{G}(\lambda)\Big(B_{n,1}(\lambda^{-1})\Big)^T,$$

where

$$B_{i,i+1} = \begin{array}{c} \\ \vdots \\ i \\ i+1 \\ \vdots \\ \\ \end{array} \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & G^{(0)}_{i,i+1} & -1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}.$$

and

$$B_{n,1}(\lambda) = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & 1 & 0 \\ -\lambda^{-1} & 0 & \dots & 0 & G^{(1)}_{n,1} \end{pmatrix}.$$

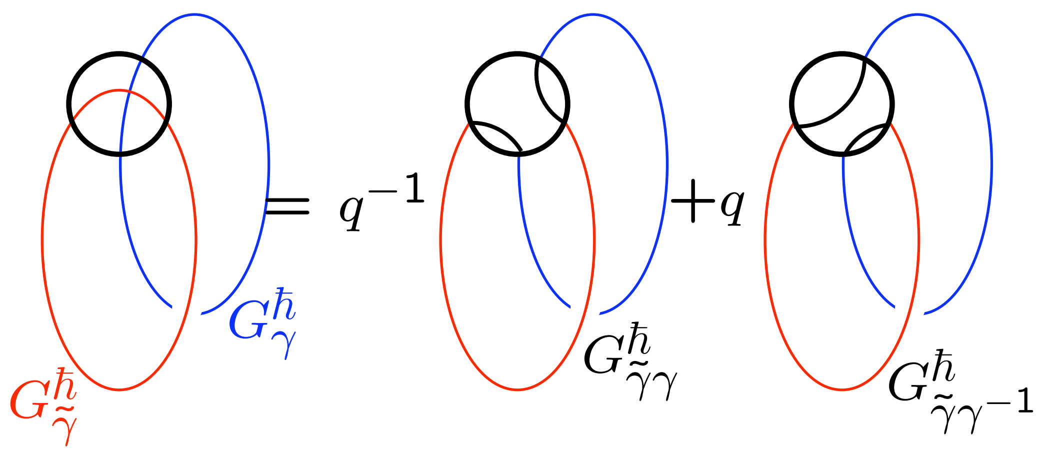
Quantization on the level of Teichmüller space coordinates is the mere correspondence principle:

$$[Z_i^\hbar, Z_j^\hbar] = 2\pi i\hbar\{Z_i, Z_j\}.$$

For the quantum geodesic functions $G_\gamma^\hbar$, we have (${}^\times_\times \bullet {}^\times_\times$ is the quantum ordering):

$$G_\gamma^\hbar \equiv {}^\times_\times tr\, P_{Z_1\dots Z_n}{}^\times_\times \equiv \sum_{j\in J} \exp\left\{\frac{1}{2}\sum_{\alpha\in E(\Gamma_{g,\delta})}\Big(m_j(\gamma,\alpha)Z_\alpha^\hbar + 2\pi i\hbar c_j(\gamma,\alpha)\Big)\right\},$$

all the $G_\gamma^\hbar$ are Hermitian operators, and we have the quantum skein relation:



to be applied simultaneously at all the intersections (the empty loop is $-q-q^{-1}$).

Quantum braid-group action for $\mathfrak{D}_n^\hbar$.

Quantum version $\mathfrak{D}_n^\hbar = Y_q'(\mathfrak{o}_n)$ – the (full) twisted Yangian algebra.

For $\mathcal{G}_{i,j}^\hbar(\lambda) = \mathcal{A}_{i,j}^\hbar + \sum_{k=1}^{\infty} G_{i,j}^{(k)^\hbar} \lambda^{-k}$, where all the $G_{i,j}^{(s)^\hbar}$, $s = 0, 1, \ldots$, are Hermitian operators and $\mathcal{A}_{i,i}^\hbar = q^{-1}$, $q^\dagger = q^{-1}$, the braid-group action is

$$\beta_{i,i+1}\mathcal{G}^\hbar(\lambda) = B_{i,i+1}^\hbar \mathcal{G}^\hbar(\lambda) \Big(B_{i,i+1}^\hbar\Big)^\dagger, \qquad \beta_{n,1}\mathcal{G}^\hbar(\lambda) = B_{n,1}^\hbar(\lambda) \mathcal{G}^\hbar(\lambda) \Big(B_{n,1}^\hbar(\lambda^{-1})\Big)^\dagger,$$

where

$$B_{i,i+1}^\hbar = \begin{array}{c} \\ \vdots \\ i \\ i+1 \\ \vdots \\ \\ \end{array} \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & qG_{i,i+1}^{(0)^\hbar} & -q^2 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}.$$

and

$$B_{n,1}^\hbar(\lambda) = \begin{pmatrix} 0 & 0 & \ldots & 0 & \lambda \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & 1 & 0 \\ -q^2\lambda^{-1} & 0 & \ldots & 0 & qG_{n,1}^{(1)^\hbar} \end{pmatrix}.$$

Finite-dimensional reductions and Casimirs

Central elements of $\mathfrak{D}_n$ are generated by $\det \mathcal{G}(u)$, identically to the twisted Yangian case.

