New results on the integrable structure of CFT.

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1. Previous results. On the space $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$, acts the Hamoltonian

$$H = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left(\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3 \right), \quad \Delta = \frac{1}{2}(q + q^{-1}).$$

We consider $q = e^{\pi i \nu}$.

Introduce the Matsubara space $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbb{C}^2$, and the operator

$$T_{\mathrm{S},\mathbf{M}} = \overset{\curvearrowright}{\prod_{j=-\infty}^{\infty}} T_{j,\mathbf{M}}\,,$$

where

$$T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1), \quad T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowleft}{\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}} L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}})\,.$$

The L-operator is:

$$L_{j,\mathbf{m}}(\zeta) = q^{\frac{1}{2}} \left(q^{\frac{1}{2}\sigma_j^3 \sigma_{\mathbf{m}}^3} \zeta - q^{-\frac{1}{2}\sigma_j^3 \sigma_{\mathbf{m}}^3} \zeta^{-1} + (q - q^{-1}) \left(\sigma_j^+ \sigma_{\mathbf{m}}^- + \sigma_j^- \sigma_{\mathbf{m}}^+ \right) \right)$$

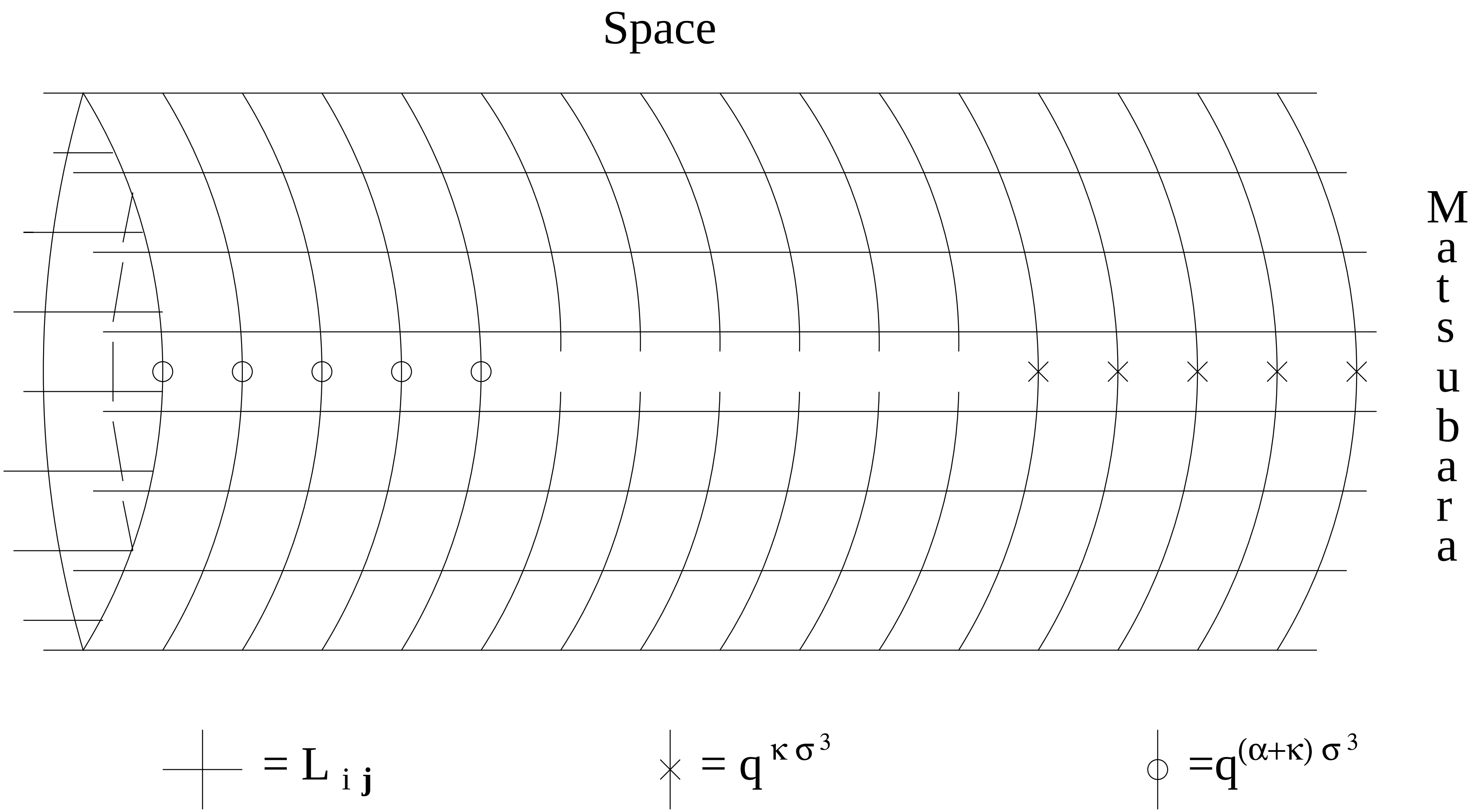
Set $S(k) = \frac{1}{2} \sum_{j=-\infty}^{k} \sigma_k^3$. We consider the quasilocal operators with tail α:

