On Confining Interactions in 1+1

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Conference in the Memory of Aliosha Zamolodchikov
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”It’s great that it [some result] is exact, but is it correct?”

Aliosha

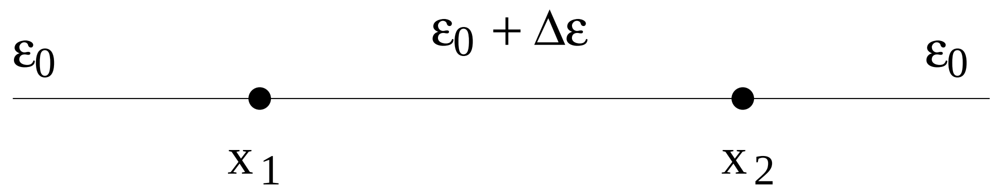
- P.Fonseca, AZ, 2001

- P.Fonseca, AZ; 2006

- V.Fateev, S.Lukyanov, AZ, 2009

Confinement is rather common phenomenon in 1+1 models

Its mechanism is relatively simple:



$$V = \Delta\varepsilon \ |x_1 - x_2|$$

Confining potential $\rightarrow$ Tower of "Meson" states (stable & resonances)

May occure due to:

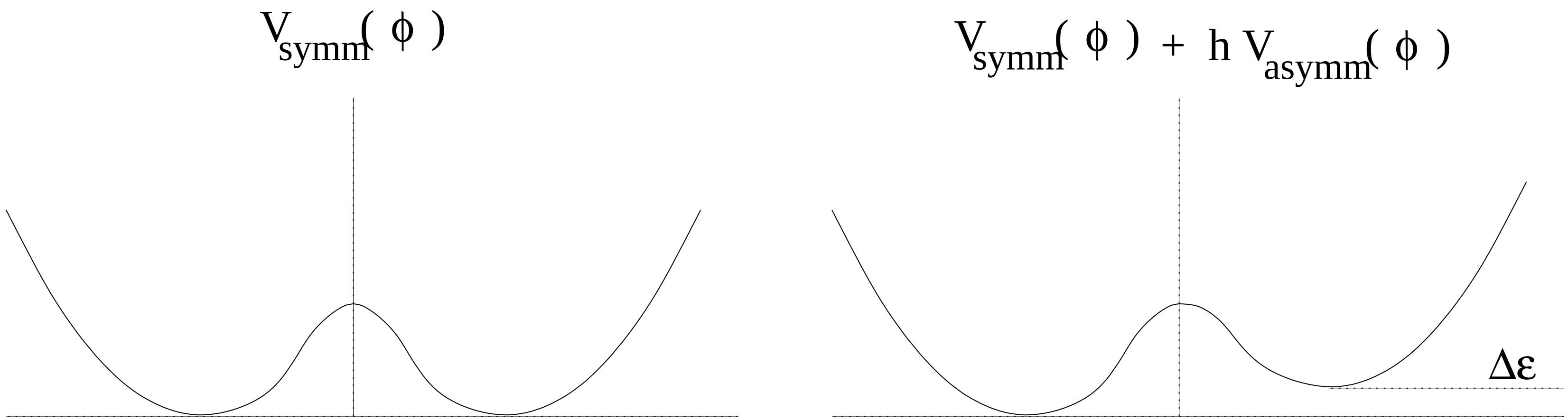
- Adding perturbation which lifts vacuum degeneracy from spontaneously broken symmetry; "Quarks" are domain walls.

- Presence of gauge field (abeelian or non-abelian), $\Delta\varepsilon \sim E$.

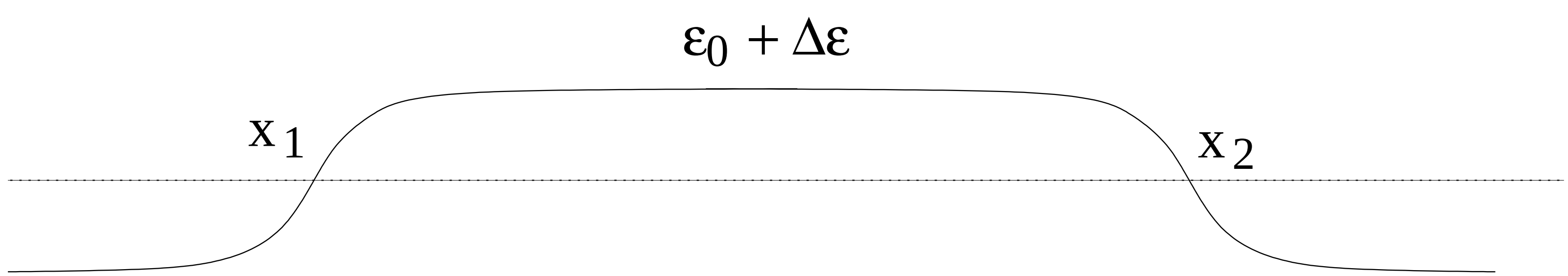
The two may be related through bosonization (QED_2)

Typical model:

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 - V(\phi)$$



Confining interaction between the kinks ("quarks")



Details may depend on specific model, but basic physics is controlled by one (dimensionless) parameter

$$\xi = \frac{\Delta\varepsilon}{m_q^2}$$

The "string tension" $\Delta\varepsilon$ may deepend on h (as well as on other parameters of the model), but at small h

