Physical aspects

of supergroup sigma models

a quick review related with integrability
and conformal invariance

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Supergroup sigma models (1+1=2)

(apart from string theory and the AdS/CFT conjecture)

play an important role in several problems of condensed matter `physics'

via the Efetov/Parisi Sourlas (79) SUSY trick (Brezin, Hikami, Wegner, Zirnbauer...)

- Geometrical problems: percolation, polymers (dilute/dense/theta/branched),...
- Non interacting disordered systems (random potentials): random bond Ising, integer quantum Hall effect (2+1), ...
- SUSY spin chains, graded Yang-Baxter equation, ...

but present fundamental oddities/ difficulties

- Strong non unitarity issues: logarithmic CFTs with indecomposable representations of chiral algebras

- and left right indecomposability

- continuous symmetry does not imply Kac Moody symmetry any longer

- Strong non unitarity issues: probabilities $p < 0$ or $p > 1$ appear in S matrix approach

a quick tour

Spontaneous symmetry breaking

- Supersphere sigma models: field

$$\phi \equiv (\phi_1, \ldots, \phi_{m+2n}, \psi_1, \ldots, \psi_{2n})$$

and invariant bilinear form

$$\phi \cdot \phi' = \sum \phi_a \phi'_a + \sum J_{\alpha\beta} \psi_\alpha \psi'_\beta$$

where $J_{\alpha\beta}$ is symplectic form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Unit supersphere

$$\phi . \phi = 1$$

Supersphere sigma model $\left(\frac{\mathrm{OSp}(m+2n|2n)}{\mathrm{OSp}(m-1+2n|2n)}\right)$ action

$$S = \frac{1}{2g_\sigma^2} \int d^2 x \partial_\mu \phi . \partial_\mu \phi$$

- Perturbative beta function depends only on m (to all orders)

$$\beta(g_\sigma^2) = (m-2)g_\sigma^4 + O(g_\sigma^6)$$

- In bosonic case $m > 2$ model flows to strong coupling, symmetry is restored at large distance (Mermin Wagner thm).

- In supersphere case $m < 2$ model flows to weak coupling, symmetry is spontaneously broken.

This gives rise to a Goldstone phase with logarithmically divergent correlation functions (eg to lowest order:

$$\langle\phi(R)\phi(0)\rangle = \left[1 - \mathrm{cst}\, g_\sigma^2 (m-2) \ln \frac{R}{a}\right]^{\frac{m-1}{m-2}}$$

(equivalently, this can be seen as a `metallic phase', in contrast with insulating/localized phase)

- Integrable spin chains based on the fundamental of $\mathrm{OSp}(m+2n|2n),\ m<2$ flow to weakly coupled sigma model.

This is in contrast with the ordinary (bosonic) case where they flow to WZW theories.