$$q^{2\alpha S(0)} \mathcal{O} \in \mathcal{W}_{\alpha,0} \, .$$

The problem is to compute

$$Z^\kappa \left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\mathrm{Tr}_{\mathrm{S}} \mathrm{Tr}_{\mathbf{M}} \left(T_{\mathrm{S},\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \mathcal{O} \right)}{\mathrm{Tr}_{\mathrm{S}} \mathrm{Tr}_{\mathbf{M}} \left(T_{\mathrm{S},\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \right)} \, .$$

Graphically, the numerator is



Our main achievement. Consider the space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}\ ,$$

By purely algebraic construction we introduced action of creation -annihilation operators on the space $\overline{\mathcal{W}}^{(\alpha)}$:

$$\mathbf{t}^*(\zeta),\ \mathbf{b}^*(\zeta),\ \mathbf{c}^*(\zeta)\,; \qquad \mathbf{b}(\zeta),\ \mathbf{c}(\zeta)$$

such that $\mathcal{W}^{(\alpha)}$ is created form the primary field $q^{2\alpha S(0)}$ by Taylor coefficients of $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ at $\zeta^2 = 1$.
The expectation values are computed in this basis as

$$Z^\kappa\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_q^+)\mathbf{c}^*(\zeta_q^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{2\alpha S(0)}\big)\Big\} =$$

$$= \prod_{i=1}^p \rho(\zeta_i^0)\det\left|\omega(\zeta_i^+,\zeta_j^-)\right|_{i,j=1,\cdots q}\ .$$

The functions $\rho(\zeta)$ and $\omega(\zeta,\xi)$ are defined by Matsubara data only. Consider the transfer-matrices in Matsubara direction $T_{\mathbf{M}}(\zeta,\kappa)$ and $T_{\mathbf{M}}(\zeta,\kappa+\alpha)$, for example,

$$T_{\mathbf{M}}(\zeta,\kappa)=(-1)^{S_{\mathbf{M}}}\mathrm{Tr}_j\bigl(T_{j,\mathbf{M}}(\zeta)q^{\kappa\sigma_j^3}\bigr),$$

Introduce the eigenvectors $|\kappa\rangle$, $|\kappa+\alpha\rangle$ with eigenvalues $T(\zeta,\kappa)$, $T(\zeta,\kappa+\alpha)$ such that $T(1,\kappa)$, $T(1,\kappa+\alpha)$ are of maximal absolute value. The functions $\rho(\zeta)$ and $\omega(\zeta,\xi)$ are constructed from $T(\zeta,\kappa)$, $T(\zeta,\kappa+\alpha)$. ρ is simple:

$$\rho(\zeta|\kappa,\kappa+\alpha)=\frac{T(\zeta,\kappa+\alpha)}{T(\zeta,\kappa)}\,.$$

