# Higher Equations of Motion in Boundary Liouville Field Theory

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## Abstract

In addition to the ordinary Al.Zamolodchikov's bulk Higher equations of motion in the Boundary Liouville field theory an infinite set of relations containing the boundary operators is found. The possible applications in the context of Boundary Minimal Liouville gravity are discussed.

## Boundary Liouville field theory

Consider the conformal invariant Liouville theory on the unit disk(or the upper half-plane) [V.Fateev,A.Zamolodchikov,Al.Zamolodchikov]

$$A_{\text{bound}} = \frac{1}{4\pi}\int_{\Gamma}\left[g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\right]\sqrt{g}d^2x + \int\limits_{\partial\Gamma}\left(\frac{QK}{2\pi}\phi + \mu_B e^{b\phi}\right)g^{1/4}dx\,,$$

here $R$ is the scalar curvature, $K$ is the curvature of the boundary of the boundary, $\mu$ is the bulk cosmological constant, $\mu_B$ is the boundary cosmological constant, $Q = b + 1/b$ is the background charge, $c_L = 1 + 6Q^2$ is the central charge of the theory.

In the bulk in the holomorphic component of stress tensor

$$T(z) \;=\; -(\partial\phi)^2 + Q\partial^2\phi$$

The boundary value of the classical stress tensor is

$$T_{\mathsf{cl}}(x) = -\frac{1}{16}\varphi_x^2 + \frac{1}{4}\varphi_{xx} + \pi b^2(\pi\mu_B^2 b^2 - \mu)e^{\varphi}$$

where $\varphi = 2b\phi$. This is equivalent to the following equation [FZZ]

$$\left(\frac{d^2}{dx^2} + T_{\mathsf{cl}}\right) e^{-\varphi/4} = \pi b^2(\pi\mu_B^2 b^2 - \mu)e^{3\varphi/4}$$

This is an example of the classical limit of of the quantum relations between boundary primary operators.

The bulk primary fields $V_\alpha(z,\bar{z})$ have conformal weights $\Delta_\alpha = \alpha(Q-\alpha)$. The structure constant $C(\alpha_3,\alpha_2,\alpha_1)$ of the bulk OPE

$$C(\alpha_3,\alpha_2,\alpha_1) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \frac{\Upsilon_0}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)}$$
$$\frac{\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_3-\alpha_2)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)}$$

where the special function $\Upsilon_b$ is such defined:

$\Upsilon_b^{-1}(x) \equiv \Gamma_b(x)\Gamma_b(Q-x)$

$\Upsilon_0 = \frac{d\Upsilon_b(x)}{dx}|_{x=0}$

$\Gamma_b(x)$ is the Barnes double gamma function.

The boundary operators $B_{\beta}^{\sigma_2\sigma_1}(x)$ have conformal weight $\Delta_\beta = \beta(Q-\beta)$.
They are labelled by two left and right boundary conditions $\sigma_1$ and $\sigma_2$ related to $\mu_{B_1}$ and $\mu_{B_2}$ by the relation [FZZ]

$$\cos\left(2\pi b(\sigma - \frac{Q}{2})\right) = \frac{\mu_B}{\sqrt{\mu}}\sqrt{\sin(\pi b^2)}.$$

The correlation function of the bulk operators $V_{\alpha_1}V_{\alpha_2}\dots V_{\alpha_n}$ and the boundary operators $B_{\beta_1}^{\sigma_1\sigma_2}\dots B_{\beta_m}^{\sigma_m\sigma_1}$ scales as follows

$$\mathcal{G}(\alpha_1,\dots\alpha_n,\beta_1\dots\beta_m) \sim$$
$$\mu^{(Q-2\sum_i\alpha_i-\sum_j\beta_j)/2b}F\left(\alpha_1,\dots\alpha_n,\beta_1\dots\beta_m,\frac{\mu_{B_1}^2}{\mu},\frac{\mu_{B_2}^2}{\mu},\dots,\frac{\mu_{B_m}^2}{\mu}\right)\ ,$$

To define LFT on the upper half plane[FZZ], one needs to know a few additional structure constants besides the bulk three point-function $C(\alpha_1, \alpha_2, \alpha_3)$.

