Boundary transitions of the O(n) model on a dynamical lattice

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Based on :   J. -E. Bourgine, I. Kostov and KH, arXiv:0910.1581

The O(n) model

Classical O(n) spin $\vec{S}(r)$ at each site r of a lattice Γ

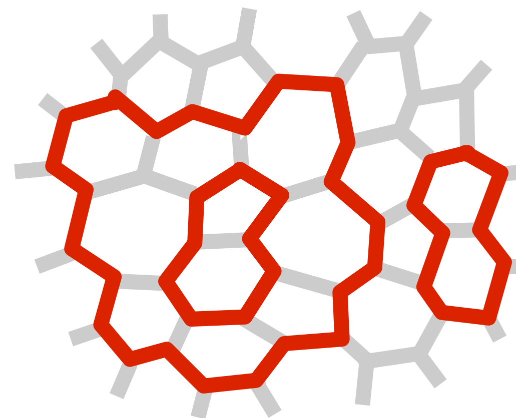
with nearest-neighbor interaction

$$\frac{1}{T}\vec{S}(r)\cdot\vec{S}(r') \qquad T \text{ : temperature}$$

$$Z(n,T;\Gamma) \equiv \int \prod_{r\in\Gamma}[d\vec{S}(r)] \prod_{\langle rr'\rangle} \Big(1+\frac{1}{T}\vec{S}(r)\cdot\vec{S}(r')\Big)$$

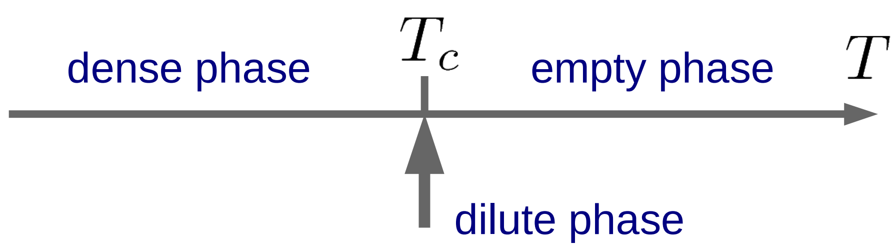
Critical behavior at $n = 2\cos(\pi/p) \in [-2,2]$

Loop gas picture



$$Z = \sum_{\text{loops on } \Gamma} n^{\sharp(\text{loops})} T^{-(\text{length})}$$

For general (irrational) p , the model describes the thermal flow between two (irrational) CFTs.

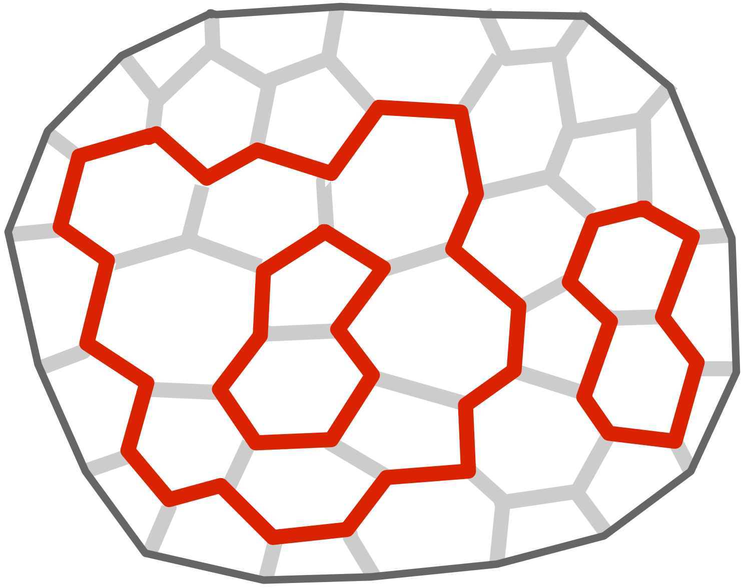


$$c_{\text{dense}} = 1 - \frac{6}{p(p-1)}, \quad c_{\text{dilute}} = 1 - \frac{6}{p(p+1)}$$

($(p-1, p)$ and $(p, p+1)$ unitary minimal models for integer p .)

