Wilson polygon-amplitude duality; a way to integrability

Alexander Gorsky

ITEP

Paris, November 5, (A.Zhiboedov,A.G. 0904.0381 and to appear , A.G..0905.2058, V.Mikhailov, A.G. to appear)

Introduction

- Gauge theory equivalently is the theory of Wilson loops and to describe the amplitudes contours with cusps have to be considered (Polyakov 79-80)
- IR singularities of QCD amplitudes = UV singularities of the Wilson contours with cusps (Korchemsky-Radyushkin, 87)
- MHV amplitudes in N=4 SYM are in exact correspondence with Wilson polygons including the finite terms(Alday-Maldacena, 06)
- MHV amplitudes ($++$——) are the simplest objects to discuss within the gauge/string duality
- Holomorphy of trees - depends only on the "‘half"’ of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions on the sphere

- The tree MHV amplitude has very simple form

$$A(1^-,2^-,3^+\ldots,n^+) = g^{n-2}\frac{<12>^4}{<12><23>\cdots<n1>}$$

- The on-shell momenta of massless particle in the standard spinor notations read as $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors.
- Inner products in spinor notations $<\lambda_1,\lambda_2> = \epsilon_{ab}\lambda_1^a\lambda_2^b$ and $[\tilde{\lambda}_1\tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$.

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Twistor space - $CP(3\|4)$
- Twistor space emerges if we make a Fourier transform with respect to the "half" of the momentum variables $\int d\lambda e^{i\mu\lambda} f(\lambda\bar{\lambda})$. Point in the Minkowski space corresponds to the plane in the twistor space
- Localization follows from the holomorphic property of the tree MHV amplitude. Possible link to integrability via fermionic representation
- Stringy interpretation - auxiliary fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.

Properties of the loop MHV amplitudes

- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

- It involves only two main ingredients - one-loop amplitude and all-loop $\Gamma_{cusp}(\lambda)$
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively
- The conjecture fails perturbatively starting from six external legs at two loop (Bern-Dixon-Kosower, Drummond-Henn-Korchemsky-Sokachev), at large number of legs at strong coupling(Alday-Maldacena) and in the Regge limit (Lipatov-Bartels)

- One more all-loop conjecture - $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Wilson polygon built from the external light-like momenta p_i.
- The conjecture was formulated at strong coupling (Alday-Maldacena, 06) upon the T-duality at the worldsheet of the string in the AdS_5 geometry
- Checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- Important role of Ward identities with respect to the dual conformal transformation in determination of the Wilson polygon (Drummond-Henn-Korchemsky-Sokachev). It fixes the form of the amplitudes at small number of legs
- The T-duality in the radial AdS direction supplemented by the fermionic T-duality is the symmetry of the sigma model (Berkovits- Maldacena, Beisert, Tseytlin, Wolf) hence it restricts the amplitudes.

Main Questions

- Is there fermionic representation of the loop MHV amplitudes and simultaneously the Wilson polygons similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability behind the amplitudes is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- What is the geometrical origin of the BDS conjecture, if any, which would suggest the way of its improvement?
- What is the physical origin of MHV amplitude-Wilson polygon duality?

MHV amplitude - Wilson loop correspondence

- It is possible to derive the correspondence at one-loop level (A.Zhiboedov, A.G.)
- Since one-loop answer is expressed it terms of 2 mass easy diagrams it is sufficient to get it for 2me box
- Start with D=4 2me box $\rightarrow$ introduce Feynman parametrization $\rightarrow$ integrate over momentum in the loop $\rightarrow$ make a change of variables$\rightarrow$ Wilson polygon in D=6
- The second ingredient of derivation - relation between D=6 and D=4 integrals (Tarasov, Bern-Dixon-Kosower)

- The change of variables is quite simple

$$
\begin{aligned}
x_1 &= \sigma_1(1-\tau_1) \qquad (1)\\
x_2 &= \sigma_1\tau_1\\
x_3 &= \sigma_2\tau_2\\
x_4 &= \sigma_2(1-\tau_2)\\
\left|\frac{\partial(x_i)}{\partial(\sigma_i,\tau_i)}\right| &= \sigma_1\sigma_2
\end{aligned}
$$

- IR divergence in the amplitude explicitly get mapped into UV divergence of the Wilson polygons

$$
\begin{aligned}
d_{UV} + d_{IR} &= 10 \qquad (2)\\
\epsilon_{IR} &= -\epsilon_{UV}\\
(\mu_{UV}^2\pi)^{\epsilon_{UV}} &= (\mu_{IR}^2)^{\epsilon_{IR}}
\end{aligned}
$$

