Differential equations and elliptic conformal blocks in Liouville field theory

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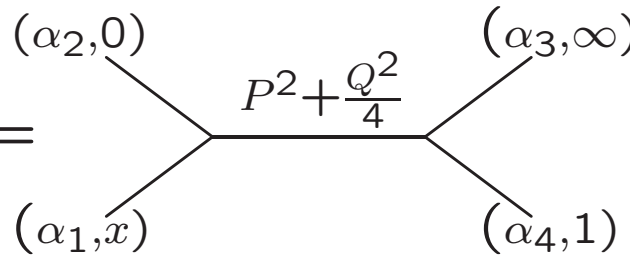
Liouville theory

- Lagrangian: $\mathcal{L} = \frac{1}{4\pi}(\partial_a \varphi)^2 + \mu e^{2b\varphi}$

- Central charge: $c_L = 1 + 6Q^2$ where $Q = b + \frac{1}{b}$

- Primary fields: $V_\alpha = e^{2\alpha\varphi}$ have conformal dimensions $\Delta(\alpha) = \alpha(Q-\alpha)$

- Three-point function (Dorn-Otto-Zamolodchikov-Zamolodchikov):

$$
C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(Q-\alpha)}{b}} \times
$$
$$
\times \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha - Q)\Upsilon(\alpha - 2\alpha_1)\Upsilon(\alpha - 2\alpha_2)\Upsilon(\alpha - 2\alpha_3)},
$$

- Four-point function: $\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_4}(z_4, \bar{z}_4) \rangle \sim$

$$\sim \int_{\mathcal{C}} C\left(\alpha_1, \alpha_2, \frac{Q}{2} + iP\right) C\left(\frac{Q}{2} - iP, \alpha_3, \alpha_4\right) \left| \mathfrak{F}_P\left(\begin{matrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{matrix} \middle| x\right) \right|^2 dP,$$

- Conformal block: $\mathfrak{F}_P\left(\begin{matrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{matrix} \middle| x\right) =$

is not known in a closed form

- Elliptic block (Al. Zamolodchikov):

$$\mathfrak{F}_P\left(\begin{matrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{matrix} \middle| x\right) = (16q)^{P^2} x^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} (x-1)^{\frac{Q^2}{4} - \Delta_1 - \Delta_4} \times$$

$$\times \theta_3(q)^{3Q^2 - 4\sum_k \Delta_k} \mathfrak{H}_P\left(\begin{matrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{matrix} \middle| q\right),$$

where $q = e^{i\pi\tau}$ with $\tau = i\frac{K(1-x)}{K(x)}$, satisfies a recursive relation which leads to an effective algorithm for calculation of its expansion in power series of q (which is more convenient for numerical studies than the ordinary x expansion)

- Degenerate fields V_α with $\alpha = \alpha_{mn} = -\frac{mb}{2} - \frac{n}{2b}$ have a null-vector in their Verma module at level $(m+1)(n+1)$ and hence four-point function satisfies Fuchsian ordinary differential equation of the same order (Belavin-Polyakov-Zamolodchikov 1984). An explicit integral representaion for the solution to this equation can be obtained. For example in the case $n = 0$ one has

$$\langle V_{-\frac{mb}{2}}(x, \bar{x}) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle = \Omega_m(\alpha_1, \alpha_2, \alpha_3) \, |x|^{2mb\alpha_1} |x-1|^{2mb\alpha_2}$$

$$\times \int \prod_{k=1}^{m} |t_k|^{2A} |t_k - 1|^{2B} |t_k - x|^{2C} \prod_{i<j} |t_i - t_j|^{-4b^2} \, d^2 t_1 \ldots d^2 t_m$$

  with parameters

$$A = b\,(\alpha - 2\alpha_1 - Q + mb/2)\,, \quad B = b\,(\alpha - 2\alpha_2 - Q + mb/2)\,,$$
$$C = b\,(Q + mb/2 - \alpha)$$

- However, for several important purposes one needs the differential operator for the four-point correlation function in explicit form.