We define ω in TBA style. Let

$$\mathfrak{a}(\zeta,\kappa)=\left(\frac{1-q^{-1}\zeta}{1-q\zeta}\right)^{\mathbf{n}}\frac{Q(\zeta q,\kappa)}{Q(\zeta q^{-1},\kappa)}\,.$$

It satisfies the Destri-De Vega equations:

$$\log \mathfrak{a}(\zeta,\kappa) = -2i\nu\kappa + \log\left(\frac{d(\zeta)}{a(\zeta)}\right) - \int\limits_C K(\zeta/\xi)\log\left(1+\mathfrak{a}(\xi,\kappa)\right)\frac{d\xi^2}{\xi^2}\,,$$

We define

$$\psi(\zeta,\alpha) = \zeta^\alpha \frac{\zeta^2+1}{2(\zeta^2-1)}\,,\quad K(\zeta,\alpha) = \frac{1}{2\pi i}\left(\psi(\zeta q,\alpha)-\psi(\zeta q^{-1},\alpha)\right),$$

in particular, $K(\zeta) = K(\zeta,0)$.

Introduce the resolvent:

$$R(\zeta,\xi) - \int\limits_C K(\zeta/\eta,\alpha)R(\eta,\xi)dm(\eta) = K(\zeta/\xi,\alpha)\,,$$

$$dm(\eta) = \frac{d\eta^2}{\eta^2\rho(\eta|\kappa,\kappa+\alpha)\left(1+\mathfrak{a}(\eta,\kappa)\right)}\,.$$

Introduce the notations

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1}), \quad \delta^-_\zeta f(z) = f(\zeta q) - \rho(\zeta) f(\zeta)\,,$$

and two kernels

$$f_{\text{left}}(\zeta,\xi) = \delta^-_\zeta \psi(\zeta/\xi,\alpha), \quad f_{\text{right}}(\zeta,\xi) = \delta^-_\xi \psi(\zeta/\xi,\alpha)$$

Now we are ready to define:

$$\omega(\zeta,\xi|\kappa,\alpha) = -\left(f_{\text{left}} \star f_{\text{right}} + f_{\text{left}} \star R \star f_{\text{right}}\right)(\zeta,\xi) + \delta^-_\zeta \delta^-_\xi \Delta^{-1}_\zeta \psi(\zeta/\xi,\alpha)\,,$$

where $\star$ stands for integration with dm. We use

$$\Delta^{-1}_\zeta \psi(\zeta,\alpha) = -\int_0^\infty \frac{1}{2\nu(1+(\zeta/\eta)^{\frac{1}{\nu}})}\psi(\eta,\alpha)\frac{d\eta^2}{2\pi i \eta^2}\,.$$

Vacuum expectation values in infinite volume:

$$Z_\infty\{q^{2\alpha S(0)}\mathcal{O}\} = \frac{\langle \text{vac}|q^{2\alpha S(0)}\mathcal{O}|\text{vac}\rangle}{\langle \text{vac}|q^{2\alpha S(0)}|\text{vac}\rangle}\,.$$

Our operators are such that under Z_∞ effectively $\mathbf{b}^*(\zeta) = 0$, $\mathbf{c}^*(\zeta) = 0$, $\mathbf{t}^*(\zeta) = 2$.

This is an evidence of their similarity with Virasoro algebra in CFT.

