

Facets of Integrability

5/11/9

...in memory of Aliosha Zamolodchikov

Exact g-functions

work in progress with

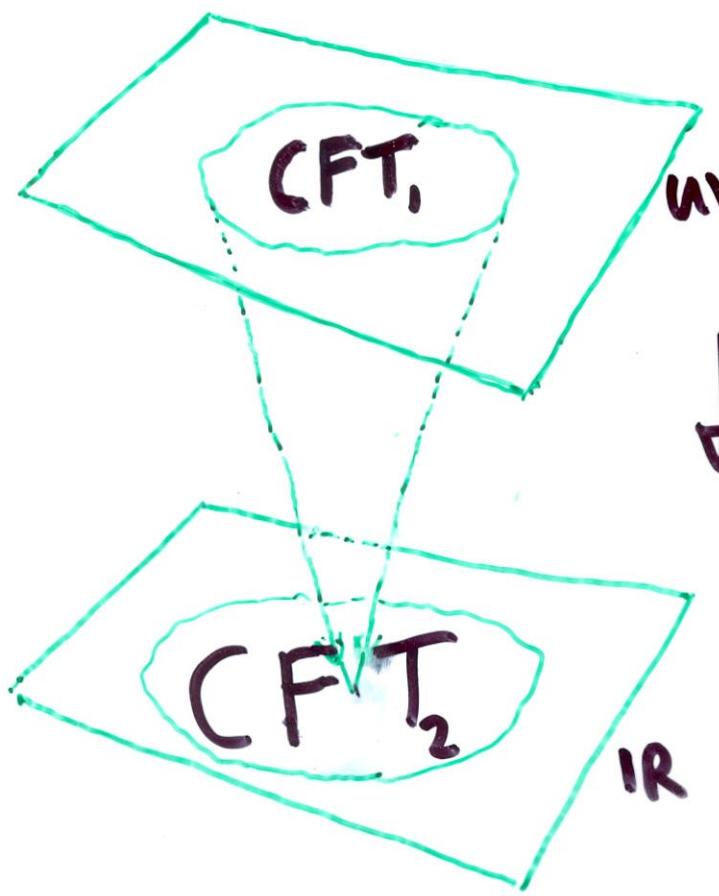
Chaiho Rim & Roberto Tateo

following earlier work, also with

Davide Fioravanti & Anna Lishman

The problem:

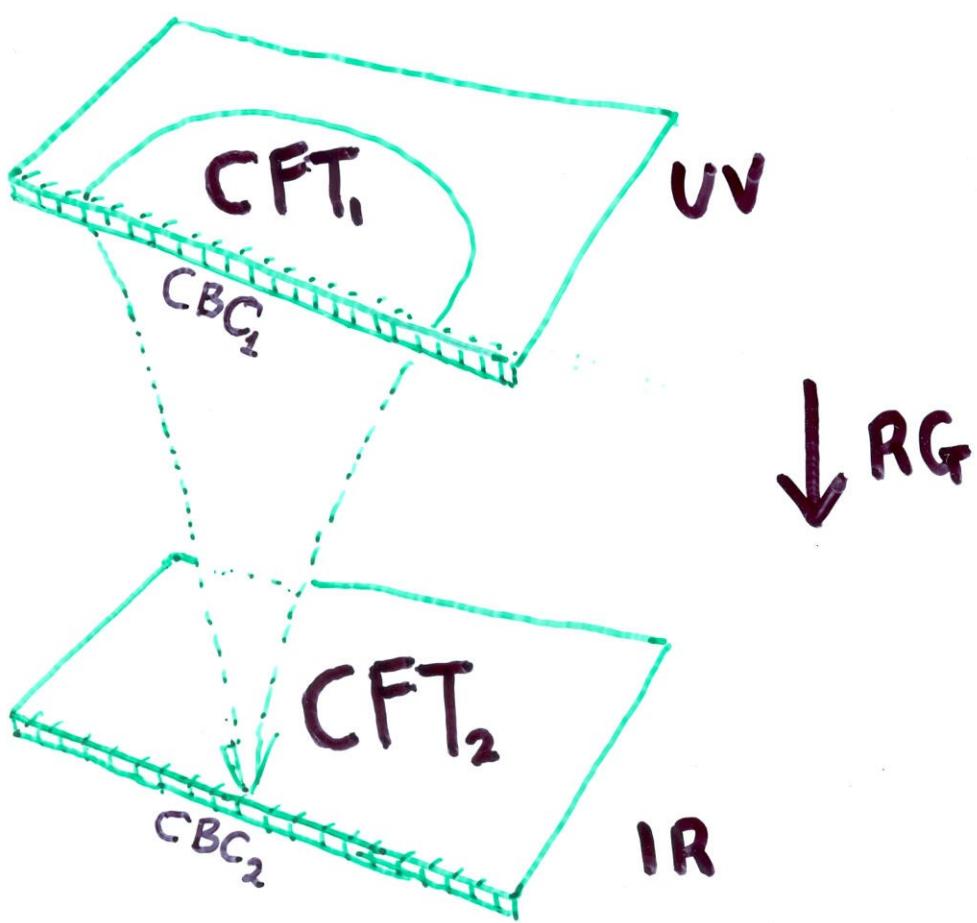
In 2d, starting from work by Sasha Zamolodchikov ~20 years ago, many examples known of QFTs which interpolate between different CFTs (RG fixed pts) in UV & IR:



$$A_{\text{PCFT}} = A_{\text{CFT}_1} + \int \phi(x, \bar{x}) d^2x$$

bulk coupling
relevant bulk field,
in the spectrum of
the UV theory CFT_1

If such a theory is placed on a manifold with a boundary, then the boundary conditions must also flow:



At the UV fixed point the boundary condition must be one of the Conformal Boundary Conditions of CFT_1 ; in the IR this must flow to a CBC of CFT_2 .

(no choice!)

- Thus for each bulk

RG flow $CFT_1 \rightarrow CFT_2$ (A)

we expect a map between the sets

$$\{CBC_s \text{ of } CFT_1\} \rightarrow \{CBC_s \text{ of } CFT_2\} \quad (\text{B})$$

- In fact there's more structure yet: the combined bulk-and-boundary theory is described by an action

$$A_{PBCFT} = A_{BCFT_1} + \lambda \int d^2x \phi(x, z) + \mu \int dx \phi(x)$$

↑
 bulk coupling ↑
 bulk field ↑
 in CFT_1 (relevant)
 boundary field
 in CFT_1 with CBC,
 (if it exists)

boundary coupling

Even for $\lambda=0$ (no bulk flow), μ will induce flows between members of $\{CBC_s \text{ of } CFT_1\}$; when λ is switched back on these should mesh in with the combined bulk-and-boundary flows.

- Even though problem **(A)** has been well mapped out in many examples by now, the full (bulk-and-boundary) case of **(B)** is surprisingly tricky.
- Some extra mathematical tools to obtain exact information for integrable cases would be welcome.
- **Claim:** (& subject of the rest of this talk)
Exact off-critical g-functions* can help!
(* to be explained later...)

...but first a reminder...

Bulk tools

- Sasha Zamolodchikov's original discoveries exploited the fact that in the minimal models $M_{p,p+1}$, the (relevant) operator ϕ_{13} becomes more and more nearly marginal as $p \rightarrow \infty$.
- This allowed a nearby fixed point, reached by perturbing by ϕ_{13} , to be detected, perturbatively in $1/p$, in this limit (a bit like the ϵ expansion).
- To figure out which fixed point is reached, he computed (again, perturbatively) the value of his c -function* there.
(* C-function: an off-critical extension of the central charge, equal to it at fixed points)

The conclusion:

- For p large, there's a bulk

slow $M_{p,p+1} \rightarrow M_{p-1,p}$
"CFT₁" "CFT₂"

induced by the bulk $\phi_{13}(x, \bar{x})$ operator.

? Can one say more?

- The bulk ϕ_{13} perturbation is known to be integrable for all p , so one might hope for an exact treatment to settle the picture even away from the $p \rightarrow \infty$ limit.
- This was realised by Aliosha Zamolodchikov, who proposed to use the **TBA** technique on the problem.

But what is TBA, and what is it set up for?

- The answer depends on where you google...

à la commission

Blair ouvre son cabinet de consultants

TBA : Tony Blair Associates. C'est le nom du cabinet de conseil en stratégie économique et politique que l'ancien premier ministre britannique vient d'ouvrir, en marge de ses activités diplomatiques au Proche-Orient. Blair est déjà conseiller auprès de la banque JPMorgan et d'un groupe d'assurances suisse.

LE FIGARO·fr

Times of London 18/10/2009

Blair has been working pro bono in the Middle East as a peace envoy while amassing a fortune from the American lecture circuit. By offering himself to the Arab states as a statesman for hire, he could comfortably double his annual earnings.

His consultancy, the London-based Tony Blair Associates (TBA), emulates the New York partnership Kissinger Associates, which was founded by Henry Kissinger, the former national security adviser to President Nixon. One friend of Blair said: “TBA has been set up to make money from foreign governments and major companies. There’s a focus on the Middle East, because that’s where the money is.”

His expanding business interests as he roves across the Middle East means he flips his roles on a daily basis in official meetings: one hour, he is the official peace envoy meeting a Middle East minister or ruler; the next, he is a representative of TBA or JP Morgan. In some meetings with Arab states, where Blair is introduced as the peace envoy, he has been flanked by Jonathan Powell, his former chief of staff, who accepted a job with Morgan Stanley, another US investment bank, after leaving Downing Street. Powell has no role in the peace prc TBA and helps to win business in the Middle East.