- We consider five-point function

$$\langle V_{-\frac{1}{2b}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)V_{\alpha_4}(x)\rangle =$$

$$= z^{\frac{1}{2b^2}}(z-1)^{\frac{1}{2b^2}}\frac{(z(z-1)(z-x))^{\frac{1}{4}}}{(x(x-1))^{\frac{2\Delta(\alpha_4)}{3}+\frac{1}{12}}}\frac{\Theta_1(u)^{b^{-2}}}{\Theta_1'(0)^{\frac{b^{-2}+1}{3}}}\Psi(u|q),$$

with $u = \frac{\pi}{4K(x)}\int_0^{\frac{z-x}{x(z-1)}}\frac{dt}{\sqrt{t(1-t)(1-xt)}}$. One finds, that $\Psi(u|\tau)$ satisfies:

$$\left[\partial_u^2 - \mathbb{U}(u|\tau) + \frac{4i}{\pi b^2}\partial_\tau\right]\Psi(u|\tau) = 0, \qquad (*)$$

$$\mathbb{U}(u|\tau) = \sum_{j=1}^{4} s_j(s_j+1)\wp(u-\omega_j)$$

parameters s_k are related with α_k as $\alpha_k = \frac{Q}{2} - \frac{b}{2}\left(s_k+\frac{1}{2}\right)$ and ω_k are half periods.

- One can try to find a solution to $(*)$ in a form

$$\Psi(u|q) = u^{s_4+1}\left(\Psi(\tau) + \Psi_{-1}(\tau)u^2 + \Psi_{-2}(\tau)u^4 + \dots\right),$$

- Function $\vec{\Psi}(\tau) = \begin{pmatrix} \cdots \\ \Psi_{-1}(\tau) \\ \Psi(\tau) \end{pmatrix}$ satisfies

$$\left(-J_- + \frac{i}{\pi b^2} \frac{\partial}{\partial \tau} + \sum_{k=1}^{\infty} \frac{W^{(k+1)}(\tau)}{k!^2} J_+^k \right) \vec{\Psi}(\tau) = 0, \qquad (**)$$

where

$$J_- = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & -2s_4 - 5 & 0 & 0 \\ \cdots & 0 & -s_4 - \frac{3}{2} & 0 \end{pmatrix} \qquad J_+ = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$W^{(k)}(\tau(x)) = \left(\frac{2K(x)}{\pi} \right)^{2k} \left((-1)^{k+1}(x-1)w_1^{(k)} P_k(x) + \right.$$

$$\left. + x w_2^{(k)} P_k(1-x) - (-1)^{k-1} x(x-1) w_3^{(k)} x^{k-2} P_k(1/x) \right)$$

with $w_j^{(k)} = \left[s_j(s_j+1) + \frac{s_4(s_4+1)}{(4^k-1)} \right]$ and $\mathrm{sn}^2(t|\sqrt{x}) = \sum \frac{P_{k+1}(1-x)}{k!^2} t^{2k}$.

- One can notice that if the parameter s_4 in eq (**) from the previous slide takes the values*

$$s_4 = -m - \frac{3}{2}$$

  then the infinite chain of equations (**) has a finite sub-chain. Due to the triangle form of (**) it is easy to conclude that the function $\Psi(\tau)$ satisfies a differential equation of the order $(m+1)$. Examples are (here $W^{(k)}(x) = (\frac{d\tau}{dx})^k W^{(k)}(\tau)$)

  - For $m = 1$ $\quad \left(\partial^2 + W^{(2)}(x)\right)\Psi = 0$

  - For $m = 2$ $\quad \left(\partial^3 + 4W^{(2)}(x)\partial + 2\partial W^{(2)}(x) + W^{(3)}(x)\right)\Psi = 0$

  - For $m = 3$ $\quad \left(\partial^4 + 10W^{(2)}(x)\partial^2 + \left(10\partial W^{(2)}(x) + 6W^{(3)}(x)\right)\partial + \right.$
    $\left. + \left(9W^{(2)}(x)^2 + 3\partial^2 W^{(2)}(x) + 3\partial W^{(3)}(x) + W^{(4)}(x)\right)\right)\Psi = 0$

*It corresponds to the situation $\alpha_4 = \frac{1}{2b} - \frac{mb}{2}$ and hence in the operator product expansion $V_{-\frac{1}{2b}}(z)V_{\alpha_4}(x)$ appears the degenerate field $V_{-\frac{mb}{2}}$.