Boundary conditions

* Ordinary b.c.   --- loops are not allowed to touch



Boundary conditions

* Ordinary b.c. --- loops are not allowed to touch

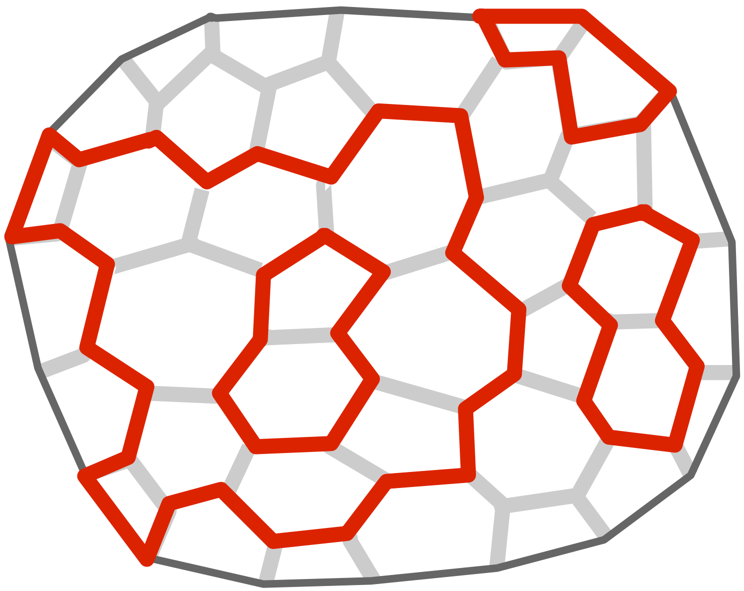
* J-S b.c. --- loops that touch the boundary are weighted by

(Jacobsen-Saleur)

$$k \cdot \lambda^{\sharp(\text{touchings})} T^{-(\text{length})}$$

Boundary spins are O(k) valued, with interaction

$$\lambda \vec{S}_B(r) \vec{S}_B(r')$$



Boundary conditions

* Ordinary b.c. --- loops are not allowed to touch

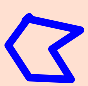
* J-S b.c. --- loops that touch the boundary are weighted by

$$k \cdot \lambda^{\sharp(\text{touchings})} T^{-(\text{length})}$$

* "Dilute" J-S b.c. --- consider loops of two colors.

Red and Blue loops are weighted by

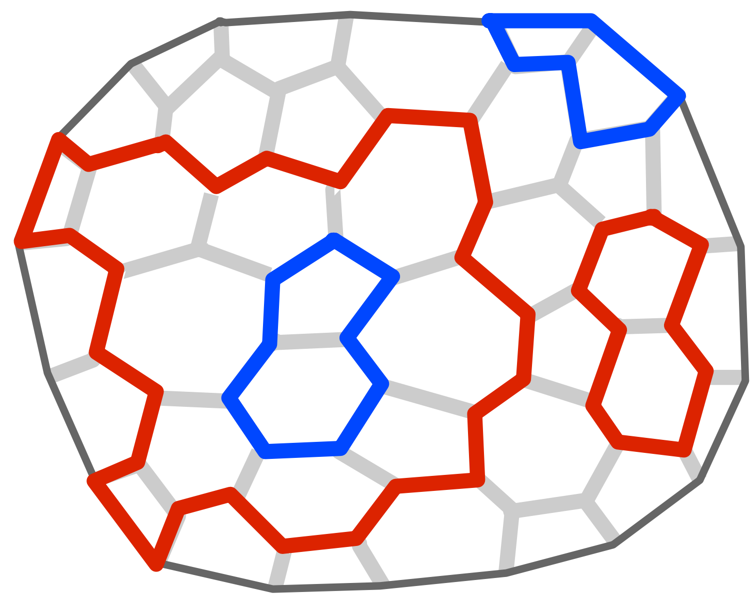
 $n_{(1)} \cdot \lambda_{(1)}^{\sharp(\text{touchings})} T^{-(\text{length})}$

 $n_{(2)} \cdot \lambda_{(2)}^{\sharp(\text{touchings})} T^{-(\text{length})}$

$$(n_{(1)} + n_{(2)} = n)$$

Interaction of boundary $O(n_{(1)}) \times O(n_{(2)})$ spins :

$$\lambda_{(1)} \vec{S}_B^{(1)}(r) \vec{S}_B^{(1)}(r') + \lambda_{(2)} \vec{S}_B^{(2)}(r) \vec{S}_B^{(2)}(r')$$



Various properties of the new boundaries have been worked out on plain lattices.