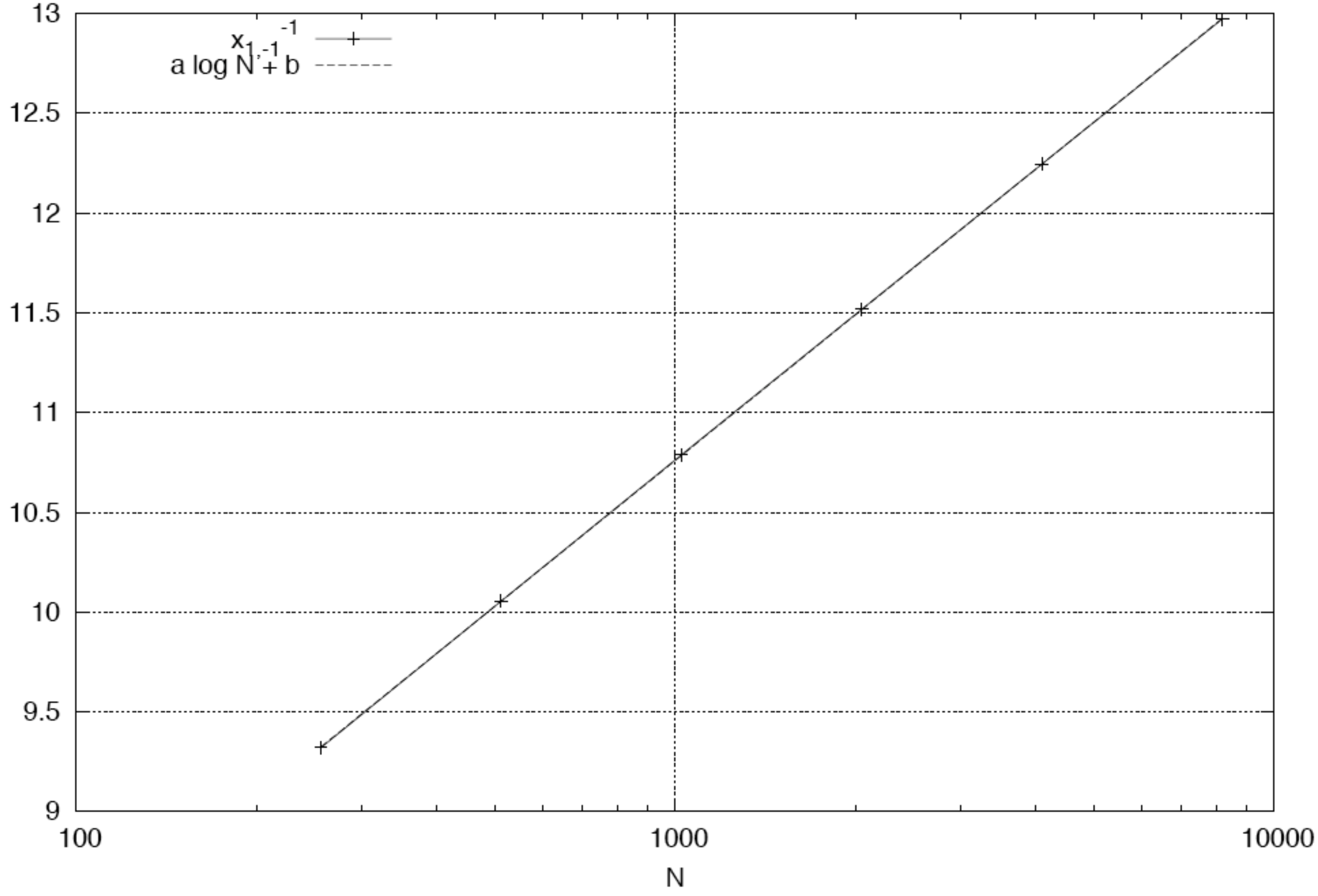
- First application: although the spin chains have a finite dim. representation on every site, the continuum limit is non compact. The effective size of the target space grows as the square root of the log of the system size. For instance for $\mathrm{OSp}(2|2)$ chain with Bethe equations

$$\left(\frac{\alpha-i}{\alpha+i}\right)^N=\prod\frac{\alpha-\beta-2i}{\alpha-\beta+2i}$$
$$\left(\frac{\beta-i}{\beta+i}\right)^N=\prod\frac{\beta-\alpha-2i}{\beta-\alpha+2i}$$

and energy

$$E=-\frac{4}{\pi}\sum\frac{1}{1+\alpha^2}+\frac{1}{1+\beta^2}$$

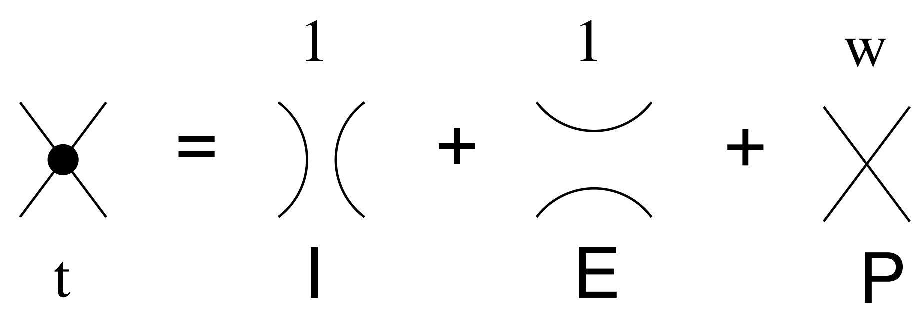
the lowest energy excitations correspond to a free boson with effective radius (deduced from beta function at m=0) $R^2 = 4 \ln L/a$