- **Bulk one point function** [FZZ]

$$\langle V_\alpha(z,\bar{z})\rangle = \frac{U(\alpha|\mu_B)}{|z-\bar{z}|^{2\Delta_\alpha}}$$

- **Boundary two point function**[FZZ]

$$\left\langle B_{\beta_1}^{\sigma_1\sigma_2}(x) B_{\beta_1}^{\sigma_2\sigma_1}(0)\right\rangle = \frac{S(\beta_1,\sigma_2,\sigma_1)}{|x|^{2\Delta_{\beta_1}}}$$

- **Bulk-boundary two point function**[Hosomichi]

$$\left\langle V_\alpha(z,\bar{z}) B_\beta^{\sigma\sigma}(x)\right\rangle = \frac{R(\alpha,\beta|\mu_B)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}\,|z-x|^{2\Delta_\beta}}$$

- **Boundary three point function**[Ponsot,Teschner]

$$\left\langle B^{\sigma_1\sigma_3}_{Q-\beta_3}(x_3) B^{\sigma_3\sigma_2}_{\beta_2}(x_2) B^{\sigma_2\sigma_1}_{\beta_1}(x_1)\right\rangle = \frac{C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}$$

$$\begin{aligned} C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1} &= \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{1}{2b}(\beta_3-\beta_2-\beta_1)} \\ &\times \frac{\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(Q+\beta_2-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_1-\beta_2)}{\Gamma_b(2\beta_3-Q)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_1)} \\ &\times \frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)}{\Gamma_b(Q)} \frac{S_b(\beta_3+\sigma_1-\sigma_3)S_b(Q+\beta_3-\sigma_3-\sigma_1)}{S_b(\beta_2+\sigma_2-\sigma_3)S_b(Q+\beta_2-\sigma_3-\sigma_2)} \\ &\times \frac{1}{i}\int_{-i\infty}^{i\infty} ds \; \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(Q+s)} \end{aligned}$$

the coefficients $U_i$, $V_i$ and $i = 1,\ldots,4$ read

$$\begin{aligned} U_1 &= \sigma_1+\sigma_2-\beta_1 & V_1 &= Q+\sigma_2-\sigma_3-\beta_1+\beta_3 \\ U_2 &= Q-\sigma_1+\sigma_2-\beta_1 & V_2 &= 2Q+\sigma_2-\sigma_3-\beta_1-\beta_3 \\ U_3 &= \beta_2+\sigma_2-\sigma_3 & V_3 &= 2\sigma_2 \\ U_4 &= Q-\beta_2+\sigma_2-\sigma_3 \end{aligned}$$

## Bulk HEM

Consider degenerate primary fields $V_{m,n}$ with conformal dimension

$$\Delta_{m,n} = \frac{Q^2}{4} - \frac{\left(mb^{-1} + nb\right)^2}{4}$$

Denote the singular vector creating operators $D_{m,n}$. Let logarithmic fields $V'_\alpha = \frac{1}{2}\frac{\partial}{\partial\alpha} V_\alpha = \phi e^{2\alpha\phi}$

It was proven by Al.Z. that $D_{m,n}\bar{D}_{m,n}V'_{m,n}$ is a primary field of dimension $\Delta_{m,n} + mn$ and

$$D_{m,n}\bar{D}_{m,n}V'_{m,n} = B_{m,n}V_{m,-n}$$

$$B_{m,n} = \left(\pi\mu\gamma(b^2)\right)^n b^{1+2n-2m}\gamma(m - nb^2) \prod_{\substack{k=1-n \\ l=1-m \\ (k,l)\neq(0,0)}}^{\substack{m-1 \\ n-1}} (lb^{-1} + kb)$$