Condition for conformal invariance (= criticality)

* In dense phase $(T < T_c)$, the couplings $\lambda_{(1)}, \lambda_{(2)}$ are irrelevant.

* In dilute phase $(T = T_c)$, we need to tune $\lambda_{(1)}, \lambda_{(2)}$ to achieve criticality.

Phase diagram for J-S boundary has

--- 3 phases, 4 RG fixed points.

ORD: no loops touch the boundary

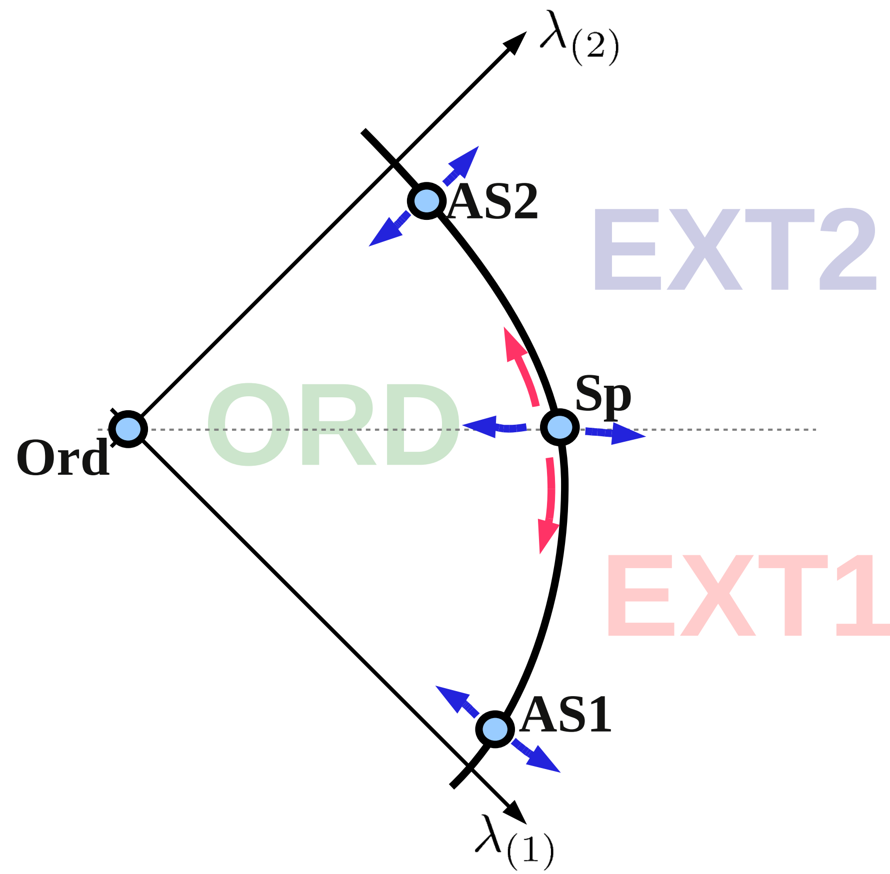
EXT1: a red loop covers the boundary

EXT2: a blue loop covers the boundary

--- arrows indicate relevant perturbations

 $\Phi_{(1,3)}$  $\Phi_{(3,3)}$

(Dubail-Jacobsen-Saleur)



Boundary-changing “L-leg” operators

$S_L^{(1)}$ $(L = 4)$

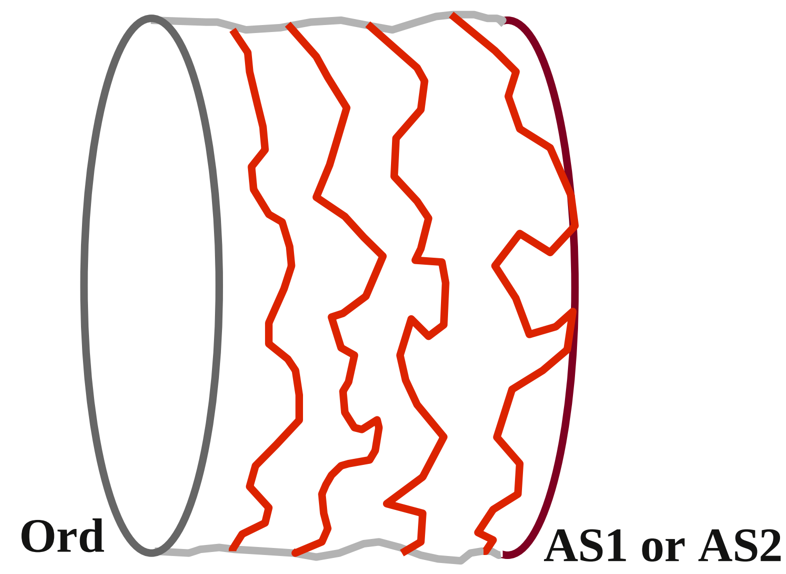
$S_L^{(1)}$ . . . sources of L red lines