Peter Brierley, 59, of Batley, West Yorkshire, whose 28-year-old son Shaun was killed in 2003 and who refused to shake Blair’s hand at a memorial service this m

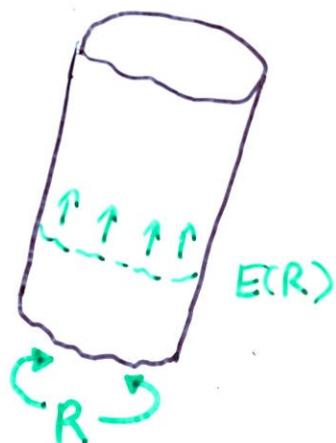
...but better today to decide that TBA stands for the **T**hermodynamic **B**ethe **A**natz, a technique whose application to relativistic 2d QFT (integrable) was pioneered by Aliosha.

- You can think of it as a "black box" which spits out a (different) off-critical c -junction, $C_{\text{eff}}(r)$, related to the ground-state energy $E(R)$ of the QFT on a circle of circumference R by

$$E(R) = -\frac{\pi}{6R} C_{\text{eff}}(r)$$

$(r = MR)$

↑
(mass/crossover
scale)



- At (unitary) fixed points $C_{\text{eff}} = \text{central charge (Cardy)}$; away from fixed points it provides an alternative to the (Sasha) Zamolodchikov c -junction.

- We'll concentrate on the furthest-from-perturbative $M_{p,\text{pt!}} \rightarrow M_{p\rightarrow p}$ flow, ie $p=4$, which would be expected to be a flow from the Tricritical Ising Model (TIM, M_{45} , $c=7/10$) to the (critical) Ising Model (IM, M_{34} , $c=1/2$):

$$\text{TIM}_{\text{UV}} \rightarrow \text{IM}_{\text{IR}}$$

- Aliosha proposed that $C_{\text{eff}}(r)$ for this flow could be found, exactly, by solving

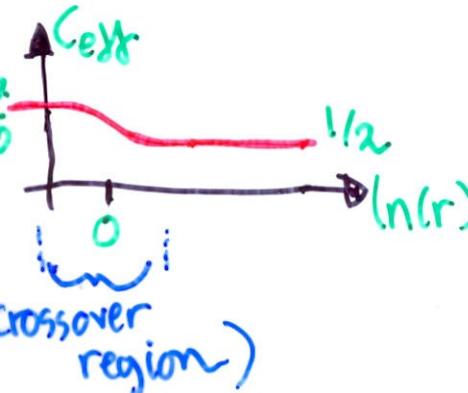
$$\mathcal{E}(\theta) = \frac{1}{2} r e^\theta - \int_R \phi(\theta + \theta') L(\theta') d\theta' \quad \leftarrow (\text{a TBA equation})$$

for $\mathcal{E}(\theta)$ (the pseudoenergy), where

$$\phi(\theta) = \frac{1}{2\pi \cosh(\theta)} \quad \text{and} \quad L(\theta) = \ln(1 + e^{-\mathcal{E}(\theta)}),$$

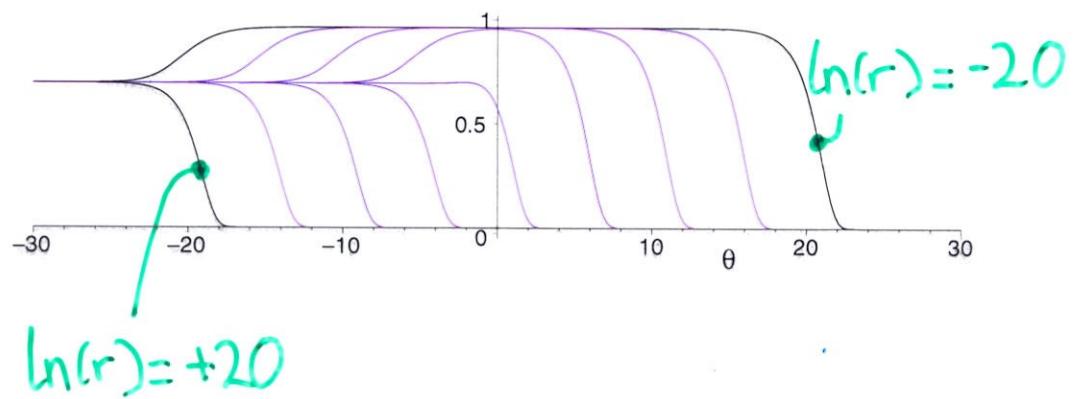
and then computing

$$C_{\text{eff}}(r) = \frac{3}{\pi^2} \int_R r e^\theta L(\theta) d\theta.$$



[Theme for later: C_{eff} is not the only thing that's encoded by the pseudoenergy.]

Plot of $\ln(1 + e^{-\epsilon(\theta)})$ at various values of $\ln(r)$:



... back to boundaries...

Boundary state of play

The boundary generalisation of Sasha Zamolodchikov's large- p computations is delicate, but was recently [0907] achieved by Fredenhagen, Gaberdiel and Schmidt-Colinet (FGS) [though, to get a full picture, FGS had to 'bolt on' some non-perturbative results for pure-boundary flows, even at $p \rightarrow \infty$.]