## Boundary HEM

Consider the action of $D_{m,n}$ on the boundary degenerate primary field $B_{m,n}^{s_1 s_2}$ with arbitrary values $s_1$and $s_2$.The analysis of the classical relation shows that in general this field is not supposed to vanish. On the other hand from the purely algebraic reasons it follows that the field $D_{m,n}B_{m,n}^{s_1 s_2}$ is a primary field whatever the boundary cosmological parameters are equal to. Then, taking into account the main assumption of LFT that there exist only one primary field of given conformal dimension, we have to identify it up to some numerical constant with the field $B_{m,-n}$

$$D_{m,n}B_{m,n}^{s_1 s_2} = \kappa_{m,n}^{s_1 s_2} B_{m,-n}^{s_1 s_2}$$

This operator-valued relation assume the corresponding relations between the correlation functions. Say,the ratio of two correlation functions including l.h.s or r.h.s. of the relation should not depend on the conformal dimesion of other fields or on the other cosmological costants.

## Free bosonic field check

Consider a simple case

$$D_{1,2}B_{1,2}^{s_1s_2} = \kappa_{1,2}^{s_1s_2}B_{1,-2}^{s_1s_2}$$

To calculate the coefficient , insert $D_{1,2}B_{1,2}^{s_1s_2}$ inside the three point correlation function and choose the two other fields in such a way to satisfy the screening relations .The correlators will be given in this case by free bosonic representation

$$\left\langle B_{\beta_1}^{s_3s_1}(0)D_{1,2}B_{-b/2}^{s_1s_2}(x)B_{Q-\beta_1-3b/2}^{s_2s_3}(\infty)\right\rangle$$
$$= \kappa_{1,2}^{s_1s_2}\left\langle B_{\beta_1}^{s_3s_1}(0)B_{3b/2}^{s_1s_2}(x)B_{Q-\beta_1-3b/2}^{s_2s_3}(\infty)\right\rangle$$

The correlators will be given in this case by free bosonic representation. Since in r.h.s the total charge balance is satisfied without any screenings, so

$$\begin{aligned}\kappa_{1,2}^{s_1s_2} &= 2(1-2b\beta_1)(1-2b\beta_1-b^2)\\ &\times \left\langle B_{\beta_1}^{s_3s_1}(0)B_{-b/2}^{s_1s_2}(1)B_{Q-\beta_1-3b/2}^{s_2s_3}(\infty)\right\rangle\end{aligned}$$

Non trivial check of the general statement should be the cancellation of the dependence on $\beta_1$ and $\mu_3$.

Using the free bosonic field representation for the correlator in r.h.s.,we obtain

$$\left\langle B^{s_3s_1}_{\beta_1}(0)B^{s_1s_2}_{-b/2}(1)B^{s_2s_3}_{Q-\beta_1-3b/2}(\infty)\right\rangle$$
$$= -\mu \int\limits_{\mathrm{Im}z>0} d^2z \left\langle e^{2b\phi(z)} B^{s_3s_1}_{\beta}(0)B^{s_1s_2}_{-b/2}(1)B^{s_2s_3}_{Q-\beta-3b/2}(\infty)\right\rangle +$$
$$+\sum_{i,j}\frac{\mu_i\mu_j}{2}\int_{C_i}\int_{C_j} dx_1 dx_2$$
$$\left\langle e^{b\phi(x_1)}e^{b\phi(x_2)} B^{s_3s_1}_{\beta}(0)B^{s_1s_2}_{-b/2}(1)B^{s_2s_3}_{Q-\beta-3b/2}(\infty)\right\rangle$$

where the contours are defined as follows: $C_1 = (-\infty, 0)$, $C_2 = (0,1)$,$C_3 = (1,\infty)$, $\mu_i$ are the corresponding values of the boundary cosmological constant,