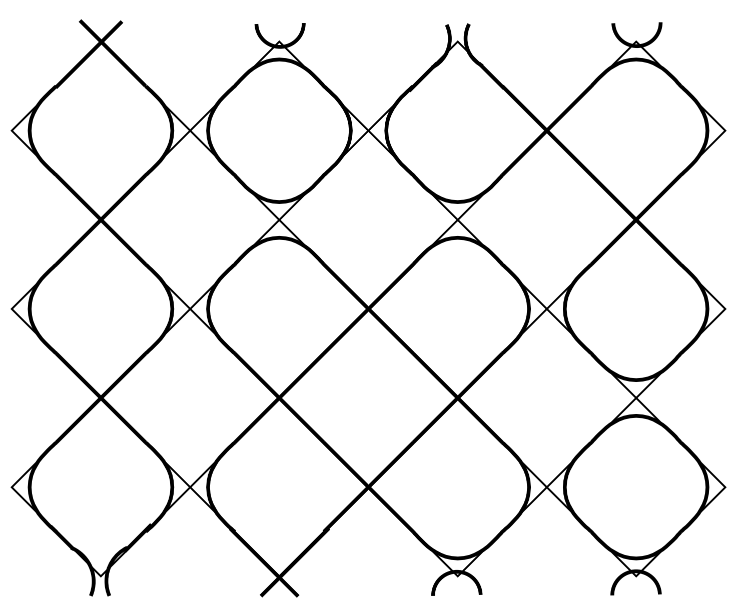
- Second application:

$$V \otimes V \;=\; 1 + \mathsf{Sym} + \mathsf{Antisym} \quad O(m)$$

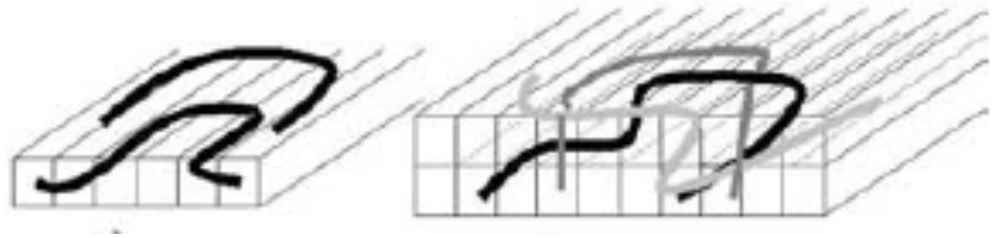
Spin chain=vertex=loop model with interaction vertices



Dense loops with fugacity m, intersecting at vertices,



The integrable case corresponds to a particular value of w but physics is independent of w positive (Goldstone phase).



Physical 2D polymer melts are not described by the `dense polymers' universality class (w=0): c=-1 instead of c=-2

Conformal invariance

- The case m=2 of $\mathrm{OSp}(2+2n|2n)$ provides the simplest example of conformal sigma model

- It is, for generic coupling g_σ^2, described by a non unitary, non rational CFT with continuous group symmetry, hence currents $J, \bar{J}$ which do not obey KM relations

- For `Neumann' boundary conditions (volume filling branes) and $\mathrm{OSp}(4|2)$

$$\mathcal{S}^{\mathrm{PCM}} = \frac{R^2}{2\pi} \int d^2 z \Big(2(1-\eta_1\eta_2)\left(\partial\eta_1\bar{\partial}\eta_2 - \partial\eta_2\bar{\partial}\eta_1\right) + (1-2\eta_1\eta_2)\left(\partial\varphi_1\bar{\partial}\varphi_1 + \cos^2\varphi_1\,\partial\varphi_2\bar{\partial}\varphi_2 + \sin^2\varphi_1\,\partial\varphi_3\bar{\partial}\varphi_3\right)\Big)$$