$$\Delta\varepsilon \sim h$$

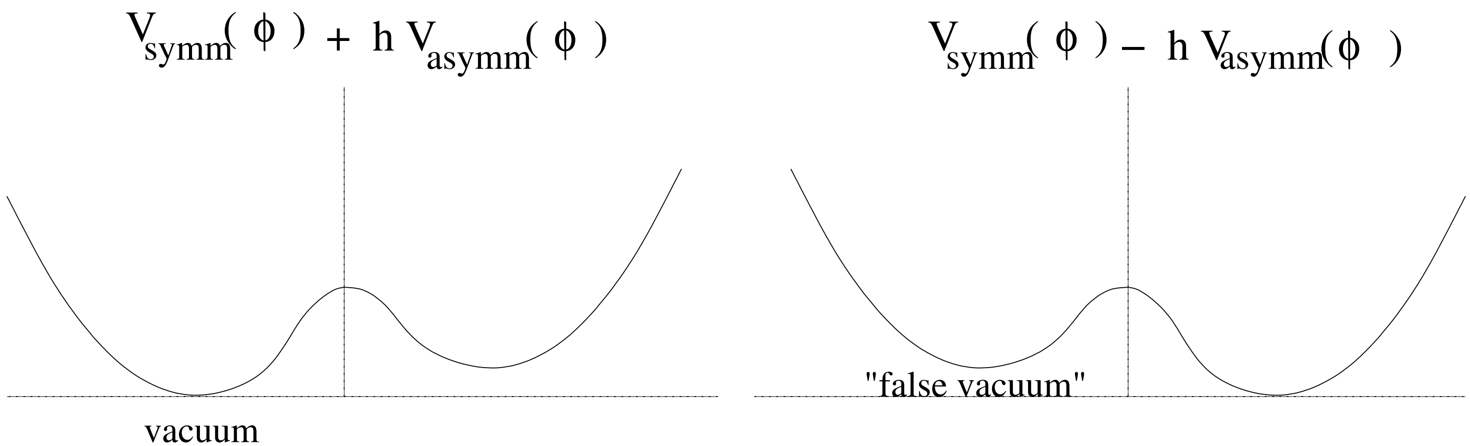
In physical system ξ is real and ≥ 0, but it is interesting to study analytic properties of physical quantities ($\varepsilon_0(\xi)$, $M_n(\xi)$, etc) as the functions of complex ξ.

Basic analytic features are expected to be universal, i.e. shared by all confining interactions in 1+1.

One is well-known: There is essential singularity at $\xi = 0$ (Andreev (1967), Fisher (1964), Langer (1967), Kobzarev, Okun, Voloshin (1975), Coleman (1977))

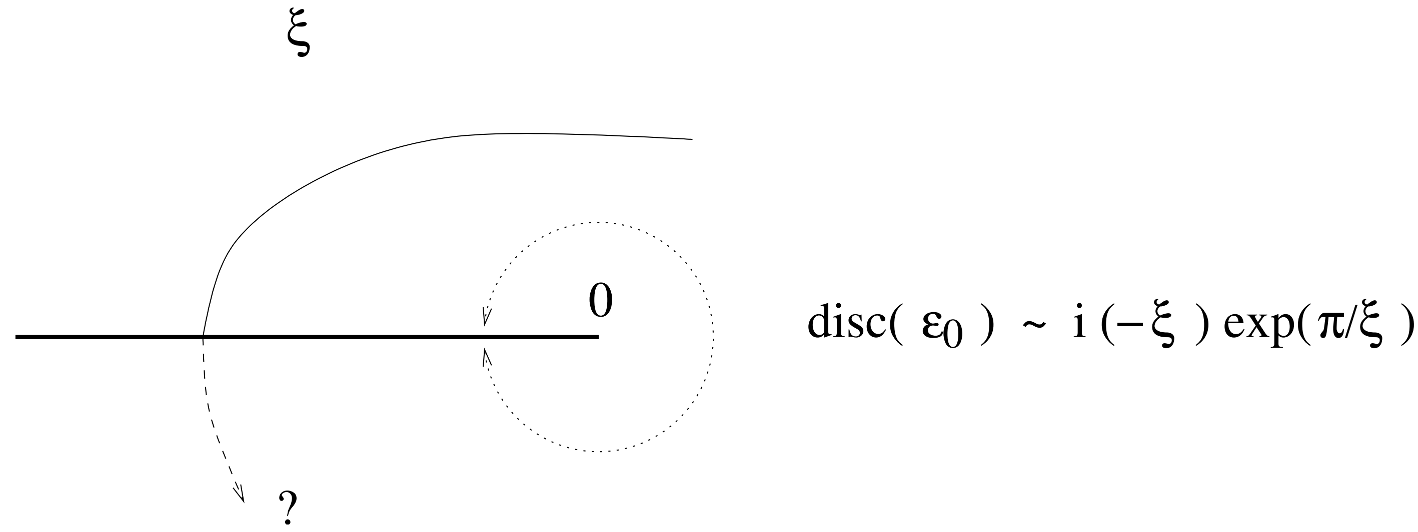
$$V_{\text{symm}}(\phi) + h\,V_{\text{asymm}}(\phi)$$

Analytic continuation to negative h turns vacuum into "false vacuum"



"False vacuum" decay:

$$\Im m\,\varepsilon_0(\xi) \sim (-\xi)\,e^{-\frac{\pi}{|\xi|}} \qquad \text{at} \quad \xi < 0\,.$$

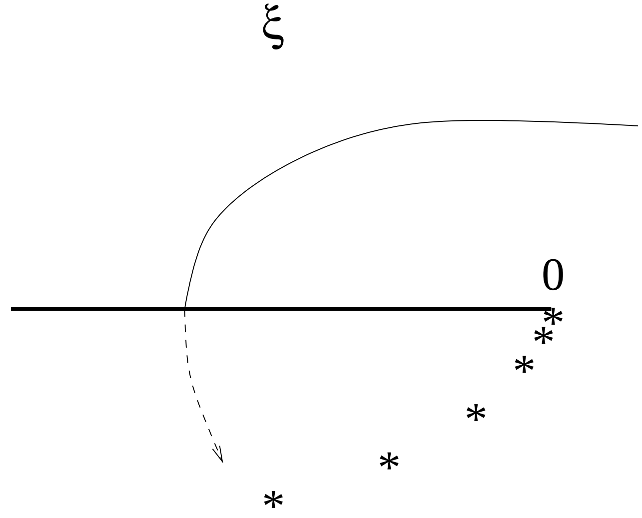


Do we encounter other singularities as we go <u>under</u> the branch cut?

Proposition:

- There are infinitely many singularities under the branch cut, accumulating towards $\xi = 0$.

