# Asymptotic behavior of the correlation functions of the XXZ spin chain

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# The spin-1/2 XXZ Heisenberg chain

The $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field is a quantum interacting model defined on a one-dimensional lattice with $M$ sites, with Hamiltonian, $H = H^{(0)} - hS_z$,

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\},$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \qquad [H^{(0)}, S_z] = 0.$$

$\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site $m$ of the chain and $\Delta = \cos(\zeta)$, $\zeta$ real, is the anisotropy parameter. We impose the **periodic** (or quasi-periodic) boundary conditions

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$ , $dim\mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$ act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

# Correlation functions of the spin chains

- Free fermion point $\Delta = 0$ : Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa,...
- Field theory methods (Luttinger liquid theory, conformal field theories and finite size effects etc) : Luther and Peschel, Haldane, Cardy, Affleck, Lukyanov, ...
- From 1984 : Izergin, Korepin, Bogoliubov, Reshetikhin,... (first attempts using Bethe ansatz for general $\Delta$)
- General $\Delta$ : multiple integral representations
  $\star$ 1996 Jimbo and Miwa $\longrightarrow$ from qKZ equation
  $\star$ 1999 Kitanine, Maillet, Terras $\longrightarrow$ from Algebraic Bethe Ansatz
- Several developments since 2000 : N.K., Maillet, Slavnov, Terras; Lukyanov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov,Takeyama; Gohmann, Klumper, Seel; Caux, Hagemans, Maillet; Affleck,... ...

# Correlation functions

$$\langle\, O \,\rangle = \frac{\mathrm{tr}_{\mathcal{H}}\left(O\, e^{-H/kT}\right)}{\mathrm{tr}_{\mathcal{H}}\left(e^{-H/kT}\right)} = \langle \psi_g |\, O \,| \psi_g \rangle \qquad \text{at} \quad T = 0$$

where $|\psi_g\rangle$ is the (normalized) ground state. Consider the correlation function of the product of two local spin operators at zero temperature and equal time:

$$g_{1,m}^{\alpha\beta} = \langle \psi_g | \sigma_1^{\alpha} \sigma_m^{\beta} | \psi_g \rangle$$

Two main strategies to evaluate such a function:

*(i)* compute the action of local operators on the ground state $\sigma_1^{\alpha}\sigma_m^{\beta}|\psi_g\rangle = |\psi_{1m}^{\alpha\beta}\rangle$ and then calculate the resulting scalar product:

$$g_{1,m}^{\alpha\beta} = \langle \psi_g | \psi_{1m}^{\alpha\beta} \rangle$$

*(ii)* insert a sum over a complete set of states $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g^{\alpha\beta}_{1,m} = \sum_i \langle\psi_g|\sigma^\alpha_1|\omega_i\rangle \cdot \langle\omega_i|\sigma^\beta_m|\psi_g\rangle$$

Main problems to be solved to achieve this (from first principles) :

- Compute exact eigenstates and energy levels of the Hamiltonian (Bethe ansatz)
- Obtain the action of local operators on the eigenstates : main problem since eigenstates are highly non-local!
- Compute the resulting scalar products with the eigenstates
- Re-sum exactly or asymptotically the form factor series

# Algebraic Bethe ansatz and correlation functions

- **Algebraic Bethe ansatz** (Faddeev, Sklyanin, Taktadjan)

  Construction of the direct map : $\sigma_m^\alpha \longrightarrow T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

  Yang-Baxter algebra : $R_{12}(\lambda_1, \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1, \lambda_2)$

  Commuting conserved charges: $t_\kappa(\lambda) = A(\lambda) + \kappa D(\lambda), \qquad [t_\kappa(\lambda), t_\kappa(\mu)] = 0$

  Hamiltonian : $H^{(0)} = 2\sinh\eta \, \frac{\partial}{\partial\lambda} \log t_\kappa(\lambda)\big|_{\lambda=\frac{\eta}{2}} + c$ for all $\xi_j = 0$.
  $\kappa = 1$ corresponds to periodic boundary conditions

  Eigenstates of $t_\kappa(\mu)$ : $|\psi(\{\lambda\})\rangle = \prod_k B(\lambda_k)|0\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

- **Ground state.** In the thermodynamic limit, the ground state of the model is the Dirac sea: the raoidities of the particles occupy the interval $[-q, q]$ with a density $\rho(\lambda)$. To find the **Fermi boundary** $q$ define the dressed energy $\varepsilon(\lambda)$:

$$\varepsilon(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \varepsilon(\mu)\, d\mu = h - 2p_0'(\lambda) \sin \zeta,$$

$$K(\lambda) = \frac{\sin 2\zeta}{\sinh(\lambda + i\zeta) \sinh(\lambda - i\zeta)}, \qquad \cos \zeta = \Delta, \qquad 0 < \zeta < \pi,$$

and $p_0(\lambda)$ is the bare momentum of one particle. Then $q$ is such that $\varepsilon(q) = 0$. The ground state density $\rho(\lambda)$ satisfies a linear integral equation

$$\rho(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \rho(\mu)\, d\mu = \frac{1}{2\pi} p_0'(\lambda).$$

Total density of particles: $D = \int_{-q}^{q} \rho(\lambda)\, d\lambda$.