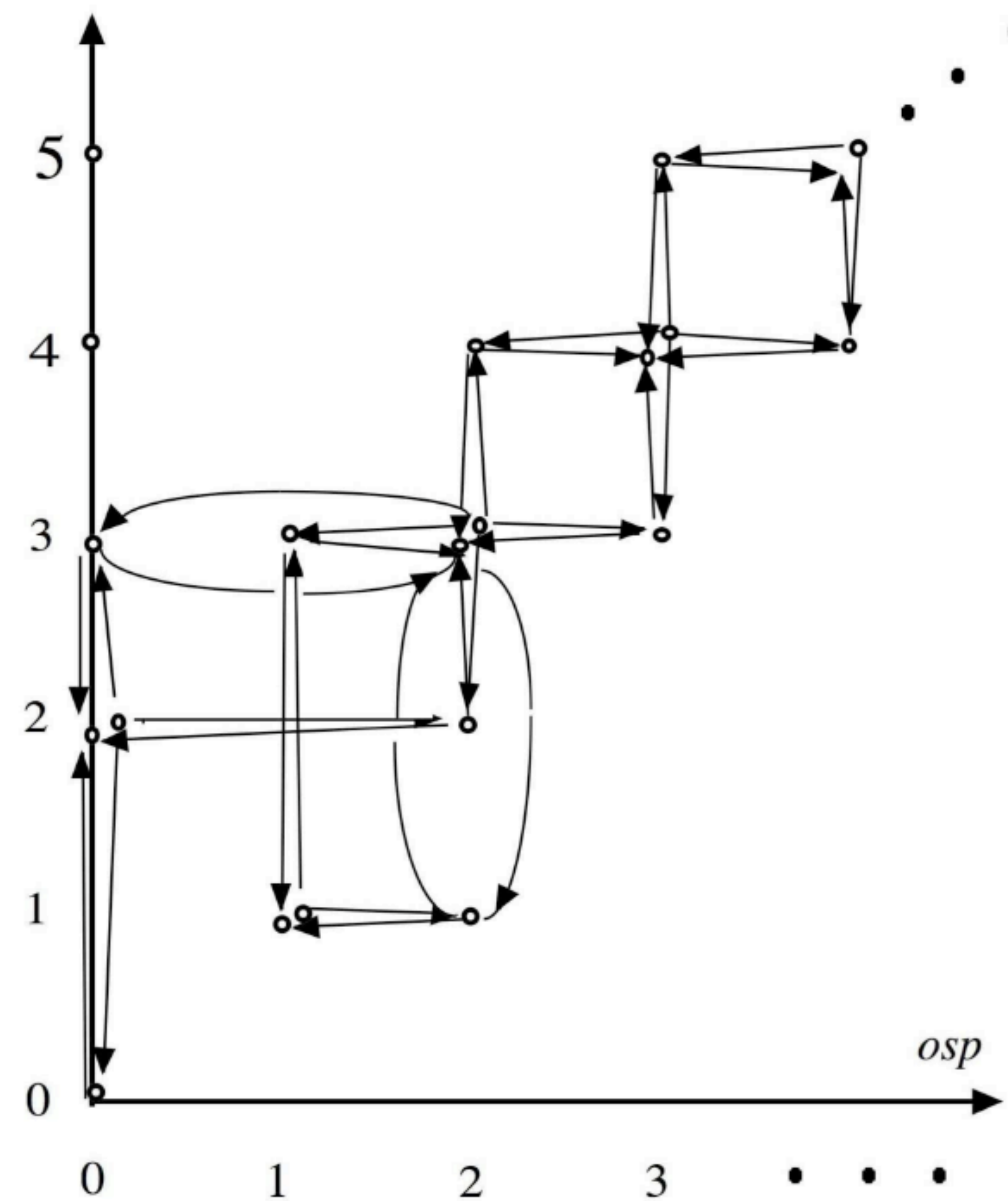
(line element on 3dim sphere $d\Omega_3 = d\varphi_1^2 + \cos^2\varphi_1\, d\varphi_2^2 + \sin^2\varphi_1\, d\varphi_3^2$)

the spectrum of the BCFT is quasiabelian

$$h(\mathcal{O}) = h_0(\mathcal{O}) + \frac{1}{2R^2} C_2(\Lambda)$$ for observable O in multiplet Λ

Of course at R=∞ the dimensions are all integers (fundamental field and its derivatives)

- The reps with vanishing Casimir are protected. Fields generate a chiral algebra with indecomposable reps. Theory has bimodule structure