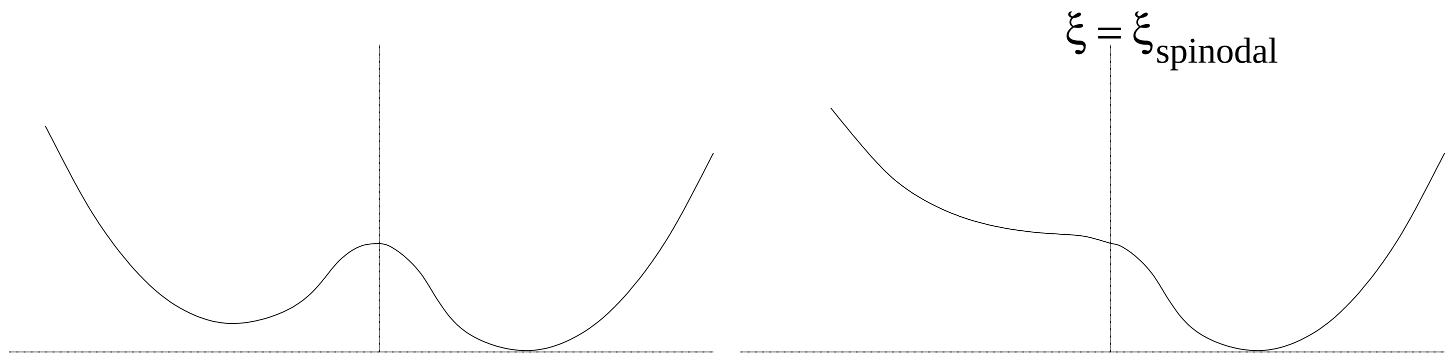
- The singularities are <u>critical points</u> (R_c diverges), with scaling, critical exponents, and all that.



Generally, physical nature of these singularities is yet to be understood.

Subject of this talk: Evidence for their presence.

Some complex singularities are expected: “Quantum spinodal”



Why infinitely many? Any one of the "meson" masses $M_n(\xi)$ may turn to zero at certain (complex) values of ξ.

Simple WKB analysis (generally valid at small ξ): Two quarks with linear confining interaction,

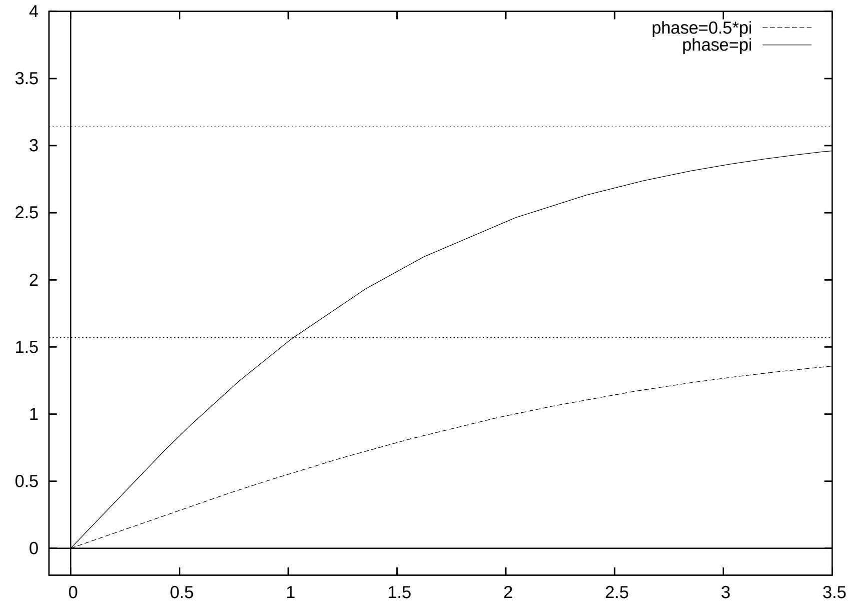
$$H = \omega(p_1) + \omega(p_2) + \Delta\varepsilon\ |x_1 - x_2|$$

Periodic motion $\rightarrow$ Quantization condition:

$$\mathrm{sinh}(2\theta) - 2\theta = \pi\,\xi\ (n + 1/2)\,, \qquad n = 0, 1, 2, ...$$

$$M_n = 2m_q\,\mathrm{cosh}\,\theta_n\,.$$

$$\xi = |\xi|\, e^{i\phi}$$



$M_n = 2m_q \cosh\theta_n$, turns to zero when

$$\theta_n \quad \text{hits} \quad i\pi/2$$

This happens at

$$\xi_n = \frac{e^{\frac{3\pi i}{2}}}{n + 1/2}$$

In fact, the leading WKB breaks down when θ gets close to $i\pi/2$.

$$\sinh(2\theta) - 2\theta = \pi\,\xi\ (n + 1/2) + \xi^2\ S_1(\theta) + \xi^3\ S_2(\theta) + ...$$

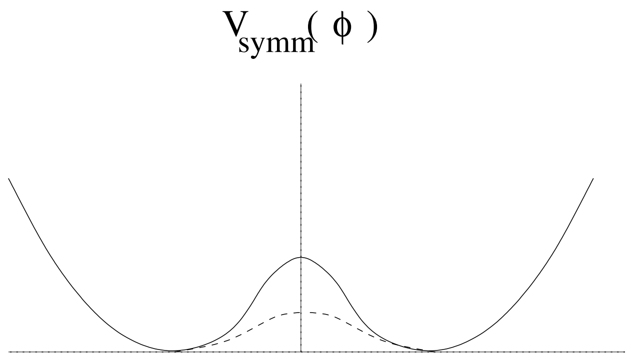
where $S_1(\theta), S_2(\theta), ...$ have poles at $\theta = i\pi/2$.

More elaborated approach is needed.

Two models:

- Ising Field Theory in a magnetic field
- QCD_2 at $N_c = \infty$

Ising Field Theory



–> Symmetry restoration transition

(Universality class of 2D Ising)

$$\mathcal{L} = -\bar{\psi}\,(\gamma\partial)\,\psi - m_q\,\bar{\psi}\psi - h\,\sigma$$

$\sigma(x)$- "spin field".