Screening operators on the lattice. Let us try to have more parameters considering the trace

$$\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(Y_{\mathbf{M}}^{(-s)}T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\Big)\,,$$

Spin of $\mathcal{O}$ must equal s, so, we take

$$q^{2\alpha S(0)}\mathcal{O}$$
$$=\mathbf{b}^*(\xi_1)\cdots\mathbf{b}^*(\xi_s)\mathbf{b}^*(\zeta_m^-)\cdots\mathbf{b}^*(\zeta_1^-)\mathbf{c}^*(\zeta_1^+)\cdots\mathbf{c}^*(\zeta_m^+)\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_m^0)(q^{2(\alpha+s)S(0)})$$

Old computation is not applicable because

$$\xi^{-\alpha}\mathbf{b}^*(\xi)(X)=\sum_{j=0}^{s-1}\xi^{2j}\mathbf{b}^*_{\infty,2j}(X)+\mathbf{b}^*_{\mathsf{reg}}(\xi)(X)\,.$$

To cut a long story short, I write the final answer:

$$Z^{\kappa,s}\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(Y^{(s)}_{\mathbf{M}} T_{\mathrm{S},\mathbf{M}} q^{2\kappa S}\mathbf{b}^*_{\infty,0}\cdots\mathbf{b}^*_{\infty,2s-2}\big(q^{2(\alpha+s)S(0)}\mathcal{O}\big)\Big)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(Y^{(s)}_{\mathbf{M}} T_{\mathrm{S},\mathbf{M}} q^{2\kappa S}\mathbf{b}^*_{\infty,0}\cdots\mathbf{b}^*_{\infty,2s-2}\big(q^{2(\alpha+s)S(0)}\big)\Big)}\,.$$

For this functional the same determinant formulae hold. The difference is that we have now

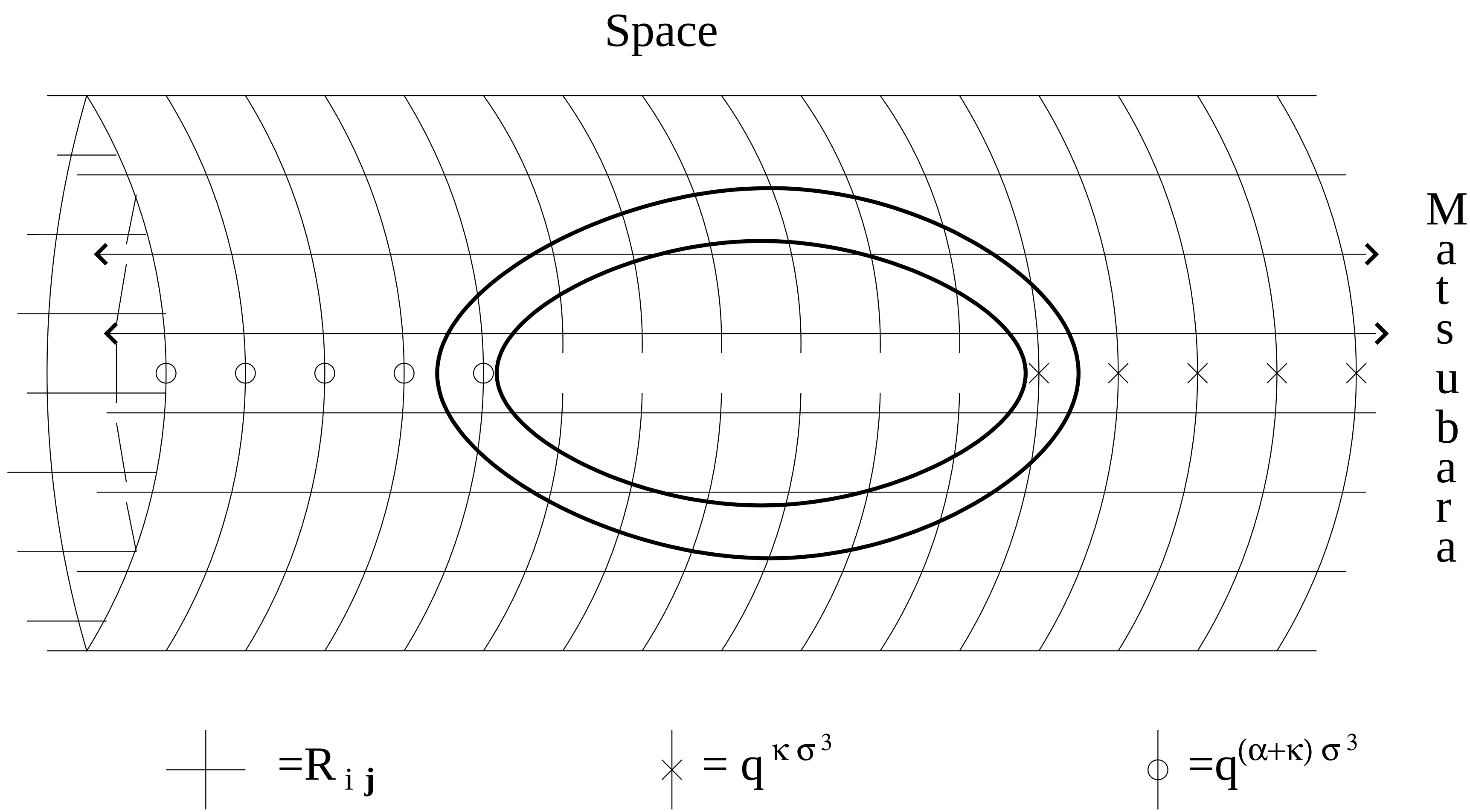
$$\rho(\zeta|\kappa,\kappa+\alpha,s) = \frac{T(\zeta,\kappa+\alpha,s)}{T(\zeta,\kappa)}\,.$$