- In O(2) case, sigma model and Gross Neveu model are dual. Now take $\mathrm{OSp}(4|2)$

$$S^{GN}_{g=0} = \frac{1}{2\pi}\int d^2 z \left[\sum_{i=1}^{4} \psi_i \bar{\partial}\psi_i + \bar{\psi}_i \partial \bar{\psi}_i + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}\right]$$

with

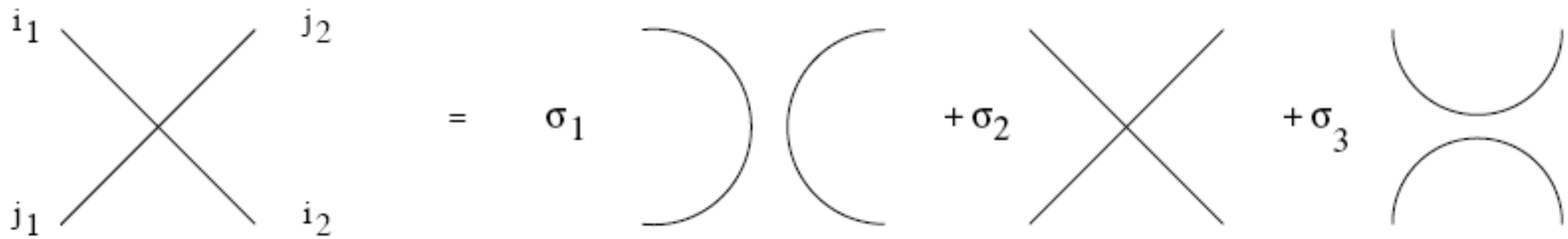
$$S^{int} = \frac{g^2}{2\pi}\int d^2 z \left[\sum_i \psi\bar{\psi}_i + \gamma\bar{\beta} - \beta\bar{\gamma}\right]^2$$

and $R^2 = 1 + g^2$. Claim: the spectra of the sigma and GN model are the same for these BC

strong coupling/weak coupling duality

S matrices and RG flows

- A priori massive directions for sigma and GN models follow from continuation of the bosonic case. Tensor structure of S matrix:



Crossing, unitarity and YB:

$$\sigma_1 = -\frac{2i\pi}{(N-2)(i\pi-\theta)}\,\sigma_2$$

$$\sigma_3 = -\frac{2i\pi}{(N-2)\theta}\,\sigma_2$$

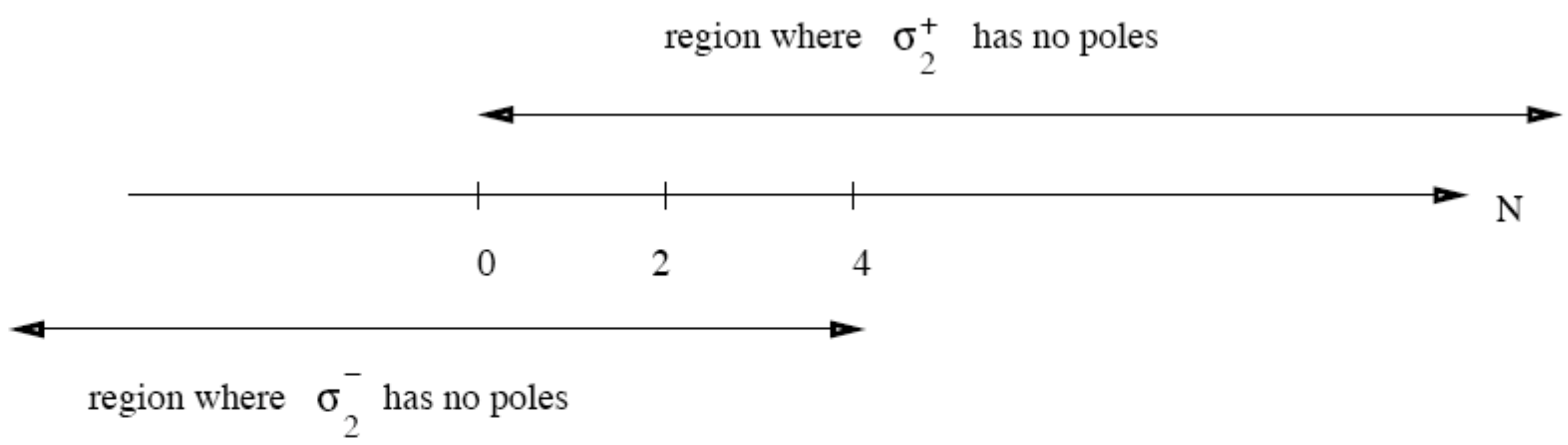
and:

$$\sigma_2^{\pm}(\theta) = \frac{\Gamma\left(1-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{\theta}{2i\pi}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\theta}{2i\pi}\right)}\frac{\Gamma\left(\pm\frac{1}{N-2}+\frac{\theta}{2i\pi}\right)}{\Gamma\left(1\pm\frac{1}{N-2}-\frac{\theta}{2i\pi}\right)}\frac{\Gamma\left(\frac{1}{2}\pm\frac{1}{N-2}-\frac{\theta}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}\pm\frac{1}{N-2}+\frac{\theta}{2i\pi}\right)}$$

σ_2^+ corresponds to sigma model,

σ_2^- to GN model. Valid for O(N), N generic, and particles in the fundamental

- There are strong indications this is true for OSp(N+2n|2n) as well



so that now it is the sigma model which exhibits bound states (for N negative enough).

- Yet difficulties lurk behind the surface

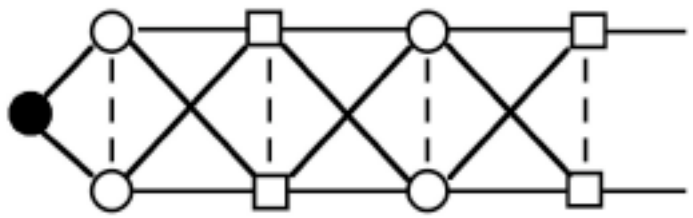
- Recall (m=N) $\beta(g_\sigma^2) = (m-2)g_\sigma^4 + O(g_\sigma^6)$

so for m<2, the massive direction corresponds to g_σ^2 negative, and the bosonic sector of the theory

$$S = \frac{1}{2g_\sigma^2}\int d^2x\partial_\mu\phi.\partial_\mu\phi$$

is naively unstable. Same holds for GN models.

- The S matrix approach might well lead to a definition via analytic continuation (as in `timelike bosons', `timelike Liouville'). For instance the $\mathrm{OSp}(2|2)$ GN model (Dirac fermion in a random scalar potential) leads to an interesting but stable TBA



with the result that, for the ground state energy in the antiperiodic sector

$$OSP(2/2)\ \mathrm{GN} \leftrightarrow SO(4)/SO(3)\ \mathrm{sigma\ Model} \equiv SU(2)\times SU(2)\ \mathrm{PCM}$$
$$OSP(2/2)/OSP(1/2)\ \mathrm{supersphere\ sigma\ Model} \leftrightarrow SO(4)\ \mathrm{GN} \equiv SU(2)\times SU(2)\ \mathrm{GN}$$

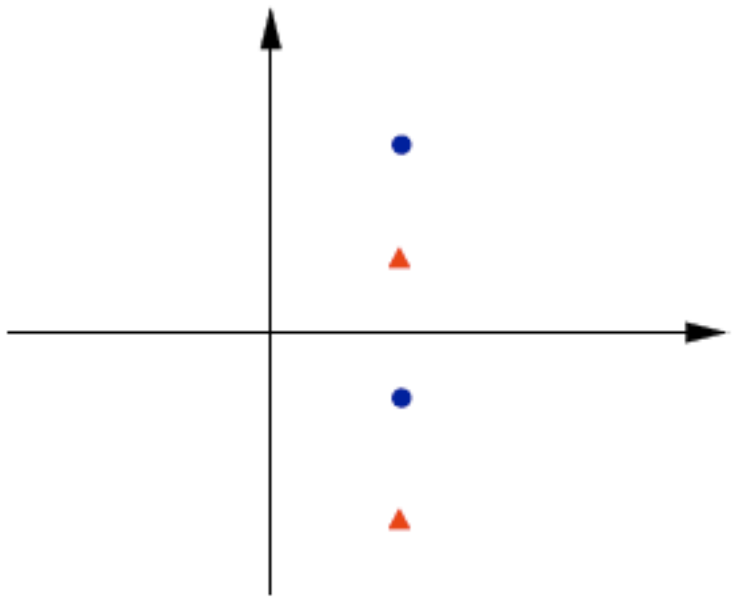
another example of GN/sigma model duality