Integrable potentials and conformal blocks

- We consider again the generalized Lamé heat equation

$$\left[\partial_u^2 - \mathbb{U}(u|\tau) + \frac{4i}{\pi b^2} \partial_\tau \right] \Psi(u|\tau) = 0,$$

with

$$\mathbb{U}(u|\tau) = \sum_{j=1}^{4} s_j(s_j+1)\wp(u - \omega_j)$$

- We propose that for $s_k = m_k + \frac{2n_k}{b^2}$ equation is integrable

- For example let $s_1 = s_2 = s_3 = 0$ and $s_4 = 1$

$$\Psi(u|q) = \int_0^\pi \left(\frac{\Theta_1(v)}{\Theta_1'(0)^{\frac{1}{3}}} \right)^{b^2} \frac{E(u+v)}{E(u)E(v)} \Psi_0(u + b^2 v|q) \, dv,$$

where Ψ_0 is the solution of the heat equation and $E(u) = \frac{\Theta_1(u)}{\Theta_1'(0)}$

- In the dual case $s_1 = s_2 = s_3 = 0$ and $s_4 = \frac{2}{b^2}$

$$\Psi(u|q) = \Theta_1'(0)^{\frac{2}{3}(1-\frac{2}{b^2})} \int_0^\pi \left(\frac{\Theta_1(v)}{\Theta_1'(0)^{\frac{1}{3}}} \right)^{\frac{4}{b^2}} \left(\frac{E(u+v)}{E(u)E(v)} \right)^{\frac{2}{b^2}} \Psi_0(u+2v|q)\, dv,$$

- For general $s_k = m_k + \frac{2n_k}{b^2}$, we expect solution is likely to be given by an integral of dimension

$$N = g + n_1 + n_2 + n_3 + n_4,$$

where g is the number of gaps for the classical potential

$$g = \frac{1}{2}\left(2\max m_k, 1 + \mathtt{m} - (1+(-1)^{\mathtt{m}})\left(\min m_k + \frac{1}{2} \right) \right),$$

here $\mathtt{m} = \sum m_k$. For example, for $s_1 = s_2 = s_3 = 0$ and $s_4 = m$

$$\Psi(u|q) = \int_0^\pi ..\int_0^\pi \prod_{k=1}^m \left(\frac{\Theta_1(v_k)}{\Theta_1'(0)^{\frac{1}{3}}} \right)^{mb^2} \prod_{i<j} \left| \frac{\Theta_1(v_i - v_j)}{\Theta_1'(0)^{\frac{1}{3}}} \right|^{-b^2}$$
$$\prod_{k=1}^m \frac{E(u+v_k)}{E(u)E(v_k)} \Psi_0(u+b^2v|q)\, dv_1...dv_m,$$

- In order to obtain the conformal block one has to take instead of Ψ_0

$$\Psi_P^\pm(u|q) = q^{P^2} e^{\pm 2b^{-1}Pu}.$$

and take the limit $u \to 0$

- Let us define:

$$\mathcal{H}_m^{(P)}(q) \stackrel{\text{def}}{=} \mathfrak{H}_P\left(\begin{array}{cc} \frac{Q}{2} - \frac{b}{4} & \frac{Q}{2} - \frac{b}{4} \\ -\frac{(2m-1)b}{4} & \frac{Q}{2} - \frac{b}{4} \end{array}\middle| q\right)$$
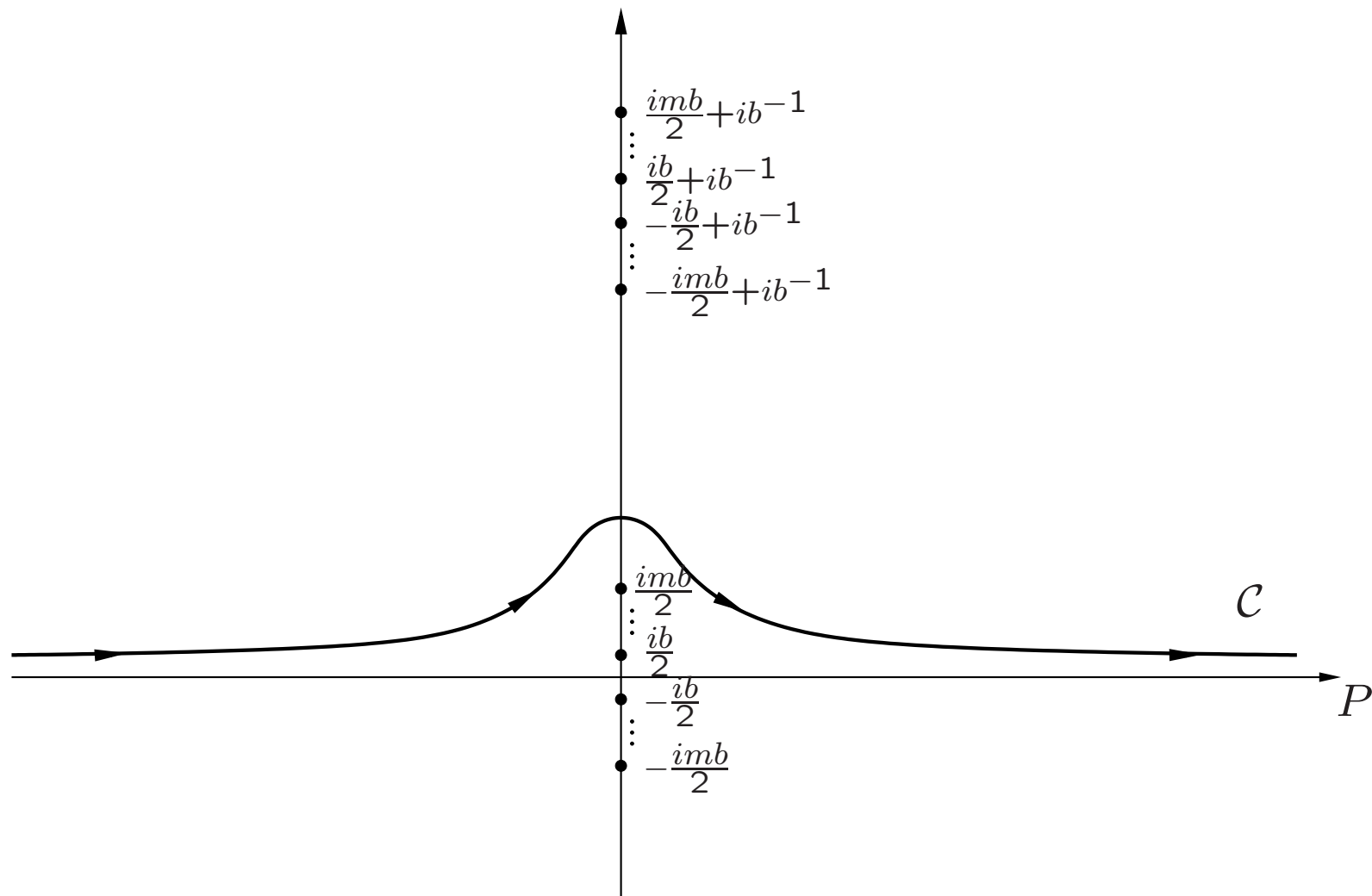
then

$$\mathcal{H}_m^{(P)}(q) = N_m^{-1} \int_0^\pi \cdots \int_0^\pi e^{2bP(u_1 + \cdots + u_m)} \prod_{k=1}^m E(u_k)^{mb^2} \prod_{i<j} |E(u_i - u_j)|^{-b^2} d\vec{u}$$