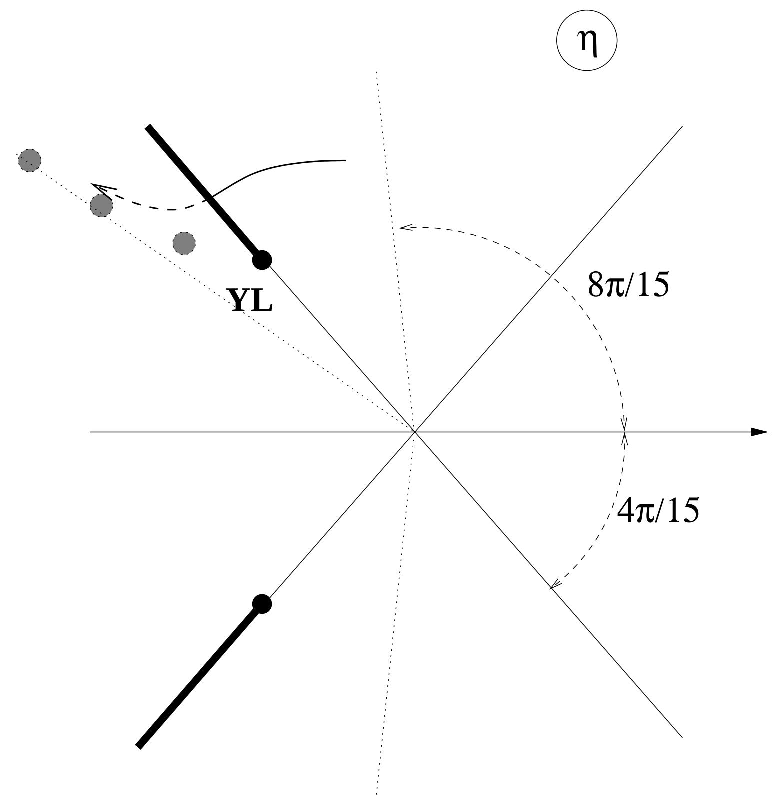
Spontaneous magnetization at $h = 0$,

$$\bar{\sigma} = \langle\,\sigma\,\rangle = \left(2^{1/12}\,e^{-1/8}\,A^{3/2}\right)\,m_q^{1/8}$$

At small h

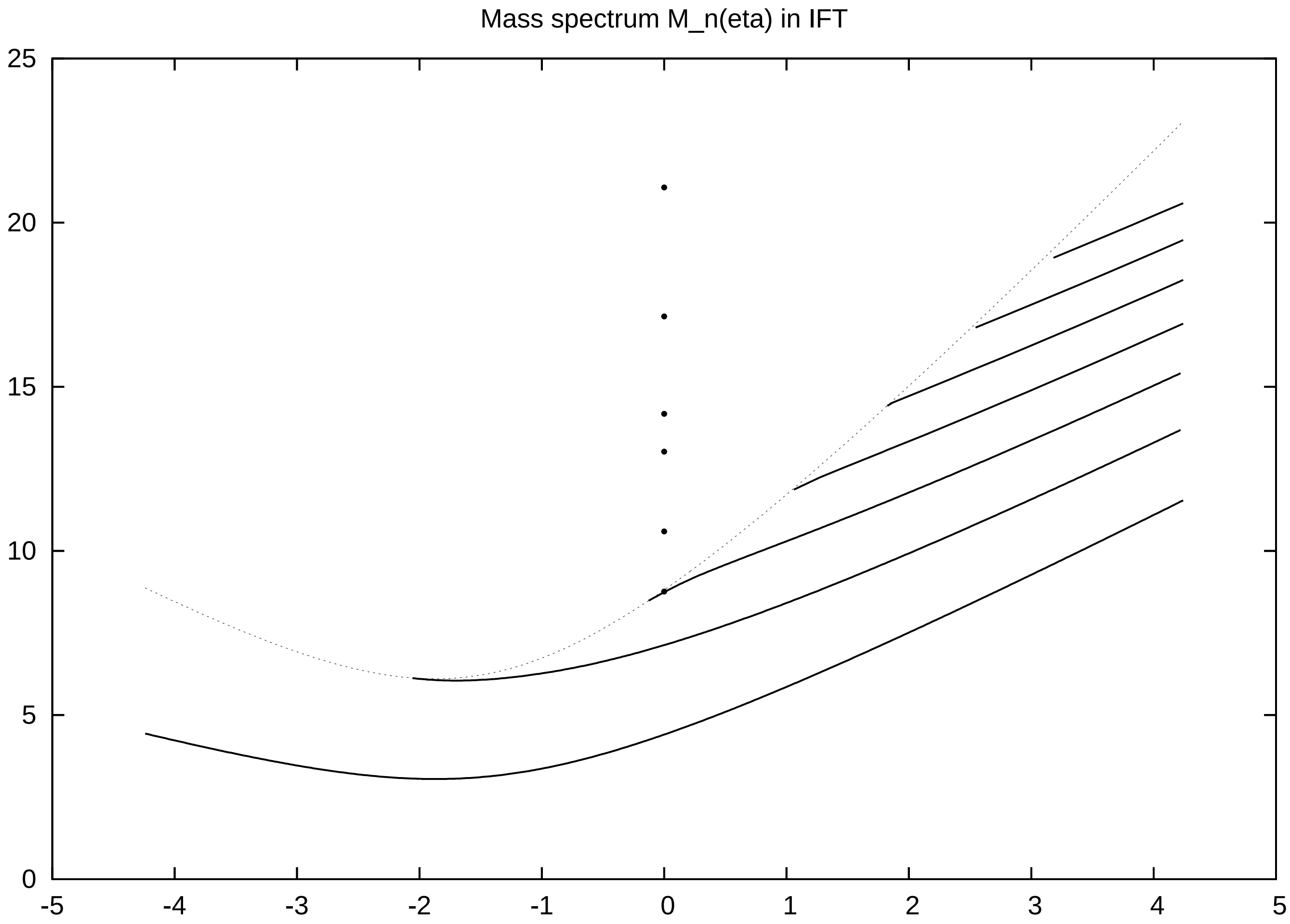
$$\Delta\varepsilon = 2h\,\bar{\sigma}$$