The Bethe equations

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{\beta=1}^{M} \frac{u_j - \gamma_\beta + i}{u_j - \gamma_\beta - i}, \quad j = 1, \ldots, N$$

$$\left(\frac{\gamma_\alpha + i/2}{\gamma_\alpha - i/2}\right)^L = \prod_{k=1}^{N} \frac{\gamma_\alpha - u_k + i}{\gamma_\alpha - u_k - i}, \quad \alpha = 1, \ldots, M,$$

now admit `strange strings' which are not invariant under complex conjugation



and in most sectors the monodromy matrix has eigenvalues whose modulus is not unity.

- In the case of $\mathrm{OSp}(4|2)$

$$h(\mathcal{O}) = h_{WZW}(\mathcal{O}) + \frac{1}{2}\left(\frac{1}{R^2} - 1\right) C_2(\Lambda) \qquad h_{WZW}(\mathcal{O}) = \frac{n_1^2}{2} + \frac{n_2^2}{2} - \frac{b-2}{2}$$

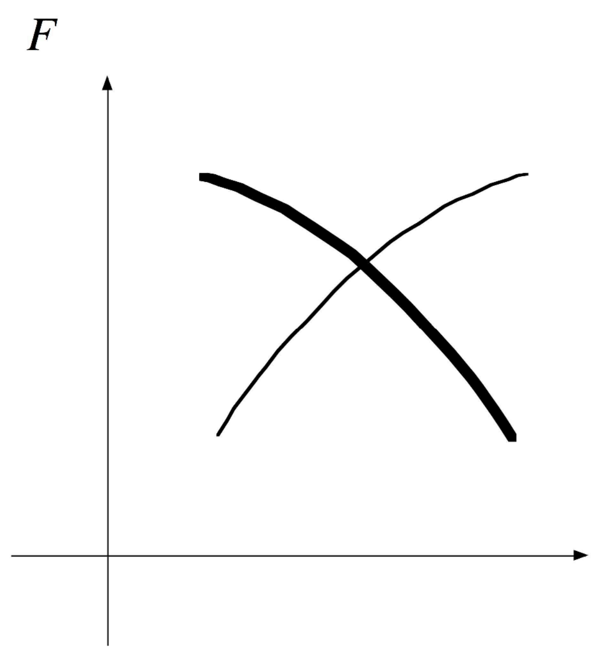
$$\Lambda = n_1 n_2 1^{b-2}$$

one has

$$C_2(\Lambda) = \frac{(n_1+n_2-1)^2}{2} + \frac{(n_1-n_2+1)^2}{2} - (b-1)^2$$

so in the `massive' direction $R^2 < 1$, theory becomes unstable in the b direction

- Seems to lead generally to first order critical points (critical points where two manifolds of free energy cross)