where N_m is the normalization constant

- The product of structure constants simplifies drastically

$$C\left(-\frac{(2m-1)b}{4}, \frac{Q}{2} - \frac{b}{4}, \frac{Q}{2} + iP\right) C\left(\frac{Q}{2} - iP, \frac{Q}{2} - \frac{b}{4}, \frac{Q}{2} - \frac{b}{4}\right) \sim$$

$$\sim 16^{-2P^2} \prod_{k=1}^m \gamma\left(ibP - \frac{kb^2}{2}\right) \gamma\left(-ibP - \frac{kb^2}{2}\right).$$

- The integral over the intermediate momentum P goes as shown

- This deformation of the contour is prescribed by the condition that the four-point correlation function is single-valued

- Surprisingly, the result of integration over the momentum P is given by a multiple integral over the torus T with periods π and $\pi\tau$

$$\int_{\mathcal{C}} \frac{|q|^{2P^2} \mathfrak{F}_m(P|\tau) \mathfrak{F}_m(-P|\tau^*)}{\prod_{k=1}^{m} \sin\left(\pi(ibP + \frac{kb^2}{2})\right) \sin\left(\pi(ibP - \frac{kb^2}{2})\right)} dP =$$

$$= \Lambda_m \Big(\text{Im}(\tau)\Big)^{-1/2} \int_T \cdots \int_T \prod_{k=1}^{m} \mathcal{E}(u_k, \bar{u}_k)^{mb^2} \prod_{i<j} \mathcal{E}(u_i - u_j, \bar{u}_i - \bar{u}_j)^{-b^2} d^2\vec{u},$$

where

$$\mathfrak{F}_m(P|\tau) \overset{\text{def}}{=} \int_0^{\pi} \cdots \int_0^{\pi} e^{2bP(u_1 + \cdots + u_m)} \prod_{k=1}^{m} E(u_k)^{mb^2} \prod_{i<j} |E(u_i - u_j)|^{-b^2} d\vec{u},$$

$$\mathcal{E}(u, \bar{u}) = E(u)\bar{E}(\bar{u}) e^{-\frac{2(\text{Im}u)^2}{\pi \text{Im}\tau}}$$

- We note that this integral representation looks like Coulomb gas representation of the one-point correlation function of the operator $V_{-mb'}$ in LFT with parameter $b' = \frac{b}{\sqrt{2}}$ on a torus

- Let us define function $\mathcal{T}(\alpha, b|q)$ in Liouville field theory with cosmological constant μ and coupling constant b on a torus

$$\mathcal{T}(\alpha, b|q) \stackrel{\text{def}}{=} \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{\alpha}{b}} |\eta(\tau)|^{-4\Delta(\alpha)} \langle V_\alpha \rangle_\tau$$

We define also the function $\mathcal{S}(\alpha, b|q)$ which is related to the four-point correlation function in LFT on sphere as (here $\zeta = \frac{Q}{2} - \frac{b}{4}$)

$$\mathcal{S}(\alpha, b|q) \stackrel{\text{def}}{=} \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{\alpha}{b}+\frac{1}{2b}-\frac{1}{4}} \times$$

$$\times |x(x-1)|^{\frac{4}{3}\Delta(\alpha)} \langle V_\alpha(x, \bar{x}) V_\zeta(0) V_\zeta(1) V_\zeta(\infty) \rangle.$$

- The correspondence between the one-point toric and the four-point spheric correlation functions states that

$$\mathcal{S}(\alpha, b|q) = \aleph\left(\left(\alpha - \frac{b}{4}\right)\sqrt{2}, \frac{b}{\sqrt{2}}\right) \mathcal{T}\left(\left(\alpha - \frac{b}{4}\right)\sqrt{2}, \frac{b}{\sqrt{2}}\middle| q\right),$$

where $\aleph(\alpha, b)$ is given by

$$\aleph(\alpha, b) = \frac{\Upsilon_b(\alpha)}{\Upsilon_b\left(\frac{1}{2b}\right)} \frac{\Upsilon_b\left(\frac{1}{b}\right)}{\Upsilon_b\left(\alpha + \frac{1}{2b}\right)}.$$