Notice that $T(\zeta,\kappa+\alpha,s)$ satisfies the same Baxter equations as

$T(\zeta,\kappa+\alpha-s)$, but the number of Bethe roots is $\mathbf{n}/2-s$. We can consider

$s\in\mathbb{Z}$ using for $s<0$ the screening operators $\mathbf{c}^*_{\infty,j}$.

Screenings on the lattice are "topological":



Scaling limit. In the limit $\mathbf{n} \to \infty$ Bethe roots scale for $1 << j << \mathbf{n}$ as

$$\zeta_j = Const \cdot \left(\frac{j}{\mathbf{n}}\right)^{\nu} .$$

The scaling limit consists in

$$\mathbf{n} \to \infty, \quad a \to 0, \quad \mathbf{n}a = 2\pi CR \ \text{ fixed}$$

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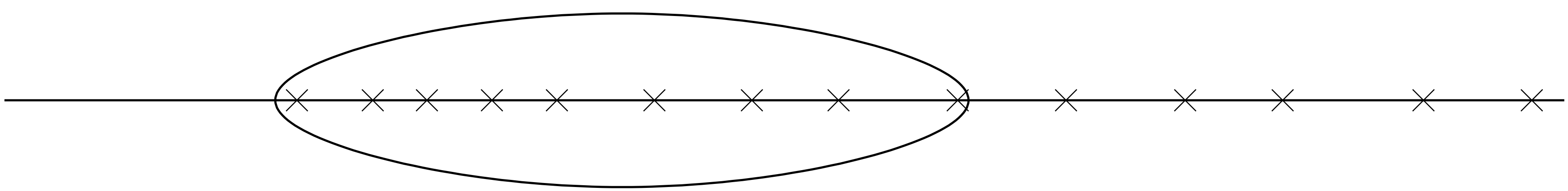

This scaling limit is chiral.