Conclusions

- Other kinds of sigma models: superprojective

$$U(n+m|n)/[U(1)\times U(n+m-1|n)] \cong \mathbf{CP}^{n+m-1|n}$$

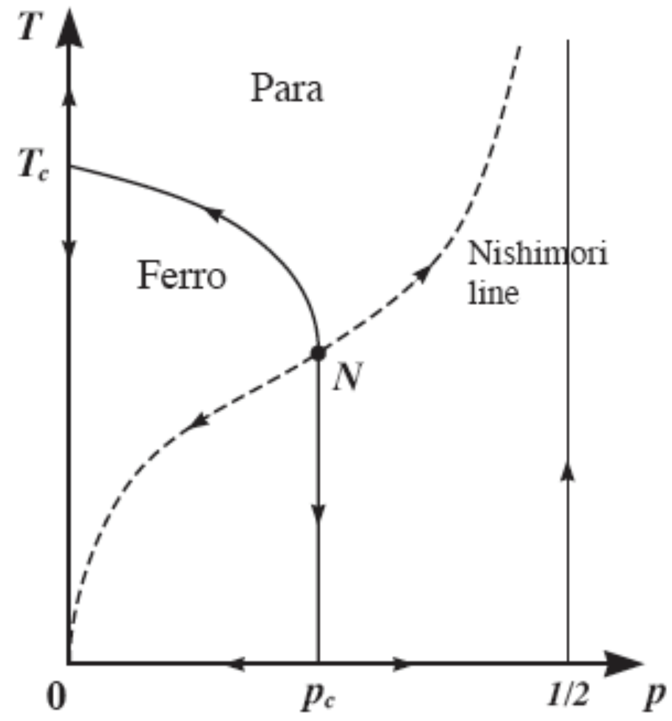
The case m=0 gives a line of CFTs at central charge c=-2

The case m=1and Θ=π describes hulls of percolation (in general the role of Θ terms and instantons in supermanifolds is intriguing)

- Other kinds of sigma models: non compact targets

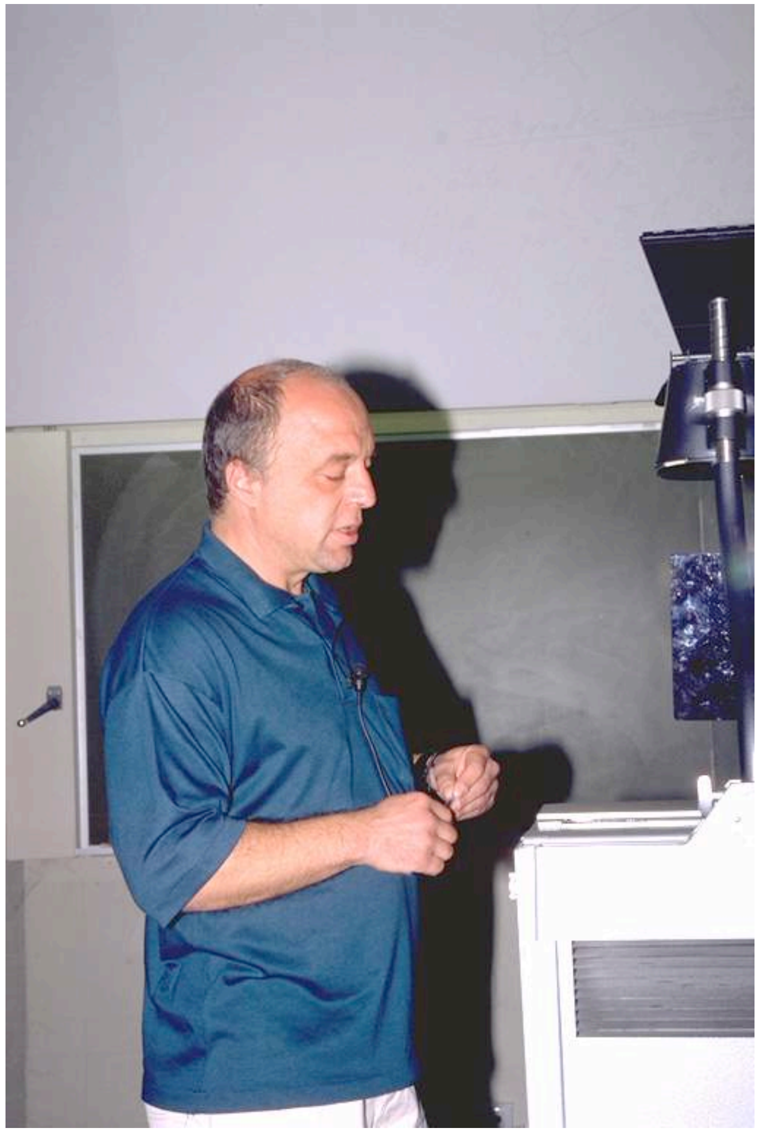
- Other applications: knot theory & topology

- Other kinds of questions: massless flows and disordered systems



RBIM: flow from N to Ising= flow from ???? to OSP(2|2) GN (in the massless direction)

exhibits Alyosha's monstron...



picture courtesy of C. Tracy

Thank you