$$\eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}}$$



YL = "Quantum spinodal" $\Rightarrow$ 2D CFT with c $= -22/5$

Mass spectrum of IFT (numerical)



$$\eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}}$$

Magnetic field $h \to$ Confining interaction between the "quarks"

$$\Delta\varepsilon = 2\bar{\sigma}\, h + O(h^3)$$

Meson states

$$|\ M_n, P\,\rangle = \int \frac{dp}{2\pi}\,\Psi_n(P,p)\ \mathbf{a}^\dagger_{P+p}\mathbf{a}^\dagger_{P-p}\ |\ 0\,\rangle + ...$$

Weak coupling (small h):

Keeping some multi-quark terms, as needed for Lorentz invariance
$\Rightarrow$ Bethe-Salpeter equation

Rapidity variables:

$$P_+ + p_+ = m_q\, e^{\beta+\theta}\,, \qquad P_+ - p_+ = m_q\, e^{\beta-\theta}$$

Lorentz invariance: Ψ_n depends only on θ.

Bethe-Salpeter equation

$$\left[m_q^2 - \frac{M_n^2}{4\,\cosh^2\theta}\right]\Psi_n(\theta) = \Delta\varepsilon\;\int_{-\infty}^{\infty} G(\theta|\theta')\,\Psi_n(\theta')\,\frac{d\theta'}{2\pi}$$

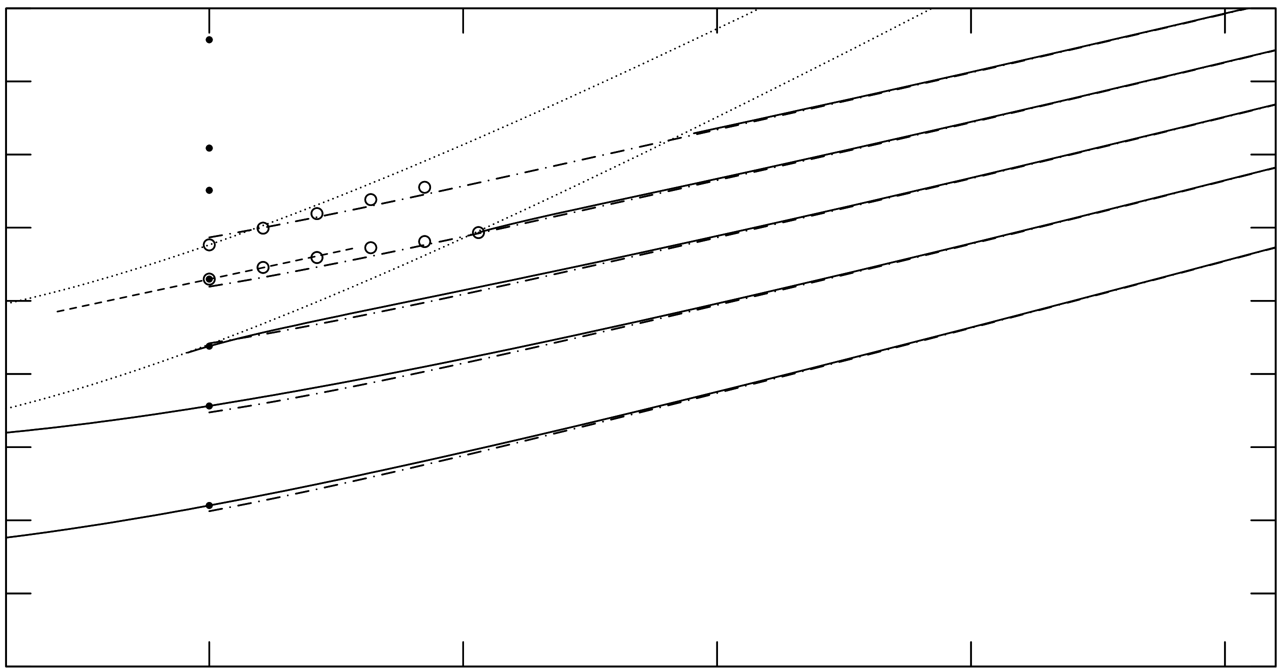
The kernel

$$G(\theta|\theta') = 2\,\frac{\cosh(\theta-\theta')}{\sinh^2(\theta-\theta')} + \frac{1}{4}\,\frac{\sinh\theta}{\cosh^2\theta}\,\frac{\sinh\theta'}{\cosh^2\theta'}$$

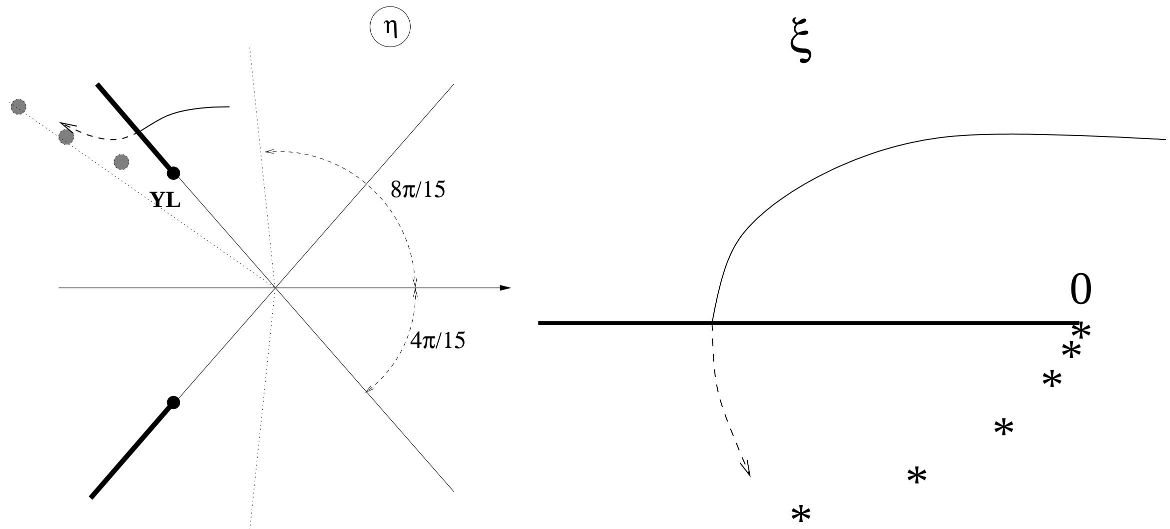
has second-order pole at $\theta = \theta' \rightarrow$ Confining interaction.