The following limit exists:

$$\rho_R(\lambda|\kappa,\kappa+\alpha,s) = \lim_{\mathbf{n}\to\infty,\ a\to 0,\ 2\pi R=\mathbf{n}a} \rho(\lambda a^{\nu}|\kappa,\kappa+\alpha,s),$$

Morevover,

$$\rho_R(\lambda|\kappa,\kappa+\alpha,s) = \rho_R(\lambda|\kappa,\kappa')\,, \quad \kappa' = \kappa+\alpha+\frac{1-\nu}{\nu}s\,.$$

Similarly,

$$\omega_R(\lambda,\mu|\kappa,\kappa',\alpha) = \lim_{\mathbf{n}\to\infty,\ a\to 0,\ 2\pi R=\mathbf{n}a} \omega(\lambda a^{\nu},\mu a^{\nu}|\alpha,\kappa,s)\,.$$

So in the weak sense the operators are defined

$$\boldsymbol{\tau}^*(\lambda) = \lim_{a\to 0}\mathbf{t}^*(\lambda a^\nu), \quad \boldsymbol{\beta}^*(\lambda) = \lim_{a\to 0}\mathbf{b}^*(\lambda a^\nu), \quad \boldsymbol{\gamma}^*(\lambda) = \lim_{a\to 0}\mathbf{c}^*(\lambda a^\nu)\,,$$

and the main identity turns into (we set $q^{2\alpha S(0)} \to \Phi_\alpha(0) = \phi_\alpha(0)\bar{\phi}_\alpha(0)$)

$$Z_R^{\kappa,\kappa'}\Bigl\{\boldsymbol{\tau}^*(\lambda_1^0)\cdots\boldsymbol{\tau}^*(\lambda_p^0)\boldsymbol{\beta}^*(\lambda_1^+)\cdots\boldsymbol{\beta}^*(\lambda_q^+)\boldsymbol{\gamma}^*(\lambda_q^-)\cdots\boldsymbol{\gamma}^*(\lambda_1^-)(\Phi_\alpha(0))\Bigr\} =$$

$$= \prod_{i=1}^{p}\rho_R(\lambda_i^0|\kappa,\kappa')\det\Bigl|\omega_R(\lambda_i^+,\lambda_j^-|\kappa,\kappa',\alpha)\Bigr|_{i,j=1,\cdots q}\ .$$

Our main claim is that $Z_R^{\kappa,\kappa'}$ describes the expectation values of descendants of the primary field ϕ_α on the cylinder with asymptotical condition defined by the primary fields $\phi_{\kappa+1}$, $\phi_{\kappa'+1}$ for the chiral CFT with the central charge $c = 1 - 6\frac{\nu^2}{1-\nu}$. The scaling dimensions are $\Delta_\alpha = \frac{\alpha(\alpha-2)\nu^2}{4(1-\nu)}$.

2. Three point functions in CFT. Consider the cylinder:

$$-\infty < \mathrm{Re}(z) < \infty, \quad -\pi R < \mathrm{Im}(z) < \pi R, \quad \mathbb{R} - \pi i R = \mathbb{R} + \pi i R\,,$$

on which we have the CFT with the energy-momentum tensor $T(z)$. At the point $z = 0$ we insert the primary field $\phi_\alpha(0)$. Define the descendants by

$$\mathbf{l}_k(O(0)) = \oint y^{k+1} T(y) O(0) \frac{dy}{2\pi i}\,.$$

We call this local description. On the other hand we have the global description

$$T(z) = \frac{1}{R^2}\Big(\sum_{j=-\infty}^{\infty} e^{\frac{nz}{R}} L_n - \frac{c}{24}\Big)$$

We set

$$T(z) = T_+(z) + T_-(z), \quad T_+(z) = \frac{1}{R^2}\Big(\sum_{j=1}^{\infty} e^{\frac{nz}{R}} L_n + \frac{L_0}{2} - \frac{c}{48}\Big)$$

Then if the primary fields $\phi_{\kappa+1}$, $\phi_{\kappa'+1}$ describe the asymptotic conditions we have

$$\lim_{z\to\infty} T_+(z) = \frac{1}{2R^2}\left(\Delta_{\kappa+1} - \frac{c}{24}\right), \quad \lim_{z\to-\infty} T_-(z) = \frac{1}{2R^2}\left(\Delta_{\kappa'+1} - \frac{c}{24}\right).$$