$S_L^{(2)}$ . . . sources of L blue lines

. . . appear in the cylinder partition function

with Ord and J-S boundaries

and L (red / blue) loops going around



We label the J-S boundaries by r.

$$n = 2\cos\frac{\pi}{p}, \quad n_{(1)} = \frac{\sin[\pi(r-1)/p]}{\sin[\pi r/p]}, \quad n_{(2)} = \frac{\sin[\pi(r+1)/p]}{\sin[\pi r/p]},$$

Then we find

$$(\mathbf{Ord}|S_L^{(1)}|\mathbf{AS1}) \iff \Phi_{(r+L,r)}$$

$$(\mathbf{Ord}|S_L^{(2)}|\mathbf{AS1}) \iff \Phi_{(r-L,r)}$$

$$(\mathbf{Ord}|S_L^{(1)}|\mathbf{AS2}) \iff \Phi_{(r+L,r+1)} = \Phi_{(p-r-L,p-r)}$$

$$(\mathbf{Ord}|S_L^{(2)}|\mathbf{AS2}) \iff \Phi_{(r-L,r+1)} = \Phi_{(p-r-L,p-r)}$$

$$h_{(r,s)} = h_{(p-r,p+1-s)} = \frac{\{r(p+1)-sp\}^2-1}{4p(p+1)}$$

We study the new boundary conditions in the O(n) model

using the methods of **2D quantum gravity**

(large N matrix model).

The O(n) matrix model

--- Integral over $(N \times N)$ matrices.

$$Z_N(T) \;=\; \int dM d^nY \exp\beta\mathrm{tr}\Big(-\tfrac{1}{2}M^2 - \tfrac{T}{2}\vec{Y}^2 + \tfrac{1}{3}M^3 + M\vec{Y}^2\Big)$$

$$\simeq\; \int dX d^nY \exp\beta\mathrm{tr}\Big(-V(X) + X\vec{Y}^2\Big)$$

$$V(x) \equiv -\tfrac{T}{3}(x+\tfrac{1}{2})^2 + \tfrac{1}{2}(x+\tfrac{1}{2})^2$$

Random lattices with loop configurations are generated as Feynman graphs with loops of Y-matrices.

Disk one-point amplitudes

Ordinary boundary

$$W(x) \;=\; \frac{1}{\beta}\Big\langle \mathrm{tr}\mathbf{W}(x)\Big\rangle, \quad \mathbf{W}(x) \equiv \frac{1}{x-X}$$

. . . solved by Eynard-Kristjansen('96), Kostov('06).

J-S boundary

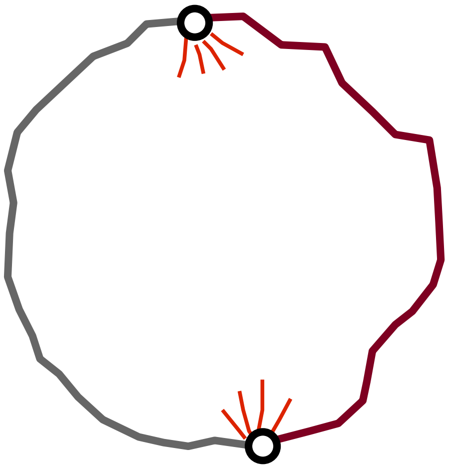
$$H(y) \;=\; \frac{1}{\beta}\Big\langle \mathrm{tr}\mathbf{H}(y)\Big\rangle, \quad \mathbf{H}(y) \equiv \frac{1}{y - X - \lambda_{(1)}\vec{Y}^2_{(1)} - \lambda_{(2)}\vec{Y}^2_{(2)}}$$

. . . we could not solve $H(y)$.