$\Rightarrow$ Tower of eigenvalues $M_n(\eta)$, $n = 1, 2, 3, \ldots$

Real $\eta = m_q/|h|^{8/15}$



Analysis of the BS equation shows infinite set of singularities (square-root) at complex η ($\equiv$ at complex $\xi = \eta^{-\frac{15}{8}}$):



E.g.

$$M_1^{(\mathsf{BS})}(\eta) \sim (\eta - \eta_{\mathsf{YL}})^{1/2} \qquad \text{in BS approximation}$$
$$M_1(\eta) \quad \sim \; (\eta - \eta_{\mathsf{YL}})^{5/12} \qquad \text{in full theory}$$

In IFT the BS equation is an approximation (uncontrolled at finite η), as it ignores multi-meson states.

Q: Do the complex singularities exist in full theory?

Physics is similar to QCD$_2$. At $N_c = \infty$ the BS approximation is exact ('t Hooft, 1974)

Q': Do similar singularities exist in 't Hooft's model of mesons?

t' Hooft's model: QCD$_2$

$$\mathcal{L} = \frac{N_c}{4g^2}\,\mathrm{tr}\left(F^2\right) - \bar{\psi}\,(\gamma D + m_q)\,\psi\,, \qquad\qquad D_\mu = \partial_\mu + A_\mu$$

At $N_c = \infty$ the Bethe-Salpeter equation is exact.

$$\left[\frac{\alpha}{x} + \frac{\alpha}{1-x}\right]\varphi(x) - \int_0^1 \mathrm{d}y\ \frac{\varphi(y)}{(y-x)^2} = 2\pi^2\,\lambda \quad \varphi(x)\ ,$$

$$\alpha = \frac{\pi m_q^2}{g^2} - 1\,, \qquad\qquad M^2 = 2\pi g^2\ \lambda\,.$$

Spectral problem for λ: $\{\lambda_n(\alpha)\}$.

Analytic properties of $\lambda_n(\alpha)$ at complex α? Singular points?

Singularity at $\alpha = -1$. Chiral limit $m_q \to 0$:

$$M_\pi^2 \sim m_q\, g \quad \to \quad \lambda_0(\alpha) \sim \sqrt{\alpha + 1}$$

Critical point. At finite N_c

$$N_c \,\mathrm{WZW}\,[G_{\text{flavor}}] + \mu\,\mathrm{tr}\left[G_{\text{flavor}} + G^\dagger_{\text{flavor}}\right]$$

For $G_{\text{flavor}} = U(N_f)$

$$\mu \sim m_q^{\frac{1}{1-\Delta}}\, g^{\frac{1-2\Delta}{1-\Delta}}, \qquad \Delta = \frac{N_f N_c + 1}{2N_c(N_f + N_c)} \sim \frac{N_f}{2N_c}$$

[Gepner, 1985; Affleck, 1986]

Other (complex) singularities?

Rapidity form: $x = \frac{1}{2}\,(1 + \tanh\theta)$

$$\left[2\alpha - \frac{\pi^2\,\lambda}{\cosh^2\theta}\right]\Psi(\theta) = \int_{-\infty}^{\infty} G(\theta - \theta')\,\Psi(\theta)\,d\theta$$

$$G(\theta - \theta') = \frac{1}{\sinh^2(\theta - \theta')}$$

has second-order pole at $\theta = \theta'$.

Yet more convenient form

$$\Psi(\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\; e^{i\nu\theta}\;\Phi(\nu)$$

$$\left[\alpha + \frac{\pi\nu}{2}\,\coth\frac{\pi\nu}{2}\right]\Phi(\nu) = \frac{\pi\lambda}{2}\int_{-\infty}^{\infty} d\nu'\;S(\nu - \nu')\,\Phi(\nu')$$

$$S(\nu) = \frac{\pi\nu}{2\sinh\frac{\pi\nu}{2}}$$

- $\Phi(\nu)$ is meromorphic function of ν, with poles at

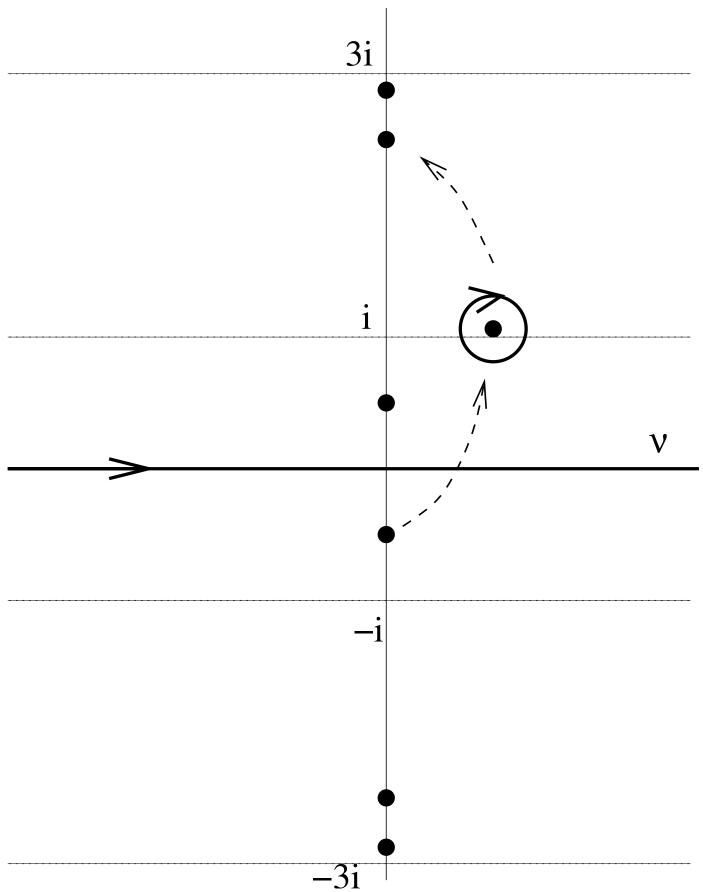
$$\nu_l + 2iN\,, \qquad -\nu_l - 2iN\,,$$