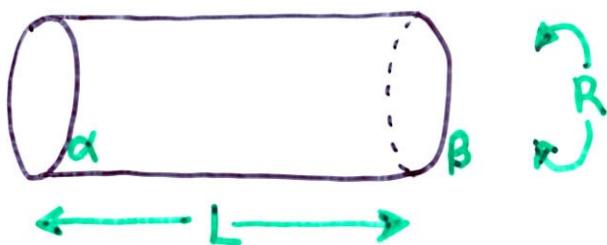
? How to identify destination boundary conditions?

- Don't just compute the c -function, but also the g -junction - an off-critical version of Affleck & Ludwig's g -junction, a useful "fingerprint" for conformal boundary conditions.

Aside: g -junctions in critical theories:

[Affleck & Ludwig 1991]

- Put a CFT on a cylinder of length L and circumference R with conformal boundary conditions α and β at the ends:



- The cylinder partition function in the limit $L \rightarrow \infty$ reveals interesting universal information:

$$\ln Z_{\alpha\beta} \sim \frac{\pi C}{6R} L + \text{constant}$$

"bulk" piece
(linear in L)

boundary bit - write it
as $\ln g_\alpha g_\beta$ and call g the
"universal noninteger ground state
degeneracy", or g -junction.

Notes: • $g_\alpha = \langle \alpha | 0 \rangle$ ~ cylinder ground state
 $\epsilon_{\text{boundary state}}$

- g can be written in terms of CFT data
- \exists "g-theorem" for pure-boundary flows. (g decreases)

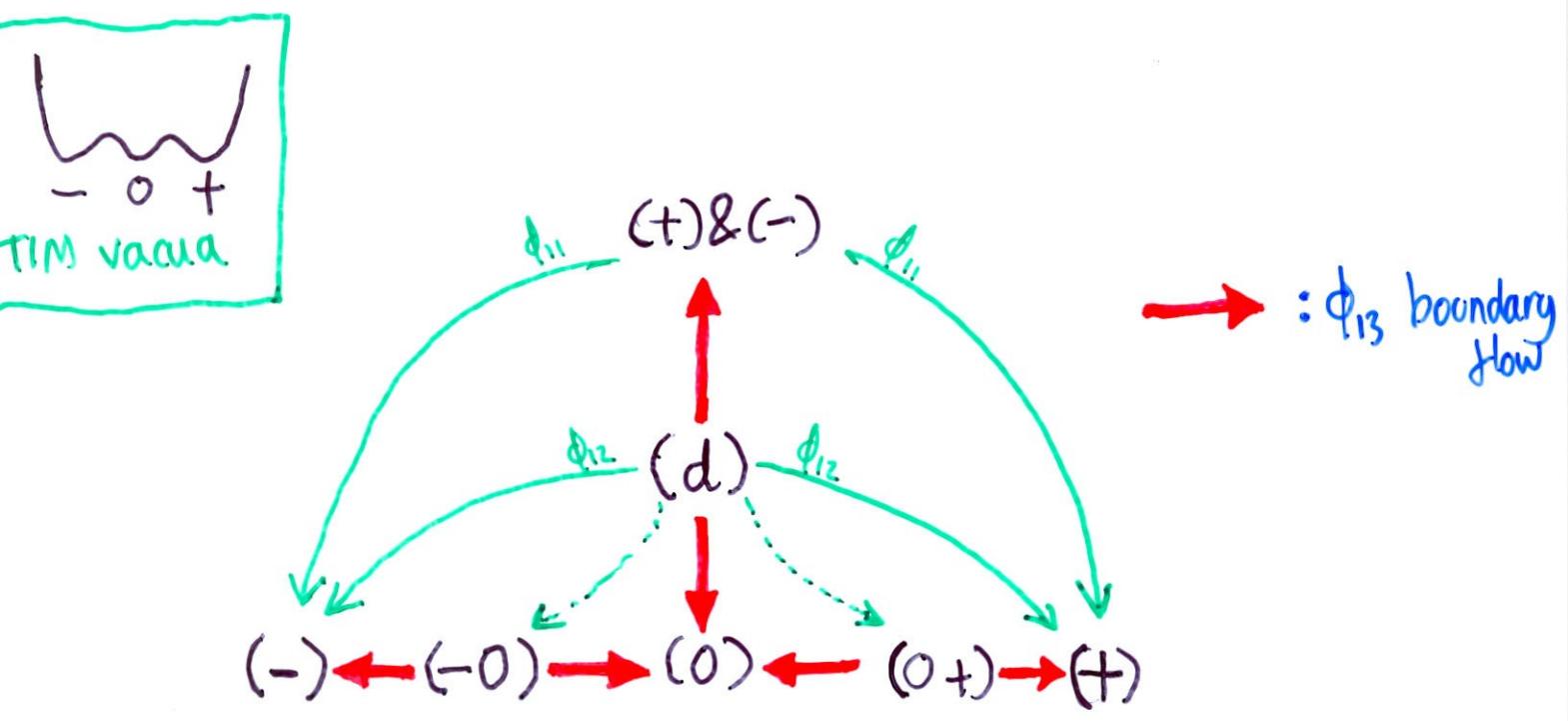
Exact boundary g off criticality.