$$\left\langle e^{2b\phi(z)} B^{s_3s_1}_{\beta}(0)B^{s_1s_2}_{-b/2}(1)B^{s_2s_3}_{Q-\beta-3b/2}(\infty)\right\rangle =$$
$$= |z|^{-4b\beta}\,|1-z|^{2b^2}\,|z-\bar{z}|^{-2b^2}$$

and

$$\left\langle e^{b\phi(x_1)}e^{b\phi(x_2)} B^{s_3s_1}_{\beta}(0)B^{s_1s_2}_{-b/2}(1)B^{s_2s_3}_{Q-\beta-3b/2}(\infty)\right\rangle =$$
$$= |x_1|^{-2b\beta}\,|x_2|^{-2b\beta}\,|1-x_1|^{b^2}\,|1-x_2|^{b^2}\,|x_1-x_2|^{-2b^2}$$

Introduce the denotation

$$I(A,B,C) = -\int\limits_{\mathrm{Im}z>0} d^2z\,(z\bar{z})^A\,|(1-z)(1-\bar{z})|^B\,|z-\bar{z}|^C$$

$$I_{ij}(A,B,C) = \int_{C_i}\int_{C_j} dx_1 dx_2\,|x_1|^A\,|x_2|^A\,|1-x_1|^B\,|1-x_2|^B\,|x_1-x_2|^C$$

Then we can write

$$\left\langle B^{s_3 s_1}_{\beta_1}(0) B^{s_1 s_2}_{-b/2}(1) B^{s_2 s_3}_{Q-\beta_1-3b/2}(\infty)\right\rangle =$$

$$= \mu I(-2b\beta, b^2, -2b^2) + \tfrac{1}{2}\sum_{i,j}\mu_i\mu_j I_{ij}(-2b\beta, b^2, -2b^2)$$

All integrations can be carried out explicitely:

$$I = -\frac{1}{2\pi^3}\sin(\frac{\pi C}{2})\sin(\pi A)\sin(\pi B)\sin(\pi(A+B+C))J(A,B,C)$$
$$I_{11} = -\frac{1}{\pi^3}\sin\frac{\pi C}{2}\sin\pi(A+B+\frac{C}{2})\sin\pi(A+B+C)J(A,B,C)$$
$$I_{12} = -\frac{1}{2\pi^3}\sin\pi C\sin\pi(A+B+\frac{C}{2})\sin\pi(A+\frac{C}{2})J(A,B,C)$$
$$I_{13} = -\frac{1}{2\pi^3}\sin\pi C\sin\pi(A+B+\frac{C}{2})\sin\pi(B+\frac{C}{2})J(A,B,C)$$
$$I_{22} = -\frac{1}{\pi^3}\sin\frac{\pi C}{2}\sin\pi(A+\frac{C}{2})\sin\pi A J(A,B,C)$$
$$I_{23} = -\frac{1}{2\pi^3}\sin\pi C\sin\pi(A+\frac{C}{2})\sin\pi(B+\frac{C}{2})J(A,B,C)$$
$$I_{33} = -\frac{1}{\pi^3}\sin\frac{\pi C}{2}\sin\pi(B+\frac{C}{2})\sin\pi B J(A,B,C)$$

$$J(A,B,C) = \Gamma(A+1)\Gamma(B+1)\Gamma(C+1)\Gamma(-C/2)\Gamma(B+C/2+1)\times$$
$$\Gamma(-A-B-C-1)\Gamma(-A-B-C/2-1)\Gamma(A+C/2+1)$$