Level-p reduction: $M_h^p = \mathbb{E}$ or $G_{i,j}^{(k+p)} = G_{i,j}^{(k)}$ for any k (the inner hole reduces to a $\mathbb{Z}_p$-orbifold point). Every such reduction is automatically Poissonian due to the Korotkin–Samtleben brackets. We have the algebra $\mathfrak{D}_n^{(p)}$ generated by $G_{i,j}^{(k)} = G_{j,i}^{(p-k)}$ for $k = 0, \dots, p-1$.

Due to the periodicity $\mathcal{G}(\lambda)$ becomes $\frac{1}{\lambda^p - 1}\mathcal{G}_p(\lambda)$, where

$$\mathcal{G}_p(\lambda) := \mathcal{A}^{(0)} + \frac{\mathcal{G}^{(1)}}{\lambda} + \cdots + \frac{\mathcal{G}^{(p-1)}}{\lambda^{p-1}} + \frac{{\mathcal{A}^{(0)}}^T}{\lambda^p},$$

The central elements of $\mathfrak{D}_n^{(p)}$ are generated by $\det \mathcal{G}_p(\lambda)$. The maximum number of independent central elements is $\left[\frac{np}{2}\right]$.

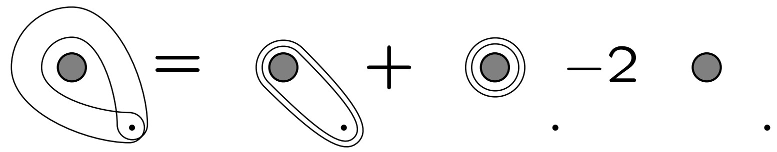
Remark: at $p = 1$, we obtain the A_n algebra generating function $\det(\mathcal{A} + \lambda^{-1}\mathcal{A}^T)$.

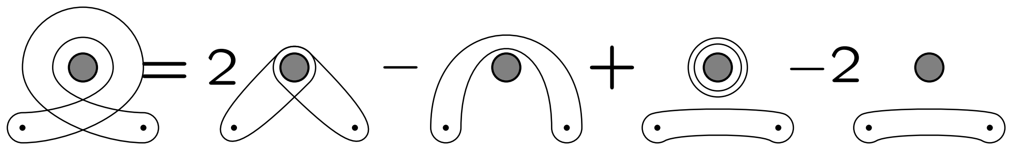
From $\mathfrak{D}_n$ to D_n

We use the skein relations to represent elements $G^{(1)}_{i,j}$ with $i \le j$ (i.e., those with self-intersections):

$$G^{(1)}_{i,i} \to \widehat{G}^2_{i,i} + \Pi^2 - 2, \qquad G^{(1)}_{i,j} \to 2\widehat{G}_{i,i}\widehat{G}_{j,j} - \widehat{G}_{j,i} + (\Pi^2 - 2)\widehat{G}_{i,j}, \quad 1 \le i < j \le n,$$

or, in the graphical form,





(In these relations we used that the empty loop is -2.) Note the appearance of the parameter related to the inner hole perimeter P_h:

$$\Pi := e^{P_h/2} + e^{-P_h/2}.$$

Very useful is to write this reduction in terms of the matrices $\widehat{\mathcal{A}}$, $\widehat{\mathcal{R}}$, and $\widehat{\mathcal{S}}$:

$$\mathcal{G}^{(1)} \to \widehat{\mathcal{R}} + \widehat{\mathcal{S}} + (\Pi^2 - 1)\widehat{\mathcal{A}} - \widehat{\mathcal{A}}^T.$$

Here $\widehat{\mathcal{R}}$ is the $n \times n$ skewsymmetric matrix of entries:

$$(\widehat{\mathcal{R}})_{i,j} := \begin{cases} -\widehat{G}_{j,i} - \widehat{G}_{i,j} + \widehat{G}_{i,i}\widehat{G}_{j,j} & j > i \\ \widehat{G}_{j,i} + \widehat{G}_{i,j} - \widehat{G}_{i,i}\widehat{G}_{j,j} & j < i \\ 0 & j = i \end{cases},$$