- By analogy with Aliosha's work for the bulk $TIM \rightarrow IM$ flow, we want to find exact equations describing the evolution of g -functions during combined bulk-and-boundary perturbations which take the boundary TIM down to the boundary IM .

Note: to keep integrability with the bulk perturbed by $\phi_{13}(x, \bar{x})$, the boundary perturbation must be $\phi_{13}(x)$, the corresponding boundary field, if it exists at all for the chosen initial conformal boundary condition. So we're looking at

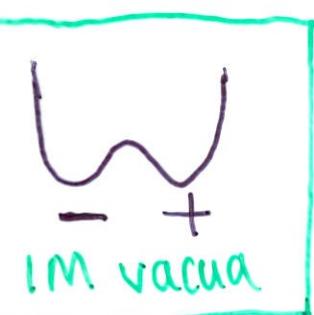
$$A_{PBCFT} = A_{\substack{\text{bdry TIM,} \\ \text{with CBC (a)}}} + \gamma \int \phi_{13}(x, \bar{x}) d^2x + \mu \int \phi_{13}(x) dx$$

Prelude: • The $\lambda^{(T/M)}$ pure boundary ($\lambda=0$) cases are already well understood (Chim, Affleck ...)



[CBCs are roughly labelled by the vacua accessible at the boundary, with $(d) \equiv (-0+)$]

• Also, "downstairs", what can happen in Ising is known:



$$(-) \xleftarrow{\quad} (f) \xrightarrow{\quad} (+)$$

→ : ϕ_{13} bndry flow

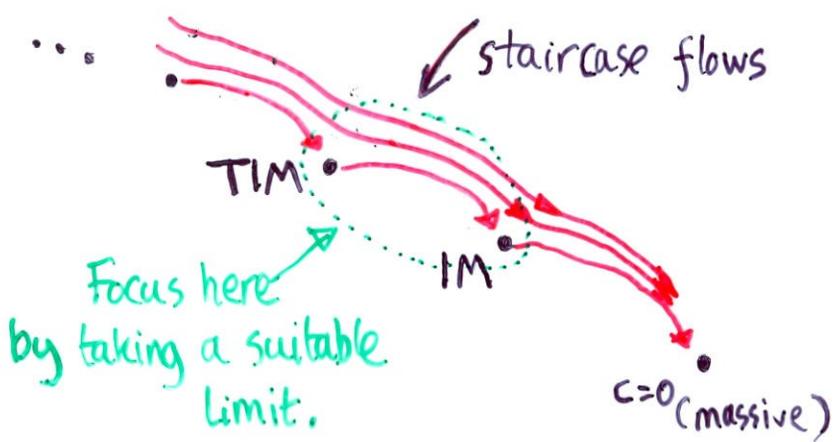
$$(+)&(-) \rightarrow (f)$$

[$(f) = \text{"free"} = (-+)$]

Question: how do these glue together for $\lambda \neq 0$?

Strategy:

1. We already know exact off-critical g-junction equations for cases where the bulk theory becomes entirely massive after perturbation [PED, DF, CR, RT 2004; PED, AL, CR, RT 2005]
2. Aliosha Zamolodchikov's staircase model shows how interpolating bulk theories can be obtained from an entirely massive theory in a suitable 'double scaling' limit.



NB: step 2 for bndry cases previously considered by Le sage-Saleur-Simonetti (98) but at that time, 1 was not known, so results weren't 100% successful.

Upshot: (one case)

- To find $g(r)$, first solve the bulk $T/M \rightarrow IM$ TBA equation for the pseudoenergy $\epsilon(\theta)$:

$$\boxed{\epsilon(\theta) = \frac{1}{2} r e^\theta - \int_R \phi(\theta + \theta') L(\theta') d\theta'}$$

$$(\phi(\theta) = \frac{1}{2\pi \cosh \theta}, L(\theta) = \ln(1 + e^{-\epsilon(\theta)}))$$

- Then, with θ_b a boundary parameter, $g(r)$ is given

by:

$$\ln g(r) = \ln g_b(r) + \ln g_o(r)$$

where

$$\ln(g_o(r)) = \sum_{j=1}^{\infty} \frac{1}{(2j-1)} \int_{R^{2j-1}} \frac{d\theta_1}{1 + e^{\epsilon(\theta)}} \cdots \frac{d\theta_{2j-1}}{1 + e^{\epsilon(\theta_{2j-1})}} \phi(\theta_1 + \theta_2) \dots \phi(\theta_{2j-1} + \theta)$$

and

$$\ln(g_b(r)) = -\frac{1}{2} \ln(2) + \int_R (\phi_b(\theta) - \phi(2\theta)) L(\theta) d\theta$$