Summing these contributions we have

$$\left\langle B^{s_3 s_1}_{\beta_1}(0) B^{s_1 s_2}_{-b/2}(x) B^{s_2 s_3}_{Q-\beta_1-3b/2}(\infty)\right\rangle =$$

$$= -\frac{1}{2\pi^3}\Big[ -\mu \sin(\pi A)\sin(\pi B)\sin(\pi(A+B+C)) +$$

$$+\mu_1^2 \sin(\pi(A+B+C/2))\sin(\pi(A+B+C)) +$$

$$+\mu_2^2 \sin(\pi(A+C/2))\sin(\pi A) + \mu_3^2 \sin(\pi(B+C/2))\sin(\pi B) -$$

$$-2\mu_1\mu_2 \cos\frac{\pi C}{2}\sin(\pi(A+B+C/2))\sin(\pi(A+C/2)) -$$

$$-2\mu_1\mu_3 \cos\frac{\pi C}{2}\sin(\pi(A+B+C/2))\sin(\pi(B+C/2)) +$$

$$+2\mu_2\mu_3 \cos\frac{\pi C}{2}\sin(\pi(A+C/2))\sin(\pi(B+C/2))\Big] \sin\frac{\pi C}{2} J(A,B,C)$$

Due to $C = -2B$ the result does not depend on $\mu_3$

$$\left\langle B^{s_3 s_1}_{\beta_1}(0) B^{s_1 s_2}_{-b/2}(x) B^{s_2 s_3}_{Q-\beta_1-3b/2}(\infty)\right\rangle =$$

$$\frac{1}{2\pi^3}\Big[ -\mu \sin(\pi B) + \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2 \cos(\pi B)\Big]$$

$$\times \sin(\pi A)\sin(\pi B)\sin(\pi(A-B)) J(A,B,C)$$

with the standard parameterization

$$\mu_i^2 = \mu \frac{\cosh^2 \pi b s_i}{\sin \pi b^2}\,, \quad \mu_i \mu_j = \mu \frac{\cosh \pi b s_i \cosh \pi b s_i}{\sin \pi b^2}$$

the correlation function reads

$$\left\langle B^{s_3 s_1}_{\beta_1}(0) B^{s_1 s_2}_{-b/2}(x) B^{s_2 s_3}_{Q-\beta_1-3b/2}(\infty) \right\rangle =$$

$$\frac{1}{2\pi^3} \sin \frac{B+is_1-is_2}{2} \sin \frac{B-is_1+is_2}{2} \sin \frac{B+is_1+is_2}{2} \sin \frac{B-is_1-is_2}{2}$$
$$\times \sin(\pi A) \sin(\pi B) \sin(\pi(A-B)) J(A,B,C)$$

$\beta$-dependence of $\kappa^{s_1 s_2}_{1,2}$ vanishes. Finally

$$\kappa^{s_1 s_2}_{1,2} = \frac{4}{\pi} \mu \gamma(b^2) \Gamma(1-2b^2) \Gamma(1-b^2) \Gamma(1+b^2) \times$$
$$\sin \pi b \frac{b+i(s_1+s_2)}{2} \sin \pi b \frac{b-i(s_1+s_2)}{2} \sin \pi b \frac{b+i(s_2-s_1)}{2} \sin \pi b \frac{b-i(s_2-s_1)}{2}$$

## General three-point case

The operator valued relation means that the following equality holds

$$\left\langle B^{s_1 s_3}_{Q-\beta_3}(0) D_{m,n} B^{s_3 s_2}_{m,n}(1) B^{s_2 s_1}_{\beta_1}(\infty) \right\rangle$$
$$= \kappa^{s_3 s_2}_{m,n} \left\langle B^{s_1 s_3}_{Q-\beta_3}(0) B^{s_3 s_2}_{m,-n}(x) B^{s_2 s_1}_{\beta_1}(\infty) \right\rangle$$

or

$$\kappa^{s_3 s_2}_{m,n} = P_{m,n}(\beta_1+\beta_3) P_{m,n}(Q-\beta_1-\beta_3) \times \frac{\left\langle B^{s_1 s_3}_{Q-\beta_3}(0) B^{s_3 s_2}_{m,n}(1) B^{s_2 s_1}_{\beta_1}(\infty) \right\rangle}{\left\langle B^{s_1 s_3}_{Q-\beta_3}(0) B^{s_3 s_2}_{m,-n}(1) B^{s_2 s_1}_{\beta_1}(\infty) \right\rangle}$$