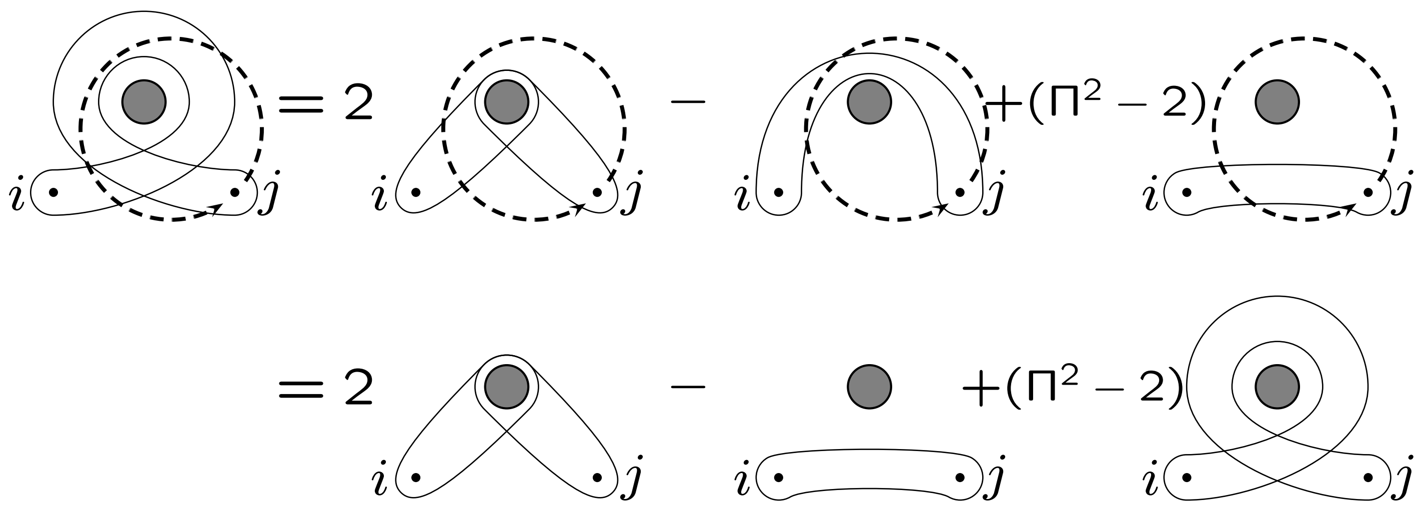
$\widehat{\mathcal{S}}$ is the symmetric matrix of entries:

$$(\widehat{\mathcal{S}})_{i,j} := \widehat{G}_{i,i}\widehat{G}_{j,j} \quad \text{for all} \quad 1 \leq i, j \leq n;$$

and $\widehat{\mathcal{A}}$ is the upper triangular matrix of entries

$$\widehat{\mathcal{A}} = \begin{pmatrix} 1 & \widehat{G}_{1,2} & \widehat{G}_{1,3} & \dots & \widehat{G}_{1,n} \\ 0 & 1 & \widehat{G}_{2,3} & \dots & \widehat{G}_{2,n} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \widehat{G}_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We then continue expressing higher $\mathcal{G}^{(k)}$ fixing the parameter i and moving j counterclockwise as below



Performing the summation over k in the resulted reduction formulas, we obtain

$$
\begin{aligned}
\mathcal{G}(\lambda) \;\; \to \;\; & \frac{\lambda}{(\lambda-1)(e^{-P_h}\lambda-1)(e^{P_h}\lambda-1)} \times \\
& \times \Big[(\lambda-1)\widehat{\mathcal{R}} + (\lambda+1)\widehat{\mathcal{S}} + (\lambda^2-1)\widehat{\mathcal{A}} - (\lambda-\lambda^{-1})\widehat{\mathcal{A}}^T\Big],
\end{aligned}
$$

Remark: This representation is consistent with the p-level reduction provided $e^{pP_h} = 1$ and $\Pi \neq 0$, or $p \geq 3$.

Central elements of D_n algebra

The braid-group representations for D_n and $\mathcal{G}(u)$ coincide. So, the central elements are again generated by $\det \mathcal{G}(\lambda)$.

Proposition. The D_n algebra admits exactly n algebraically independent central elements $c_1, \ldots, c_n$. They are generated by

$$\det\Big[(\lambda-1)\widehat{\mathcal{R}} + (\lambda+1)\widehat{\mathcal{S}} + (\lambda^2-1)\widehat{\mathcal{A}} - (\lambda-\lambda^{-1})\widehat{\mathcal{A}}^T\Big] =$$
$$= (\lambda-1)^{n-1}\Big[\lambda^{n+1} + \sum_{i=1}^{n} \lambda^i c_i + (-1)^{n+1} \sum_{i=1}^{n} \lambda^{1-i} c_i + (-1)^{n+1}\lambda^{-n}\Big].$$

[The proof follows from that $\widehat{\mathcal{S}}$ has rank one and all other summands contain $(\lambda-1)$, so the determinant contains $(\lambda-1)^{n-1}$; the remaining expansion produces n central elements.]

Outlook

Because the algebra $\mathfrak{D}_n$ is the semiclassical limit of the Yangian algebra of Molev, Sorba, and Ragoucy, the full Yangian algebra is therefore the quantum extension of the $\mathfrak{D}_n$ algebra.

2+1 gravity = Chern–Simons action $\int d^2x dt A \wedge dA + \frac{2}{3} A \wedge A \wedge A$—a topological theory.

Observables = lengths of closed geodesics: complexification of $G_{i,j}$. Describing quantum theory of 2+1 gravity.

Quantum version of a general Fuchsian system is the KZ system [Reshetikhin'90]. Quantum version of the Dubrovin system – in progress; the quantum Stokes parameters $G^{\hbar}_{i,j}$ must then be (quantum) monodromies of this system providing possibly a description of quantum Frobenius manifolds.