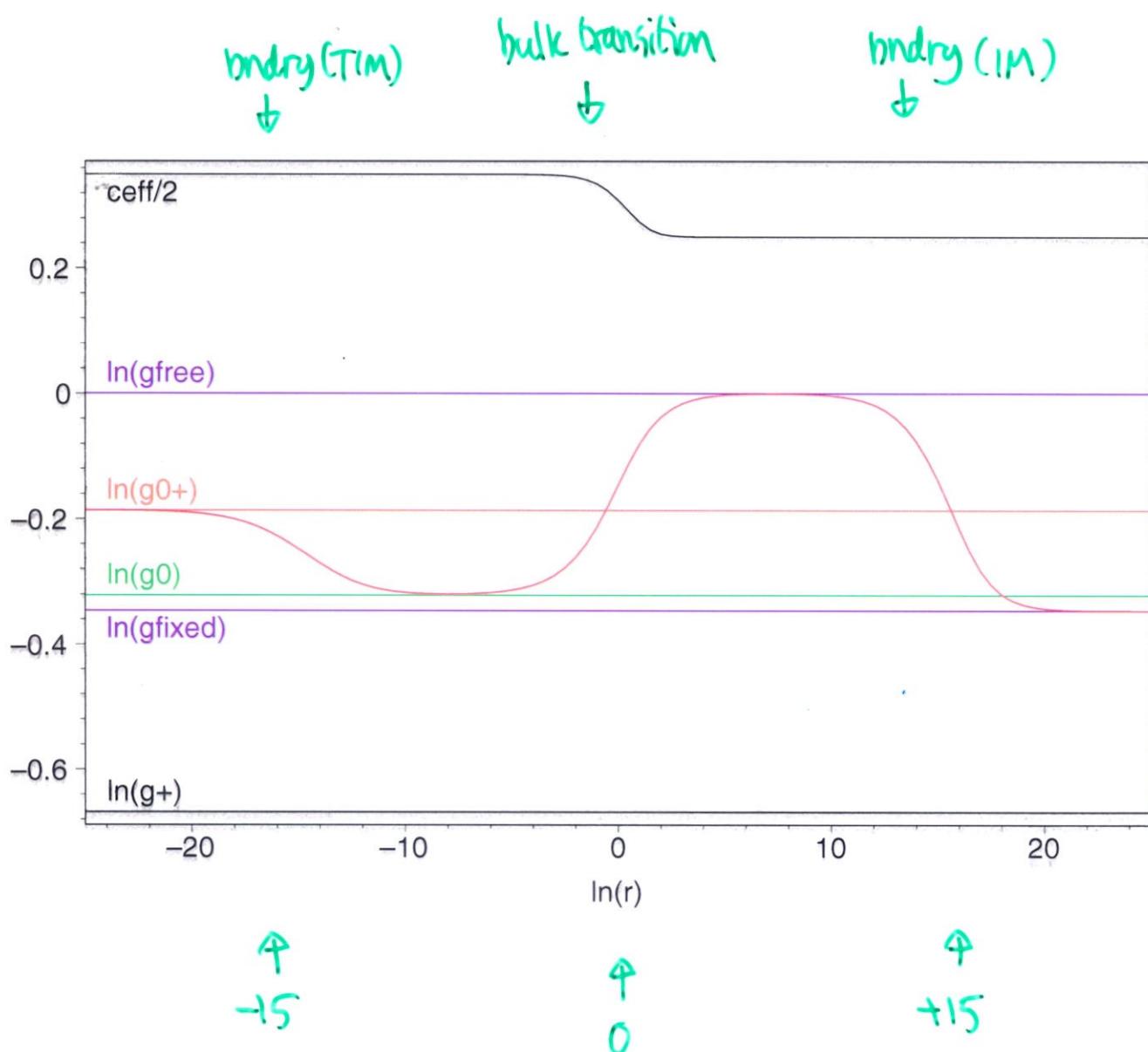
$$\text{with } \phi_b(\theta) = \phi_{(3/4)}(\theta) - \phi_{(1/2)}(\theta - \theta_b)$$

$$\text{and } \phi_{(x)}(\theta) = \frac{-\sin(\pi x)/(2\pi)}{\cosh(\theta) - \cos(\pi x)}.$$

Rather than give an analytic treatment, let's see some plots...