Remind the expression for the Boundary 3-function

$$
\begin{aligned}
&\left\langle B^{\sigma_1\sigma_3}_{Q-\beta_3} B^{\sigma_3\sigma_2}_{\beta_2} B^{\sigma_2\sigma_1}_{\beta_1} \right\rangle \\
&= \left(\pi\mu\gamma(b^2) b^{2-2b^2}\right)^{\frac{1}{2b}(\beta_3-\beta_2-\beta_1)} \\
&\times \frac{\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(Q+\beta_2-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_1-\beta_2)}{\Gamma_b(2\beta_3-Q)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_1)} \\
&\times \frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)}{\Gamma_b(Q)} \frac{S_b(\beta_3+\sigma_1-\sigma_3)S_b(Q+\beta_3-\sigma_3-\sigma_1)}{S_b(\beta_2+\sigma_2-\sigma_3)S_b(Q+\beta_2-\sigma_3-\sigma_2)} \\
&\times \frac{1}{i} \int_{-i\infty}^{i\infty} ds \; \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(Q+s)}
\end{aligned}
$$

the coefficients $U_i$, $V_i$ and $i = 1, \ldots, 4$ read

$$
\begin{aligned}
U_1 &= \sigma_1 + \sigma_2 - \beta_1 & V_1 &= Q + \sigma_2 - \sigma_3 - \beta_1 + \beta_3 \\
U_2 &= Q - \sigma_1 + \sigma_2 - \beta_1 & V_2 &= 2Q + \sigma_2 - \sigma_3 - \beta_1 - \beta_3 \\
U_3 &= \beta_2 + \sigma_2 - \sigma_3 & V_3 &= 2\sigma_2 \\
U_4 &= Q - \beta_2 + \sigma_2 - \sigma_3 & &
\end{aligned}
$$

The difference between the integral parts of 3-functions in the numerator and the denomerator comes from two Barnes S-functions only. Using that $S_b(Q-x) = 1/S_b(x)$ and also the shift relation

$$S_b(x+nb) = 2^n \prod_{k=0}^{n-1} \sin \pi b(x+kn) \cdot S_b(x)$$

we get

$$\begin{gathered} S_b(\beta_{mn}+nb+\sigma_2-\sigma_3+s)S_b(Q-\beta_{mn}-nb+\sigma_2-\sigma_3+s) = \\ \prod_{k=0}^{n-1} \frac{\sin\left[\pi b\left(\frac{(1+2k-n)b}{2}+\sigma_2-\sigma_3+s\right)+\frac{1-m}{2}\pi\right]}{\sin\left[-\pi b\left(\frac{(1+2k-n)b}{2}+\sigma_2-\sigma_3+s\right)+\frac{1-m}{2}\pi\right]} \times \\ S_b(\beta_{mn}+\sigma_2-\sigma_3+s)S_b(Q-\beta_{mn}+\sigma_2-\sigma_3+s) \\ = (-1)^{mn} S_b(\beta_{mn}+\sigma_2-\sigma_3+s)S_b(Q-\beta_{mn}+\sigma_2-\sigma_3+s) \end{gathered}$$

So the ration of the integral factors of the 3-functions is $(-1)^{mn}$.