- Integrals over σ get factorized and two integrals over τ yield the integration in the one-loop Wilson polygon
- It is possible to get the duality between three-point function and Wilson triangle

$$\begin{aligned} d_{UV} + d_{IR} &= 8 \\ \epsilon_{IR} &= -\epsilon_{UV} \\ (\mu_{UV}^2 \pi)^{\epsilon_{UV}} &= (\mu_{IR}^2)^{\epsilon_{IR}} \end{aligned} \tag{3}$$

- In this case we have analogue of 2mass hard diagram and on the Wilson polygon side we have insertion of the particular vertex operator

$$< Tr\mathcal{P} q^{\mu} A_{\mu}(x_b) \exp[ig \oint_{\mathcal{C}} d\tau \dot{x}^{\mu}(\tau) A_{\mu}(x(\tau))] > \tag{4}$$

where q^{μ} can be chosen as be arbitrary vector which is not orthogonal to p_3 in Minkowski sense, $(p_3 q) \neq 0$.

- Generalization of one-loop derivation for 2mh, 3mh,4mh boxes is possible. Say, 4mh diagram has all massive external legs. The same steps work , Change of variables and relation between D and D-2 boxes

$$I_4(6+2\epsilon,m)=\frac{1}{(1+2\epsilon)z_0}(I_4(4+2\epsilon,m)-\sum_{i=1}^{4}z_iI_3(4+2\epsilon,m;1-\delta_{ki})) \tag{5}$$

where

$$\begin{aligned} z_0=\sum_{i=1}^{4}z_i &= 2\frac{s+u-4m^2}{su-4m^4} \\ z_1 &= z_3=\frac{u-2m^2}{su-4m^4} \\ z_2 &= z_4=\frac{s-2m^2}{su-4m^4} \end{aligned} \tag{6}$$

- In the case of the 4mh box the answer is IR convergent and the relation similar to the 2me case reads as follows

$$\frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)}\mathcal{W} = \mathcal{M}_4 \leftrightarrow \frac{1}{1-\frac{4m^4}{su}}\mathcal{W} = \mathcal{M}_4 \qquad (7)$$

- However the interpretation of the dual object is more complicated. Instead of "Wilson polygon"- "Wilson simplex" when we add one additional point and relate it by a propagator with the "polygon". Looks like the correlator $< W(C)O >$ where O is the local operator and $W(C)$ is the Wilson polygon.

BDS anzatz, fermionic representation of amplitudes and Integrability

- Conjecture (A.G. 0905) All-loop MHV amplitude is the peculiar correlator in the discrete Liouville theory or equivalently the function on the Teichmueller space
- Natural framework -topological strings and effective gravity in the target space description. Effective target space description - fermions on the Riemann surface.
- The "fermions" represent the proper branes. Lagrangian branes in the Kahler gravity description of A-model. Noncompact branes in the Kodaira-Spencer description of B-model. FZZT branes in the Liouville. IR regulators in YM!

- The generating function for the amplitude is expected to have the structure

$$\tau(t_k) =< 0|exp(\sum t_k V_k) exp \int (\bar{\psi} A \psi) exp(\sum t_{-k} V_{-k})|0 > \tag{8}$$

- That is scattering amplitudes can be described in terms of the fermionic currents on the Fermi surface
- Riemann Fermi surface reflects the hidden moduli space of the theory (chiral ring) and it gets quantized. Equation of the Riemann surface becomes the operator acting on the wave function(the analogue of the secondary quantization). The following commutation relation is implied for the "Fermi surface" $P(x,y) = 0$

$$[x, y] = i\hbar$$

- This procedure of the quantization of the Riemann surface is familiar in the theory of integrable systems. Quantum Riemann surface =so-called Baxter eqution

$$P(x, \partial_x)Q(x) = 0$$

- Degrees of freedom on the Riemann surface - Kodaira-Spencer gravity reduced to two dimensions (Dijkgraaf-Vafa,07)
- Solution to the Baxter equation - wave function of the single separated variable - Lagrangian brane (Nekrasov-Rubtsov-A.G. 2000)
- Polynomial solution to the Baxter equation - Bethe equations for the roots x_i

$$Q(x) = \prod (x - x_i)$$

- Why moduli space? Naively we have set of external momenta which yield a set of points in the momentum space. This set of points provides the moduli space of the complex structures. More carefully - the marked points in the rapidity space yield the desired moduli space in the B model. Kahler modulus of the ideal tetrahedron-A model
- From the Feynman diagrams - integration over the loop Schwinger parameters in the first quantized language amounts to the integration over $M_{0,n}$ (Gopakumar. Aharony et.al)
- At strong coupling. To have the proper interpretation of the Wilson loop as the wave function the integration over the diffeomorphisms $F(s)$ of the contour is necessary (Polyakov). In the amplitude case infinite dimensional integral over F is reduced to finite dimensional integration at the vertexes.