Disk two-point amplitudes

$$D_L^{(i)}(x,y) \;\equiv\; \frac{1}{\beta}\Big\langle \mathrm{tr}\,\mathbf{W}(x)\mathbb{S}_L^{(i)}\mathbf{H}(y)\mathbb{S}_L^{(i)}\Big\rangle$$

$$D_0(x,y) \;\equiv\; \frac{1}{\beta}\Big\langle \mathrm{tr}\,\mathbf{W}(x)\mathbf{H}(y)\Big\rangle$$



Here $\mathbb{S}_L^{(1)}$ is the red L-leg operator,

$$\mathbb{S}_L^{(1)} \overset{\text{shorthand}}{=} \mathbb{S}^{(1)}_{a_1 a_2 \cdots a_L} \;\equiv\; Y_{[a_1} Y_{a_2} \cdots Y_{a_L]}$$

$$(a_1, \cdots, a_L \in \{1, \cdots, n_{(1)}\})$$

and likewise $\mathbb{S}_L^{(2)}$ is the blue L-leg operator.

Loop equation for W(x)

$$W(x)^2 - V'(x)W(x) + nW \star W(x) + \frac{1}{\beta}\left\langle \mathrm{tr}\left(\frac{V'(x) - V'(X)}{x - X}\right)\right\rangle \;=\; 0$$

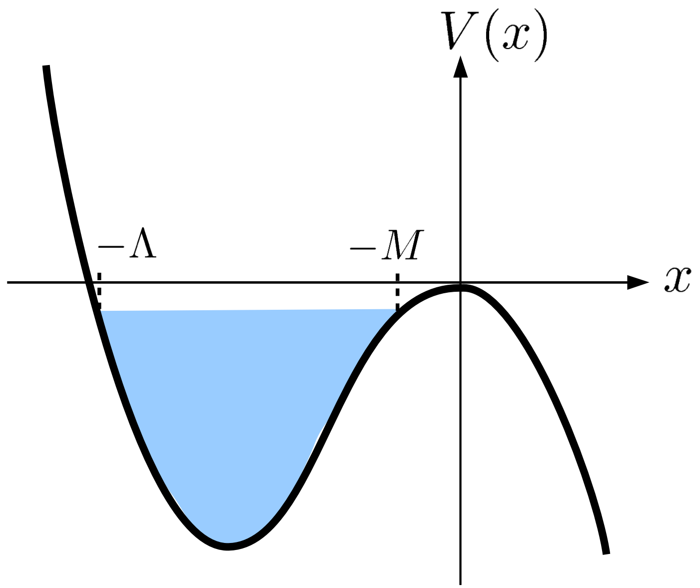
$$F \star G(x) \equiv -\oint_{\mathrm{cut}} \frac{dx'}{2\pi i}\,\frac{F(x) - F(x')}{x - x'} G(-x')$$

Reduces to 1MM when $n \equiv 0$.

Solve this with the assumption of

a single cut $x \in [-\Lambda, -M]$

along negative real axis.



Continuum limit : $\Lambda \to \infty,\;\; M$ fixed.

Solution in the continuum limit

$$
\begin{aligned}
x &\equiv M \cosh \tau \\
W_c(x) &\equiv W(x) - \frac{2V'(x) - nV'(-x)}{4-n^2} \\
&= M^{\frac{p+1}{p}} \cosh \frac{p+1}{p}\tau - tM^{\frac{p-1}{p}} \cosh \frac{p-1}{p}\tau \qquad (t \equiv T_c - T)
\end{aligned}
$$

At $t = 0$ or ∞, it agrees with the result of

(p,q) minimal models coupled to Liouville gravity

$$
S = S_{\mathrm{minimal}} + \int_\Sigma \left(\partial\phi\bar{\partial}\phi + \mu e^{2b\phi} \right) + \oint_{\partial\Sigma} \mu_B e^{b\phi}
$$

$$
\mu_B = \mu^{\frac{1}{2}} \cosh \tau, \quad \left\langle \oint_{\partial\Sigma} e^{b\phi} \right\rangle = \mu^{\frac{q}{2p}} \cosh \frac{q\tau}{p}
$$