The contribution to the ration of the non-integral factor can be suitably divided to two parts. The first part consists of $S_b$ functions

$$M_1 = \frac{S_b(\beta_{mn}+nb+\sigma_2-\sigma_3)S_b(Q+\beta_{mn}+nb-\sigma_2-\sigma_3)}{S_b(\beta_{mn}+\sigma_2-\sigma_3)S_b(Q+\beta_{mn}-\sigma_2-\sigma_3)} =$$
$$2^{2n}\prod_{k=0}^{n-1}\sin\left[\pi b\left(\frac{(1+2k-n)b}{2}+\sigma_2-\sigma_3\right)+\frac{1-m}{2}\pi\right]$$
$$\prod_{k=0}^{n-1}\sin\left[\pi b\left(\frac{(3+2k-n)b}{2}-\sigma_2-\sigma_3\right)+\frac{3-m}{2}\pi\right]$$

or changing variables $\sigma = Q/2 + is/2$

$$M_1 = 2^{2n}\prod_{k=0}^{n-1}\sin\left[\pi b\frac{(1-m)b^{-1}+(1+2k-n)b+i(s_2-s_3)}{2}\right]$$
$$\prod_{k=0}^{n-1}\sin\left[\pi b\frac{(1-m)b^{-1}+(1+2k-n)b+i(s_2+s_3)}{2}\right]$$

The second part of the ratio consists of $\Gamma_b$ functions

$$M_2 = \frac{\Gamma_b(2Q-\beta_1-\beta_{mn}-\beta_3)\Gamma_b(\beta_{mn}+\beta_3-\beta_1)}{\Gamma_b(2Q-\beta_1-\beta_{mn}-nb-\beta_3)\Gamma_b(\beta_{mn}+nb+\beta_3-\beta_1)}$$
$$\frac{\Gamma_b(Q+\beta_{mn}-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_{mn}-\beta_1)\Gamma_b(Q-2\beta_{mn}-2nb)}{\Gamma_b(Q+\beta_{mn}+nb-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_{mn}-nb-\beta_1)\Gamma_b(Q-2\beta_{mn})}$$

Using the relation

$$\Gamma_b(x+nb) = \frac{(2\pi)^{\frac{n}{2}} b^{n(bx-\frac{1}{2})} b^{\frac{n(n-1))}{2}b^2}}{\prod_{k=0}^{n-1}\Gamma\left[b(x+kb)\right]}\Gamma_b(x)$$

The part of $M_2$depending of $\beta_1, \beta_3$ $M_2$ is product of two similar factors

$$\frac{\Gamma_b(\beta_{mn}+\beta)\Gamma_b(Q-\beta_{mn}+\beta)}{\Gamma_b(\beta_{mn}+nb-\beta)\Gamma_b(Q-\beta_{mn}-nb+\beta)} = \frac{b^{-nb(2\beta_{mn}+nb-Q)}b^{-mn}}{P_{mn}(\beta)}$$

where $\beta$ is either $\beta_3-\beta_1$ or $Q-\beta_3-\beta_1$. from here we obtain

$$M_2 = \frac{b^{-2nb(2\beta_{mn}+nb-Q)}b^{-2mn}}{P_{mn}(\lambda_3-\lambda_1)P_{mn}(\lambda_3+\lambda_1)}$$
$$\frac{\Gamma_b(Q-2\beta_{mn}-2nb)}{\Gamma_b(Q-2\beta_{mn})}$$

Using these expressions for $M_1$,$M_2$ finally we get the general form of BHEM:

$$D_{m,n}B_{m,n}^{s_1 s_2} = \kappa_{m,n}^{s_1 s_2} B_{m,-n}^{s_1 s_2}$$

and the coefficients

$$\kappa_{m,n}^{s_1 s_2} = (-1)^{mn} 2^n \left(\frac{\mu\gamma(b^2)}{\pi}\right)^{\frac{n}{2}} b^{2n(1-m)}$$
$$\prod_{k=0}^{2n-1} \Gamma(m-(n-k)b^2) \times$$
$$\prod_{k=0}^{n-1} \sin \pi b \frac{(1-m)b^{-1}+(1+2k-n)b+i(s_1+s_2)}{2}$$
$$\prod_{k=0}^{n-1} \sin \pi b \frac{(1-m)b^{-1}+(1+2k-n)b+i(s_1-s_2)}{2}$$