The OPE's in this setting read as

$$\begin{aligned} T(x)T(y) &= \frac{1}{R}\frac{d}{dy}T(y)\chi(x-y) - 2T(y)\frac{1}{R}\frac{d}{dx}\chi(x-y) \\ &\quad - \frac{c}{12R}\frac{d^3}{dx^3}\chi(x-y) + :T(x)T(y):, \\ T(x)\phi(y) &= \frac{1}{R}\frac{d}{dy}\phi(y)\chi(x-y) - \Delta\phi(y)\frac{1}{R}\frac{d}{dx}\chi(x-y) + :T(x)\phi(y):, \end{aligned}$$

with $\chi(x) = \frac{1}{2}\coth\frac{x}{2R}$, which brings a frightening number of Bernoulli numbers into the three-point functions of descendants.

The integrable structure of CFT is based on Zamolodchikov's local integrals of motion. They are described by densities $h_{2k}(z)$.
Important warning. Local integrals in our setting play double role.

- $$\mathbf{i}_{2k-1}(O(0)) = \oint h_{2k}(z)O(0)\frac{dz}{2\pi i}\,, \qquad \mathbf{i}_1 = \mathbf{l}_{-1}, \ \ etc$$

- $$I_{2k-1}(u) = \int\limits_{u-\pi iR}^{u+\pi iR} h_{2k}(z)\frac{dz}{2\pi i}\,,$$
  $$I_{2k-1}(\infty) = I_{2k-1}(\kappa), \qquad I_{2k-1}(-\infty) = I_{2k-1}(\kappa')$$

Obviously,

$$\langle \mathbf{i}_{2k-1}(O(0))\rangle = (I_{2k-1}(\kappa') - I_{2k-1}(\kappa))\langle O(0)\rangle\,.$$

Comparing scaling limit with CFT.

On the lattice the local operators were found around

$$\zeta^2 = 1\,.$$

The formula

$$\zeta^2 = \lambda^2 a^{2\nu}$$

shows that after the scaling limit they have to be looked for at

$$\lambda^2 = \infty\,.$$

The dimensional arguments imply that if the entire construction is consistent with CFT they have to be found in series in $\lambda^{-\frac{1}{\nu}}$. Let us investigate that.

One thing which is clear from the very beginning: $\boldsymbol{\tau}^*(\lambda)$ must describe the action of $\mathbf{i}_{2k-1}$. How to see that?

First,

$$T_{\mathbf{M}}(\zeta,\kappa)\to T_{\mathbf{H}}^{\mathsf{BLZ}}(\lambda,\kappa)\,,$$

where

$$T_{\mathbf{H}}^{\mathsf{BLZ}}(\lambda,\kappa)=\mathrm{Tr}\;q^{\kappa\sigma^3}\mathcal{P}\exp\Bigl(\lambda\int\limits_{-\pi iR}^{\pi iR}\Bigl(\sigma^+e^{-2\varphi(x)}+\sigma^-e^{2\varphi(x)}\Bigr)\frac{dx}{2\pi i}\Bigr)\,.$$

According to BLZ

$$\log\left(T_{\mathbf{H}}^{\mathsf{BLZ}}(\lambda,\kappa)\right)\simeq R\,C_0(\nu)\lambda^{\frac{1}{\nu}}+\sum_{k=1}^{\infty}\lambda^{-\frac{2k-1}{\nu}}C_k(\nu)I_{2k-1}\,.$$

Together with

$$\rho_R(\lambda|\kappa,\kappa')=\frac{T(\lambda,\kappa')}{T(\lambda,\kappa)},\qquad \langle\mathbf{i}_{2k-1}(O(0))\rangle=(I_{2k-1}(\kappa')-I_{2k-1}(\kappa))\langle O(0)\rangle$$

this means

$$\boldsymbol{\tau}^*(\lambda)\simeq\exp\sum_{k=1}^{\infty}\lambda^{-\frac{2k-1}{\nu}}C_k(\nu)\mathbf{i}_{2k-1}\,.$$