with $N = 0, 1, 2, ...$, and ν_l - roots of

$$\alpha + \frac{\pi\nu}{2}\coth\frac{\pi\nu}{2} = 0$$

with $\Im m\,\nu_l \geq 0$ at real $\alpha > -1$

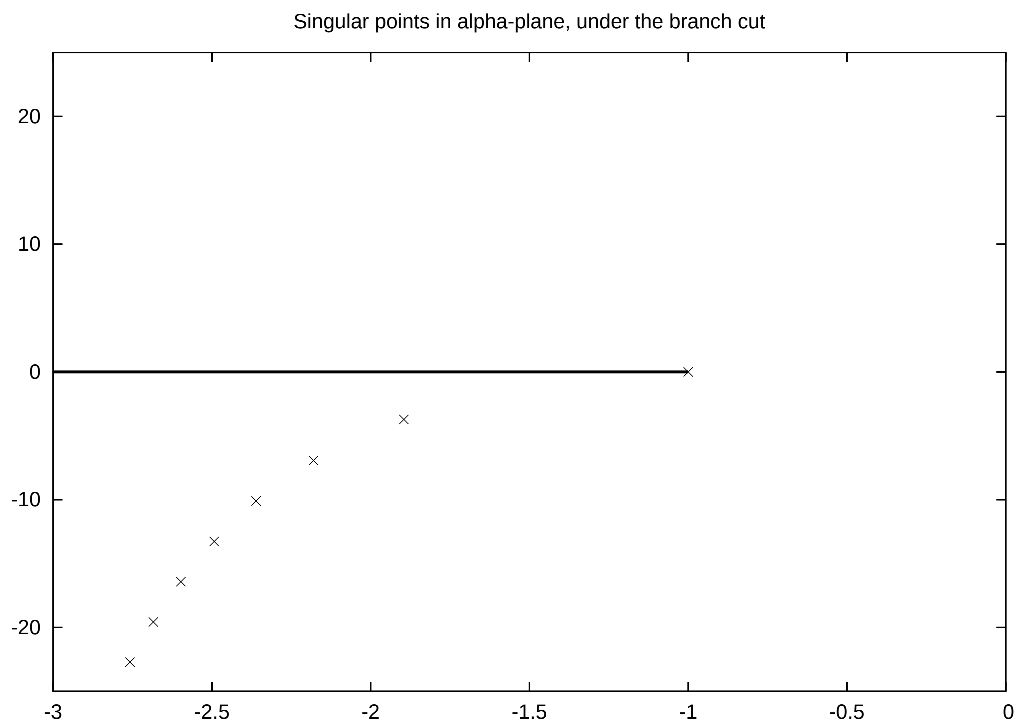
• At complex α the pole $-\nu_0$ can wander into the upper half-plane, and at special α_k it hits another pole there.



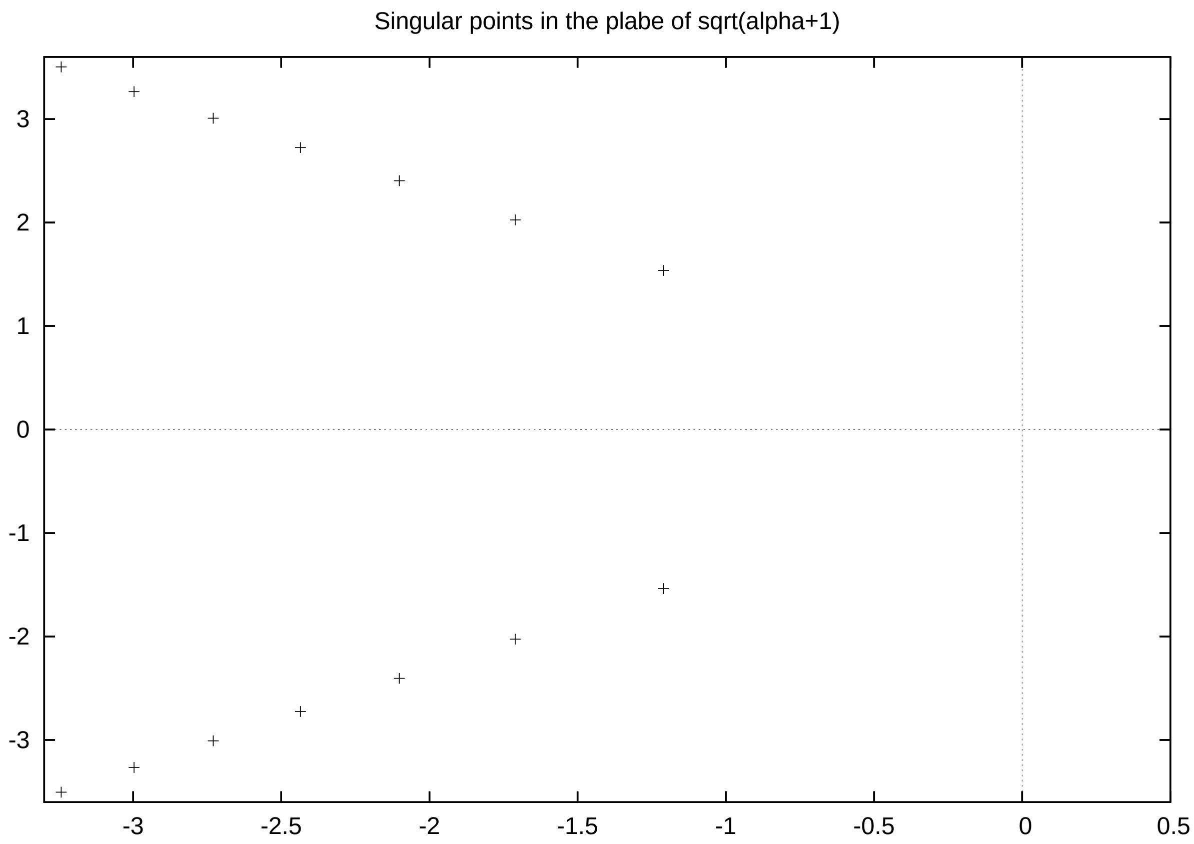
• This gives rise to singularities (square-root branching points of $\lambda_n(\alpha)$) at