- The discrete Liouville system is related to the Teichmueller space of the disc like surface with n-points at the boundary. The mapping class group generator is identified with the Hamiltonian of the discrete Liouville system (Faddeev-Kashaev-Teschner).
- Hence we can claim that the transition from the tree to loop amplitude involves the proper dressing by the discrete Liouville modes

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- Quantum mechanically there is operator of the "'duality"' K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "'quantum spectral curve operator"' Classically this curve looks as

$$e^u + e^v + 1 = 0$$

and gets transformed quantum mechanically into the Baxter equation

$$(e^{i\hbar\partial_v} + e^v + 1)Q(v) = 0$$

- The pair of Baxter equations for the discrete Liouville reads as (Kashaev-Teschner-Bytsko))

$$Q(x + ib^{\pm}/2) + (1 - e^{4\pi x b^{\pm}})^N Q(x - ib^{\pm}/2) = t(x)Q(x)$$

- Let us use the representation for the finite part of the one-loop amplitude as the sum of the following dilogariphms. The whole amplitude is expressed in terms of the sums of the so-called two easy-mass box functions

$$\sum_i \sum_r Li_2\left(1 - \frac{x^2_{i,i+r} x^2_{i+1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}\right), \qquad x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta of gluons

- One-loop amplitude with n-gluons is described in terms of the "fermions" living on the spectral curve=Fermi surface which is embedded into the four dimensional complex space! MHV loop amplitude - fermionic current correlator on the spectral curve. Fermi surface lives in the space $T^*M_{0,4}$
- BDS conjecture for all-loop answer=quasiclassics of the fermionic correlator with the identification

$$\hbar^{-1} = \Gamma_{cusp}(\lambda)$$

- Is any ground behind this identification?
- In the limit describing the operators with large Lorentz spin the ground state energy of the corresponding string O(6)(Alday-Maldacena) behaves as

$$E \propto \Gamma_{cusp} logS \propto TL$$

that is Γ_{cusp} plays the role of the effective tension when the boundary of the string worldsheet is light-like
- For the Wilson loop with cusps and without self-intersections the loop equations reads as

$$\Delta W(c) = \sum_i \Gamma_{cusp}(\lambda, \theta_i) W(c)$$

that is Γ_{cusp} plays the same role

- Link with AGT Conjecture
- For the conformal N=2 or N=4 SYM theory the Nekrasov partition function integrated over Coulomb branch moduli equals to the Liouville correlator for $N_f = 4, N_c = 2$ and $N = 2^*$

$$< V_1(0)V_2(1)V_3(\infty)V_4(q) >= \int da|Z(a, \epsilon_1, \epsilon_2, q)|^2$$

- $$< V_m >_{torus}= \int da|Z(a, \epsilon_1, \epsilon_2, m, q)|^2$$

- It is convenient to match integrable structures on the both sides of the correspondence(Mikhailov,A.G. to appear)
- First, matching on the level of partition functions. At the gauge side Superconformal Ward identity=Whitham dynamical system

$$\frac{\partial Z}{\partial \tau} = H_{Calogero} Z, \qquad \omega = da \wedge da_D$$

on the Liouville side KZB equations for the conformal block

$$\frac{\partial \Omega}{\partial \tau} = H_{Hitchin} \Omega$$

- Some intermediate step; from Liouville to the H_3^+ WZW model and generically maximal winding number (k-2) violating correlators(Teschner-Ribout)

$$< V_1......V_k >_{Liouv} = < 0|V_1.....V_k|k-2 >_{WZW}$$

- In the spherical case first lift the Liouville (or generally W_n) conformal block to WZW conformal block then use the relation between the WZW conformal block and Gaudin model via KZB and finally map the Gaudin at few sites to the integrable model with many degrees of freedom
- General role of integrability. For the fundamental string probe quantum self-consistency $\Leftrightarrow$ EOM for the bulk fields(Fradkin-Tseytlin)
- For the D-brane probe more complicated situation, since there is gauge theory on its worldvolume. To get self-consistency more severe conditions have to be imposed. Integrability provides the matching of the RG flows on the probe and in the bulk.
- Example of this kind (Nekrasov-Shatashvili,Dorey). Positions of the probe in the bulk are fixed by the Bethe-Anzatz equations in the worldsheet theory on the D-brane probe

- AGT conjecture for Wilson loops (Alday et.al, Drukker et.al.)

$$< W >_{SYM} \propto < L_1.....L_k >_{Liouv}$$

where L_i-are the length operators. That is MHV amplitude is the correlator of the product of the length operators in Teichmueller or Liouville (for SU(2)) theory.

$$Tr_{1/2} exp(\int A)$$

where A is $SL(2, R)$ connection.