- **Action of local operators on Bethe states** (N.K., Maillet, Terras)

  Solution of the quantum inverse scattering problem : $\sigma_m^\alpha \longleftarrow T(\lambda)$

$$\begin{aligned}
\sigma_j^- &= \prod_{k=1}^{j-1} t(\xi_k) \cdot B(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\
\sigma_j^+ &= \prod_{k=1}^{j-1} t(\xi_k) \cdot C(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\
\sigma_j^z &= \prod_{k=1}^{j-1} t(\xi_k) \cdot (A - D)(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k),
\end{aligned} \tag{1}$$

$+$ Yang-Baxter algebra for A, B, C, D to get the action on arbitrary states, for example

$$\langle 0| \prod_{k=1}^{N} C(\lambda_k)\, A(\lambda_{N+1}) = \sum_{a'=1}^{N+1} \Lambda_{a'} \, \langle 0| \prod_{\substack{k=1 \\ k\neq a'}}^{N+1} C(\lambda_k)$$

- Scalar products (Slavnov; N.K., Maillet, Terras)

$$\langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{N} B(\lambda_k)\, |0\rangle = \frac{\det U(\{\mu_j\},\{\lambda_k\})}{\det V(\{\mu_j\},\{\lambda_k\})}$$

for $\{\lambda_k\}$ a solution of Bethe equations and $\{\mu_j\}$ an arbitrary set of parameters, :

$$U_{ab} = \partial_{\lambda_a}\tau(\mu_b,\{\lambda_k\}), \qquad V_{ab} = \frac{1}{\sinh(\mu_b-\lambda_a)}, \qquad 1 \leqslant a,b \leqslant N,$$

where $\tau(\mu_b,\{\lambda_k\})$ is the eigenvalue of the transfer matrix $t(\mu_b)$

**The results :** explicit determinant representations for the form factors of spin operators and multiple integral representations for the elementary blocks of the correlation functions on infinite $XXZ$ chain with magnetic field. Recent extension to open chains. Numerical resummation of the form factor series leading to accurate determination of the dynamical structure factors for large chains and successful comparison with neutron scattering experiments on magnets.

# The spin-spin correlation functions

→ A priori, the problem is solved:

- expression for any elementary block $\langle\psi_g|\, E_1^{\epsilon_1',\epsilon_1} \dots E_m^{\epsilon_m',\epsilon_m}\, |\psi_g\rangle$
- any correlation function $= \sum$ (elementary blocks)

→ However ...

$$\langle\psi_g|\sigma_1^z\,\sigma_m^z|\psi_g\rangle \equiv \langle\psi_g|(E_1^{11}-E_1^{22})\underbrace{\prod_{j=2}^{m-1}(E_j^{11}+E_j^{22})}_{\text{propagator}}(E_m^{11}-E_m^{22})|\psi_g\rangle$$

$$= \sum_{2^m \text{ terms}} \text{(elementary blocks)} \underset{m\to\infty}{\sim} ?$$

# Generating function.

- Generating function (more convenient that two point function)

$$Q_\kappa(m) = \langle\psi_g|\prod_{n=1}^{m}\left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2}\cdot\sigma_n^z\right)|\psi_g\rangle$$
$$= \langle\psi_g|\prod_{a=1}^{m}(A+\kappa D)(\xi_a)\prod_{b=1}^{m}(A+D)^{-1}(\xi_b)|\psi_g\rangle,$$

- Two-point function:

$$\frac{1}{2}\langle(1-\sigma_1^z)(1-\sigma_{m+1}^z)\rangle = \frac{\partial^2}{\partial\kappa^2}\left(Q_\kappa(m+1) - 2Q_\kappa(m) + Q_\kappa(m-1)\right)\Big|_{\kappa=1}$$

# Master equation.

- Master equation for the generating function:

$$Q_\kappa(m) = \frac{1}{N!} \oint\limits_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{b=1}^{N} e^{im\left(p_0(z_b)-p_0(\lambda_b)\right)}$$

$$\times \prod_{a,b=1}^{N} \sinh^2(\lambda_a - z_b) \cdot \frac{\det_N\left(\frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial z_k}\right) \cdot \det_N\left(\frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j}\right)}{\prod\limits_{a=1}^{N} \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N\left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j}\right)}$$

$$\mathcal{Y}_\kappa(\mu|\{z\}) = a(\mu)\prod_{k=1}^{N}\sinh(z_k - \mu - i\zeta) + \kappa\, d(\mu)\prod_{k=1}^{N}\sinh(z_k - \mu + i\zeta)$$