$$\alpha_k = -\frac{1}{2}\left[1 + \cosh(\pi\nu_k)\right]$$

$$\sinh(\pi\nu_k) - \pi\nu_k = 0 \ , \qquad \Re e\, \nu_k \leq 0$$



$$\eta = \sqrt{\alpha + 1} \;\sim\; m_q/g = 1/\sqrt{\xi}$$



Proposition: η_k are critical points:

$$\lambda_{2k}(\alpha) \;\sim\; \sqrt{\eta - \eta_k}$$

The operator

$$\widehat{S} : \Phi(\nu) \to \int_{-\infty}^{\infty} d\nu' \; S(\nu - \nu')\Phi(\nu')$$

is inverse to a finite-difference operator $\Rightarrow$ Finite difference equation

$$Q(\nu + 2i) + Q(\nu - 2i) - 2\,Q(\nu) = U(\nu)\,Q(\nu)$$

for

$$Q(\nu) = \left[\alpha \,\sinh \frac{\pi\nu}{2} + \frac{\pi\nu}{2} \;\cosh \frac{\pi\nu}{2}\right] \Phi(\nu)$$

with

$$U(\nu) = 2\pi^2 \lambda \; \left[\alpha + \frac{\pi\nu}{2} \coth \frac{\pi\nu}{2}\right]^{-1}$$

Baxter's TQ equation (with $T(\nu) = 2 + U(\nu)$)

Analytic results for $\lambda_n(\alpha)$

1. Systematic large-n expansions of $\lambda_n(\alpha)$:

$$n = 2\lambda - \frac{2\alpha}{\pi^2}\ \log(2\lambda) - C_0(\alpha) + \frac{\alpha^2}{\pi^4\lambda} + \frac{C_2(\alpha)}{\lambda^2} +$$
$$\frac{1}{\lambda^3}\left[C_3(\alpha) - \frac{(-)^n\,(1+\alpha)}{\pi^6}\ \Big(\log\Big(2\pi\lambda e^{\gamma_E}\Big) + C_3'(\alpha)\Big)\right] + \ldots$$

where

$$C_0(\alpha) = \frac{3}{4} + \frac{2\alpha}{\pi^2}\ \log\Big(4\pi e^{\gamma_E}\Big) -$$
$$\frac{\alpha^2}{2\pi^2}\int_{-\infty}^{\infty}\ \frac{dt}{t}\ \frac{\sinh(t)\,(\,\sinh(2t) - 2t)}{\cosh^2(t)\Big(\,\alpha\,\sinh(t) + t\,\cosh(t)\,\Big)}\ ,$$

$$C_2(\alpha) = \frac{1}{2\pi^6}\left[\,\alpha^3 + (-1)^n\,\pi^2\,(1+\alpha)\right]\ ,$$

$$C_3(\alpha) = \frac{1}{12\pi^8}\left[\,5\alpha^4 + \pi^2\,(1+\alpha)^2\right]\ ,$$

$$C_3'(\alpha) = -\frac{1+3\alpha}{3} + \frac{\alpha}{8}\ \int_{-\infty}^{\infty} dt\ \ \frac{\sinh(2t) - 2t}{t\,\sinh(t)\Big(\,\alpha\,\sinh(t) + t\,\cosh(t)\,\Big)}\ .$$

['t Hooft, 1974; Brauer, Spence, Weis, 1979; Fateev, Lukyanov, AZ, 2009]

2. Exact sum rules:

$$G_{+}^{(s)}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m}^{s}(\alpha)}\,, \quad G_{-}^{(s)}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m+1}^{s}(\alpha)}$$

E.g.

$$G_{\pm}^{(1)}(\alpha) = \log(8\pi) - 2 \pm 1 - \frac{\alpha}{4} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t} \frac{\sinh(t)\ (\sinh(2t) \pm 2t)}{\cosh^{2}(t)\,(\alpha \sinh(t) + t\ \cosh(t)\,)}\ .$$

$$G_{+}^{(s)}(\alpha) \sim (\alpha - \alpha_k)^{-s/2}\,, \qquad G_{-}^{(s)}(\alpha) \sim (\alpha - \alpha_k)^{1/2}$$

α_k are critical points: $M_{2k}^{2}(\alpha_k) \sim \sqrt{\alpha - \alpha_k}$.

Speculation: $N_c < \infty$,

$$M_{2k}^2(\alpha) \sim (\alpha - \alpha_k)^{\beta_k}$$

with critical exponents

$$\beta_k = \frac{1}{2} + \frac{b_k}{N_c} + ...$$

α_k are likely to become non-trivial (non-unitary) CFT.

Q: What kind of criticality α_k correspond to?

Requires study of finite N_c QCD_2.

Summary:

- $N_c = \infty$ QCD$_2$ has infinitely many critical points at complex $\alpha = m_q^2/g^2 - 1$

- This phenomenon seems to be common for confining theories in 1+1 (e.g. IFT in a magnetic field).

Questions:

• What these critical points try to tell us about basic mechanism of confinement?

• Are similar singularities present in 4D?

• What would Aliosha say?