Now we set $\kappa=\kappa'$ which means working modulo action of $\mathbf{i}_{2k-1}$. The quotient space can be generated, for example, by $\mathbf{l}_{-2k}$. How to compare action of our fermions with Virasoro? We need to know the asymptotics

$$\omega_R(\lambda,\mu|\kappa,\kappa,\alpha)\simeq\sum_{k,l=1}^{\infty}\lambda^{-\frac{2k-1}{\nu}}\mu^{-\frac{2l-1}{\nu}}\omega_{2k-1,2l-1}(\kappa,\alpha)\,,$$

and to compare it with the three-point functions of descendants. For the fermions we define

$$\boldsymbol{\beta}^*(\lambda)=\sum_{k=1}^{\infty}\lambda^{-\frac{2k-1}{\nu}}\boldsymbol{\beta}^*_{2k-1},\quad \boldsymbol{\gamma}^*(\lambda)=\sum_{k=1}^{\infty}\lambda^{-\frac{2k-1}{\nu}}\boldsymbol{\gamma}^*_{2k-1},$$

Computing the asymptotics is hard, but we did it. I present the results of comparison.

Introducing

$$D_{2m-1}(\alpha) = \sqrt{\frac{2}{\pi\nu}}\,\frac{\Gamma\left(\frac{\alpha}{2}+\frac{2m-1}{2\nu}\right)}{(m-1)!\,\Gamma\left(\frac{\alpha}{2}+\frac{(2m-1)(1-\nu)}{2\nu}\right)}\Gamma(\nu)^{-\frac{2m-1}{\nu}}(1-\nu)^n\,.$$

we have on levels 2 and 4:

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^*(\phi_\alpha) = D_1(\alpha)D_1(2-\alpha)\,\mathbf{l}_{-2}(\phi_\alpha)\,,$$

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_3^*(\phi_\alpha) = \tfrac{1}{2}D_1(\alpha)D_3(2-\alpha)\left(\mathbf{l}_{-2}^2+\tfrac{2(16-c)+6d_\alpha}{9}\,\mathbf{l}_{-4}\right)(\phi_\alpha)\,,$$

$$\boldsymbol{\beta}_3^*\boldsymbol{\gamma}_1^*(\phi_\alpha) = \tfrac{1}{2}D_3(\alpha)D_1(2-\alpha)\left(\mathbf{l}_{-2}^2+\tfrac{2(16-c)-6d_\alpha}{9}\,\mathbf{l}_{-4}\right)(\phi_\alpha)\,,$$

where

$$d_\alpha = \frac{6}{\sqrt{(25-c)(24\Delta_\alpha+1-c)}}\,.$$

A formula for Alyosha.

Consider $\Phi_{1,3}$-perturbation of $c<1$ model.

Second chirality:

$\bar{\boldsymbol{\beta}}^*_{2n-1}$, $\bar{\boldsymbol{\gamma}}^*_{2n-1}$ are related to $\bar{\mathbf{I}}_{-2k}$ by the same formulae with

$$\alpha \to -\alpha\,.$$

We have

$$\frac{\langle \boldsymbol{\beta}^*_{I^+}\boldsymbol{\gamma}^*_{I^-}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\left(\Phi_\alpha\right)\rangle_{\mathrm{Pert}}}{\langle \Phi_\alpha\rangle_{\mathrm{Pert}}} = \delta_{I^+,\bar{I}^-}\delta_{I^-,\bar{I}^+}\ m^{|I^+|+|I^-|}$$

$$\times \prod_{2m-1\ \in\ I^+\cup I^-} \frac{\sin\pi\left(\frac{1-\nu}{2\nu}(2m-1)\pm\frac{\alpha}{2}\right)}{\sin\pi\left(\frac{1}{2\nu}(2m-1)\pm\frac{\alpha}{2}\right)}\,.$$