where $a(\mu) = \left(\sinh(\mu - \frac{i\zeta}{2})\right)^M$, and $d(\mu) = \left(\sinh(\mu + \frac{i\zeta}{2})\right)^M$

and, $p_0(\mu) = -\frac{i}{M}\log\left(\frac{a(\mu)}{d(\mu)}\right)$

# The general scheme

**To get the asymptotic behavior of the correlation function we apply the following steps :**

- Expand the master equation representation for the propagator at distance $m$ into the sum of multiple cycle integrals
- Obtain the asymptotic behavior of each cycle integral using Riemann-Hilbert techniques
- Decompose the cycle integral asymptotic into three pieces : non-oscillating, oscillating and sub-leading effects
- Re-sum each of these part consecutively exactly for the leading one's, perturbatively for the sub-leading effects to get the correlation function asymptotic behavior

# Asymptotic expansion of $Q_\kappa(m)$

- Non-oscillating and oscillating parts

$$Q_\kappa(m) = \sum_{\sigma=0,\pm} G^{(\sigma)}(\beta, m) \ .$$

here $\beta = \log \kappa$. The term $G^{(0)}(\beta, m)$ gives the non-oscillating part of the asymptotics, while $G^{(\pm)}(\beta, m)$ describe the oscillating part.

$$G^{(0)}(\beta, m) = \mathcal{A}(\beta) \cdot e^{\beta m D} (2\pi \sinh(2q)\rho(q) m)^{\frac{\beta^2 \mathcal{Z}^2}{2\pi^2}} G^2 \left(1, \frac{\beta \mathcal{Z}}{2\pi i}\right) e^{\frac{\beta^2}{4\pi^2}(C_0 - C_1)}$$

Here $\mathcal{Z} = Z(\pm q)$, $q$ - Fermi boundary and $Z(\lambda)$ is the **dressed charge** which satisfies the following integral equation:

$$Z(\lambda) + \int_{-q}^{q} \frac{d\mu}{2\pi} K(\lambda - \mu) Z(\mu) = 1$$

here $G(1,z) = G(1+z)G(1-z)$, $G(z)$ is the **Barnes function**, and $\mathcal{A}(\beta)$, $C_0$ and $C_1$ are computable constants :

$$C_0 = \int_{-q}^{q} \frac{Z(\lambda)Z(\mu)}{\sinh^2(\lambda - \mu - i\zeta)}\, d\lambda d\mu$$

$$C_1 = \frac{1}{2}\int_{-q}^{q} \frac{Z'(\lambda)Z(\mu) - Z(\lambda)Z'(\mu)}{\tanh(\lambda - \mu)}\, d\lambda d\mu + 2\mathcal{Z}\int_{-q}^{q} \frac{\mathcal{Z} - Z(\lambda)}{\tanh(q - \lambda)}\, d\lambda$$

The coefficient $\mathcal{A}(\beta)$ is given as the ratio of four Fredholm determinants acting on the contour $\Gamma$ surrounding the interval $[-q, q]$.

The leading oscillating is related to the non-oscillating part as :

$$G^{(\pm)}(\beta, m) = G^{(0)}(\beta \pm 2\pi i, m)$$

# Asymptotic expansion of $\langle\sigma_1^z\sigma_{m+1}^z\rangle$

- Non-oscilating part

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{non-osc} = \langle\sigma_1^z\rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2}, \qquad m \to \infty$$

- Oscillating part:

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{osc} = 8\widetilde{\mathcal{A}}\sin^2 k_F \cdot \frac{\cos(2mk_F)e^{C_1-C_0}}{(2\pi\sinh(2q)\rho(q)m)^{2\mathcal{Z}^2}}$$

Here $\langle\sigma^z\rangle = 1-2D$, $k_F = \pi D$ and $\widetilde{\mathcal{A}} = -4G^2(1,\mathcal{Z})\mathcal{A}''(2\pi i)$

Final answer :

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2|F_\sigma|^2 \cdot \frac{\cos(2mk_F)}{m^{2\mathcal{Z}^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2\mathcal{Z}^2}}\right)$$

# Correlation amplitude

Consider a finite chain of length $M$

- **Ground state**: Bethe roots $\{\lambda_1, \dots \lambda_N\}$,

- Special **excited state**, with one particle and one hole located on the opposite ends of the Fermi zone $\pm q$: Bethe roots $\{\mu_1, \dots, \mu_N\}$.

Note: $\frac{N}{M} = D$

- Correlation amplitude:

$$\lim_{N,M\to\infty} \left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2} \frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2 \cdot \|\psi(\{\lambda\})\|^2} = |F_\sigma|^2.$$

# Some open problems...

- Zero magnetic field case : Lukyanov predictions
- Asymptotic behavior of correlation functions : general case (time dependent, temperature dependent)
- Simpler method using form factors? Structure factors?
- Continuum (Field theory) models (Sine-Gordon,...) :
  - ⋆ Approach from the lattice
  - ⋆ Inverse problem
  - ⋆ Link to Q operator and SOV methods