From the expression in r.h.s. of this eq-n we see when the singular vector $D_{m,n}B_{m,n}^{s_1 s_2}$ is decoupled

## Application in minimal gravity

One of the possible applications of the BHEM in physical context is the construction of the correlation functions of physical fields in boundary minimal Liouville gravity. BMLG theory consists of the matter, Liouville and ghost sectors. The Liouville central charge is defined by the central charge balance condition

$$c_{\mathsf{M}} + c_{\mathsf{L}} = 26$$

The physical fields are difined in the framework of RRST cohomologies in respect to the BRST charge

$$\mathcal{Q} = \oint \left(C(T_{\mathsf{L}} + T_{\mathsf{M}}) + C\partial C B\right) \frac{dz}{2\pi i}$$

Here $B$ and $C$ are ghost fields (we use here the upper case letters for the ghosts for not to mix $B$ with the parameter $b$ of minimal gravity).

It is the specific property of MG that in the construction of the physical phields all matter fields are "dressed" by Liouville exponentials entering the higher equations of motion . For example there exist ghost number equal one basic boundary physical fields of the form

$$W_{m,n}^{(\alpha_1,\alpha_2|s_1,s_2)} = U_{m,n}^{(\alpha_1,\alpha_2|s_1,s_2)} C$$

where $C$ is the ghost ($\Delta_C = -1$) and

$$U_{m,n}^{(\alpha_1,\alpha_2|s_1,s_2)} = \Psi_{m,n}^{\alpha_1,\alpha_2} B_{m,-n}^{s_1,s_2}$$

Here the parameters $\alpha_1, \alpha_2$ and $s_1, s_2$ caracterize the conformal boundary conditions to the left to the right from the operator insertion in the matter and Liouville sector respectively. Due to the anomaly of the ghost current the LG correlation function of any number $N$ of fields of this kind

$$G^N = \prod_{i=4}^{N} \int dx_i \langle W_1(x_1) W_2(x_2) W_3(x_3) U_4(x_4) \ldots U_N(x_N) \rangle_{\mathsf{MG}}$$

Another important class of boundary physical fields (ghost number 0) is the boundary ground ring. The general form of the elements of the boundary ground ring (see also for more recent developments) is

$$O_{m,n} = H_{m,n}\Psi_{m,n}V_{m,n}$$

Here $H_{m,n}$ are operators of level $mn$ and ghost number 0 constructed from $L_n$, $L_n^{(\mathrm{M})}$ and ghosts. The boundary higher equations lead to the following important relation between the two types of physical fields introduced above

$$\mathcal{Q}O_{m,n}^{s_1,s_2} = k_{m,n}^{s_1,s_2}W_{mn}^{s_1,s_2}$$

which means in particular that to construct the boundary ground ring element $O_{m,n}^{s_1,s_2}$ the fusion relations for the cosmological constants $s_1$ and $s_2$ in the Liouville sector should be satisfied, $k_{m,n}^{s_1,s_2} = 0$. Another consequence of this relation is that if the fusion rules are not satisfied the field $W_{mn}^{s_1,s_2}$ is $\mathcal{Q}$ exact and the correlation functions of this field should be equal zero. This statement should be checked more carefully.

Taking into account the commutation relations $\{B_{-1}, \mathcal{Q}\} = \partial$ it is straightforward to verify that

$$\frac{d}{dx}\left(\frac{d}{ds}O^{s,\sigma}_{m,n}|_{s=s_{mn}(\sigma)}\right) = \left(\frac{d}{ds}k^{s,\sigma}_{m,n}|_{s=s_{mn}(\sigma)}\right)U^{s_{mn}(\sigma),\sigma}_{mn}$$

So,integrating by parts the integral can be reduced to boundary terms which are defined in principle by the operator product expansion.