Conformal blocks and Nekrasov partition function

One-point conformal block $\mathcal{F}_\alpha^{(\Delta)}(q)$ is defined as the contribution to the trace of the conformal family with conformal dimension $\Delta = \frac{Q^2}{4} + P^2$

$$\mathcal{F}_\alpha^{(\Delta)}(q) \stackrel{\text{def}}{=} \text{Tr}_\Delta\left(q^{L_0 - \frac{c}{24}} V_\alpha(0)\right) = 1 + \frac{2\Delta + \Delta^2(\alpha) - \Delta(\alpha)}{2\Delta} q + \cdots$$

It was proposed by Alday, Gaiotto and Tachikawa that

$$\mathcal{F}_\alpha^{(\Delta)}(q) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{2\Delta(\alpha)-1} Z(\varepsilon_1, \varepsilon_2, m, a),$$

where $Z(\varepsilon_1, \varepsilon_2, m, a)$ is the instanton part of the Nekrasov partition function in $\mathcal{N} = 2^*$ $U(2)$ SYM with

$$P = \frac{a}{\hbar}, \qquad \alpha = \frac{m}{\hbar}, \qquad \varepsilon_1 = \hbar b, \qquad \varepsilon_2 = \frac{\hbar}{b},$$

where a is VEV of scalar field, m is the mass of the adjoint hypermultiplet and ε_k are the parameters of the Ω background. Parameter q is given by

$$q = e^{2i\pi\tau}, \quad \text{where} \quad \tau = \frac{4i\pi}{g^2} + \frac{\theta}{2\pi}.$$

Nekrasov partition function

$$Z(\varepsilon_1, \varepsilon_2, m, a) = 1 + \sum_{k=1}^{\infty} q^k \mathfrak{Z}_k,$$

can be represented as a sum over partitions. Let $\vec{Y} = (Y_1, Y_2)$ be the pair of Young diagrams with the total numbers of cells equal to N. Then

$$\mathfrak{Z}_N = \sum_{\vec{Y}} \prod_{i,j=1}^{2} \prod_{s \in Y_i} \frac{(E_{ij}(s) - \alpha)(Q - E_{ij}(s) - \alpha)}{E_{ij}(s)(Q - E_{ij}(s))},$$

where

$$E_{ij}(s) = 2P\epsilon_{ij} - bH_{Y_j}(s) + b^{-1}(V_{Y_i}(s) + 1),$$

$H_Y(s)$ and $V_Y(s)$ are respectively the horizontal and vertical distance from the square s to the edge of the diagram Y.

- AGT relation can proved using Al. Zamolodchikov's recursive formula

- Seiberg-Witten prepotential can be obtained in the limit $\hbar \to 0$

$$Z(\varepsilon_1, \varepsilon_2, m, \vec{a}) \to \exp\left(\frac{1}{\hbar^2} \mathcal{F}(m, \vec{a}|q) + O(1)\right).$$

- Let us consider two-point function with one degenerate field

$$\Psi(z) \sim \langle V_{-\frac{b}{2}}(z) V_\alpha(0) \rangle$$

This function satisfies Scrödinger equation

$$\left(-\partial_z^2 + \frac{b^2 m^2}{\hbar^2} \wp(z)\right) \Psi(z) = \frac{ib^2}{\pi} \partial_\tau \Psi(z).$$

We look for the solution in the form

$$\Psi(z) = \exp\left(\frac{1}{\hbar^2} \mathcal{F}(q) + \frac{b}{\hbar} \mathcal{W}(z|q) + \ldots\right)$$

WKB approximation gives

$$\mathcal{W}(z|q) = \int \sqrt{E(q) - m^2 \wp(z)} dz,$$

where $E(q) = 4q \partial_q \mathcal{F}(q)$ is given by

$$\oint_A \sqrt{E(q) - m^2 \wp(z)} dz = 2\pi a,$$