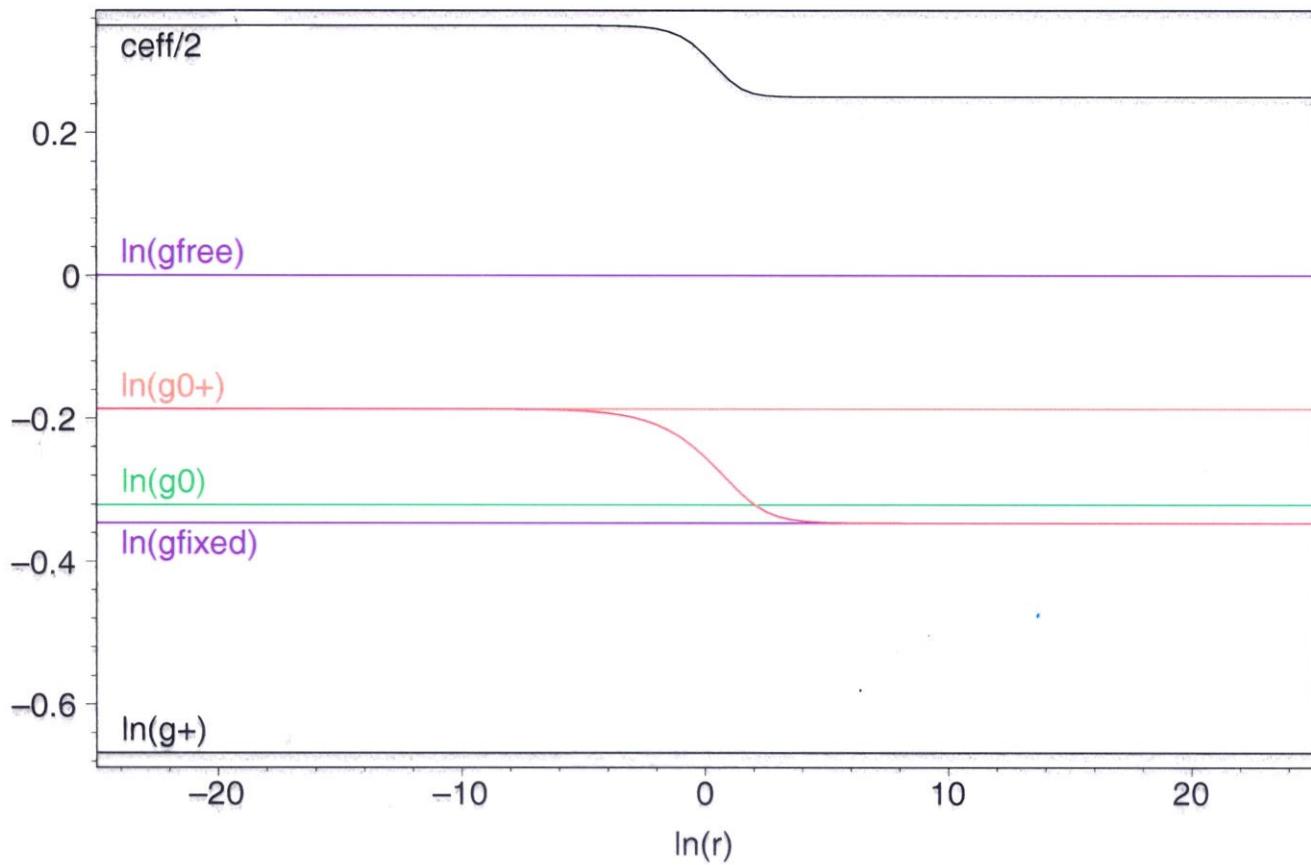
Exact g-function for TIM \rightarrow IM:

$$\Theta_b = -15$$

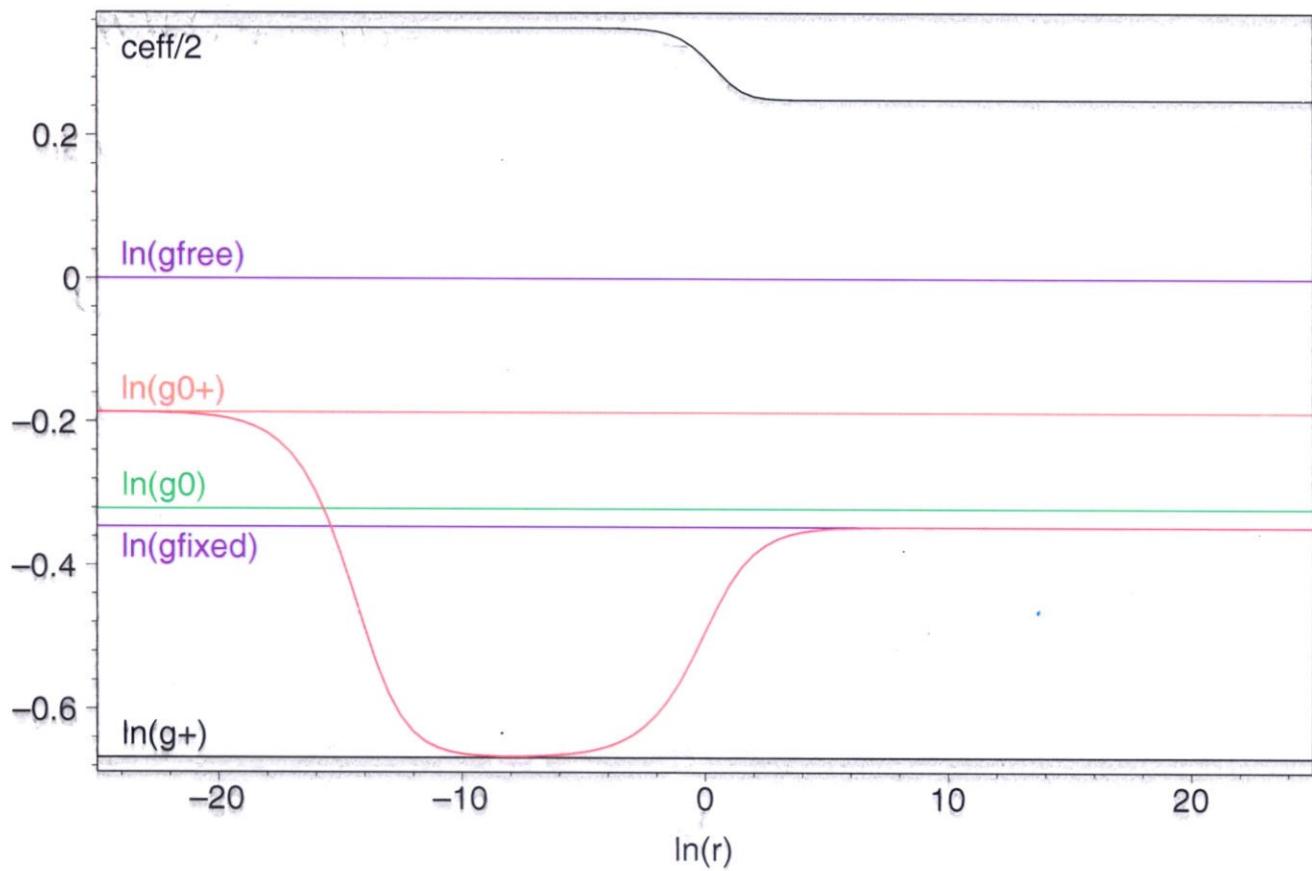


g_0^+, g_0, g^+ : TIM g-function values;
 $g_{\text{free}}, g_{\text{fixed}}$: IM g-function values.

$$\Theta_b = 0$$

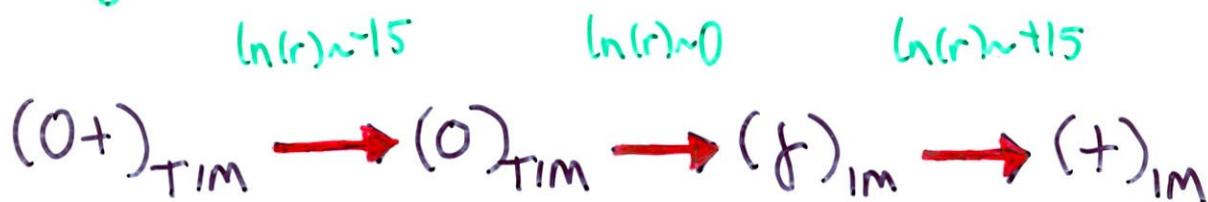


$$\Theta_b = +15$$



Interpretation: (of this case)

For $\Theta_b = -15$:



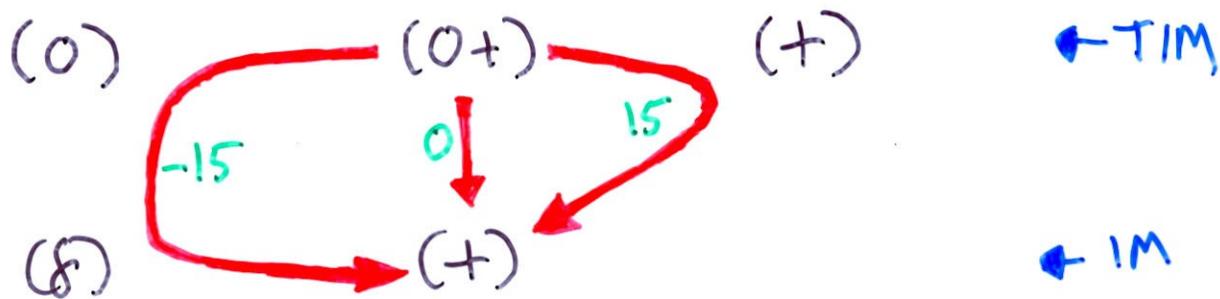
For $\Theta_b = 0$:



For $\Theta_b = +15$:



Put it all together:



This matches the pattern seen by FGS at large p , and certain limiting cases match predictions by Pearce, Chim & Ahn (2003) found by other means.

Conclusions:

- More precise and exact checks can be done – eg exact UV & intermediate resummations and checks against conformal perturbation theory.
So far the exact proposal passes all of them. But... is it correct?
(more tests would be worthwhile)
- Assuming it survives, this promises to be a powerful and relatively easy-to-use tool to map flows which are otherwise rather hard to treat.
- Start to see signs of a rich structure of flows, which should enable us to confirm perturbative predictions of FGS and then go further...

FIN.