- Correlator in the Liouville theory once again obeys the equation with respect to the variation of the moduli τ and kinematical moduli. This is the link with integrability at generic kinematic point. Multiple Wilson lines-statistical model.
- Cusp anomaly has a meaning of the vector in the Hilbert space in Teichmueller theory $\Gamma_{cusp}(\theta) \propto |\Psi>$

- Simplified case - the low-energy scattering in the Higgsed regime. The Heisenberg-Euler effective Lagrangian -generating function for the low-energy amplitudes. For N=4 theory

$$S_{eff} = \int_0^\infty e^{sm^2} \frac{ds}{s^3} \frac{f_1 f_2 s^2}{sinh(f_1 s) sinh(f_2 s)} (cosh(f_1 s) - cosh(f_2 s))^2$$

where f_1, f_2 are invariants of the external field

- It can be written as

$$< 1|e^{m^2 s}(L_1(s) - L_2(s))^2|2 >$$

and upon expanding the action in

$$F = \chi_+ + \chi_-$$

the $N^k MHV$ low-energy amplitudes are just the coefficients $A_{k,n-k}$ in front of the term $\chi_+^k \chi_-^{N-k}$

- Semi-classics for the solution to the equation of the quantized Fermi surface

$$\Psi(z,\hbar) = \int \frac{e^{ipz}}{p \times sinh(\pi p) sinh(\pi \hbar p)} dp$$

reduces to

$$\Psi(x) \rightarrow exp(\hbar^{-1} Li_2(x) + ...)$$

- Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.
- The quantum dilogariphm has the dual-symmetric form

$$\Psi(z,\hbar) = \frac{e_q(\omega)}{e_{\tilde{q}}(\tilde{\omega})}$$

where $e_q(z) = \prod(1 - zq^n)$

- It can be visualized as two "left" and "right" lattices

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|J(z_1)...J(z_n)exp(\psi_k A_{nk} \psi_n)|0 >$$

- The variables ψ_k are the modes of the fermion on the spectral curve and $J(z)$ is the fermionic current. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

Towards the Regge limit

- Candidates for reggeons - open strings between the IR regulator branes. These states have masses depending on the kinematical invariants
- Possible link with the Reggeon field theory

$$L_{int} = -\frac{1}{g}\partial_+ Pexp(-\frac{g}{2}\int_{-\infty}^{x^+} A_+ dx_-)\partial^2 V_-$$

$$-\frac{1}{g}\partial_- Pexp(-\frac{g}{2}\int_{-\infty}^{x^-} A_- dx_+)\partial^2 V_+$$

where x_+, x_- are the light-cone coordinates and A is the conventional gluon field. Reggeons are the sources for the Wilson lines in accordance with holographic approach.

Conclusion

- Wilson polygon-amplitude duality is derived at one loop for MHV case. Some duality for elements of NMHV amplitudes with more complicated dual object-Wilson simplex.
- The loop MHV amplitude as the correlator of the fermionic currents representing regulator degrees of freedom on the quantized Fermi surface is suggested.
- All-loop MHV amplitude is the correlator of the length operators in discrete Liouville theory or in the quantized Teichmueller theory
- Integrability pattern behind the generic MHV amplitudes via fermionic representation. Particular solutions to 3-KP integrable system which correspond to the Faddeev-Volkov model for discrete Liouville with the good S-duality properties. The corresponding statistical model with the positive weights is Bazhanov-Mangazeev-Segreev one (2007)

- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space with $\Gamma_{cusp}(\lambda)$ as the quantization parameter. Way to improve-take into account the cubic vertex (screening operator) on the world-sheet in the Kodaira-Spencer gravity and loops in the 2d theory. Hopefully this improves the matching with the Regge limit of the amplitudes lost in BDS anzatz

- The degrees of freedom responsible for the dual description of the gluon amplitudes - IR branes (M2 branes) providing the length operators in the Teichmueller theory. They are analogue of the D1-instantons (Witten) or IR branes of (Alday-Maldacena)
- Positions of the branes are fixed by the Bethe anzatz equations. Similarly extremization of the superpotential in the brane worldvolume theory yields their positions in the embedding space
- There are some candidates for the "reggeon" degrees of freedom - open strings between two regulator branes. They are analogue of "W-bosons" with masses depending on the momenta. This could explain the same universality class of the N=2 SQCD at $N_f = 2N_c$ and Reggeon Hamiltonian. The brane geometry is similar.