Loop equation for 2pt functions

* recursion relation

$$D_{L+1}^{(i)}(x) = W \star D_L^{(i)}(x) \quad (L \geq 1,\ i = 1, 2)$$

* algebraic relation between D_0 and $D_1^{(i)}$

$$A^{(i)}\overline{B^{(i)}} \ = \ C^{(i)} \qquad \left(\overline{f}(x) \equiv f(-x)\right)$$

$$A^{(i)} \ = \ \lambda_{(i)} D_0 - 1$$

$$B^{(i)} \ = \ \mathrm{const} \cdot D_1^{(i)} + (\mathrm{junk})$$

$$C^{(i)} \ = \ \mathrm{const} \cdot (W + n_{(i)}\overline{W}) + (\mathrm{junk})$$

Loop equation for 2pt functions

* recursion relation

$$D^{(i)}_{L+1}(x) = W \star D^{(i)}_L(x) \quad (L \geq 1,\ i = 1, 2)$$

* algebraic relation between D_0 and $D^{(i)}_1$

$$A^{(i)}\overline{B^{(i)}} \ = \ C^{(i)} \qquad \Big(\overline{f}(x) \equiv f(-x)\Big)$$

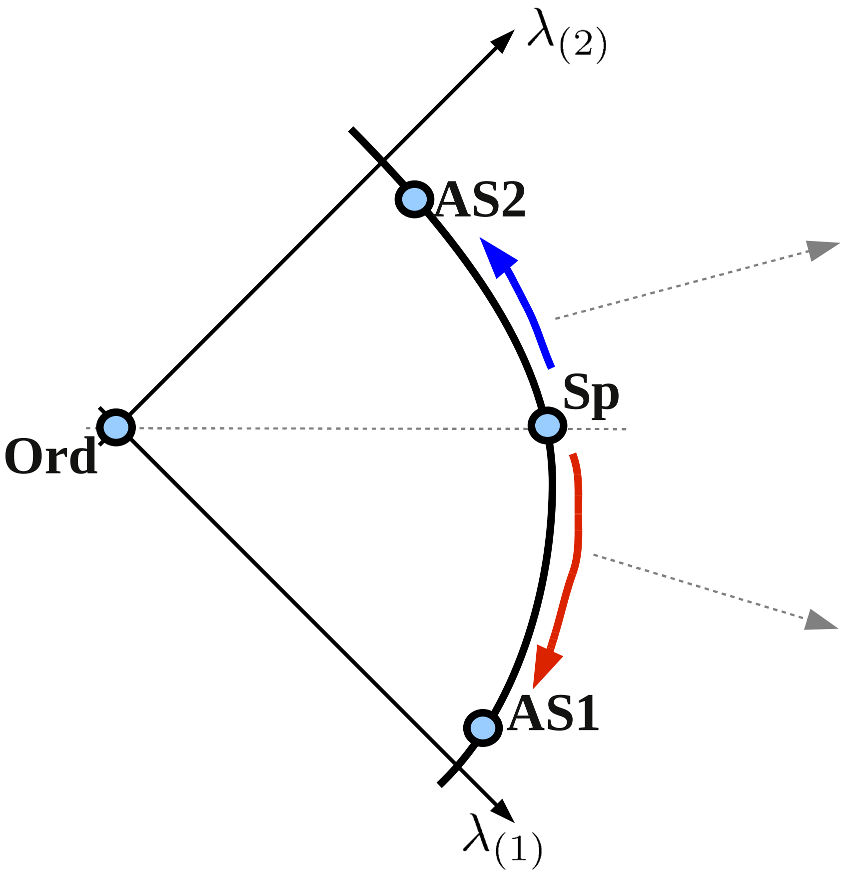
$$\begin{aligned}
A^{(1)} \ &= \ \lambda_{(1)} D_0 - 1 \\
B^{(1)} \ &= \ (\lambda_{(1)} - \lambda_{(2)}) n_{(1)} \{\lambda_{(1)} D^{(1)}_1 + W\} - \lambda_{(2)} (\overline{W} - \overline{V'} + H) - x - y \\
C^{(1)} \ &= \ \lambda_{(1)} \lambda_{(2)} P + (\lambda_{(1)} + \lambda_{(2)}) H - x + y - (\lambda_{(1)} - \lambda_{(2)}) (W + n_{(1)} \overline{W}) - \lambda_{(2)} V'
\end{aligned}$$

$$P \ \equiv \ \frac{1}{\beta} \Big\langle \mathrm{tr} \Big(\frac{V'(x) - V'(X)}{x - X} \mathbf{H}(y) \Big) \Big\rangle$$

Condition for criticality

If $(T, \beta, y, \lambda_{(1)}, \lambda_{(2)})$ are set to the critical values,

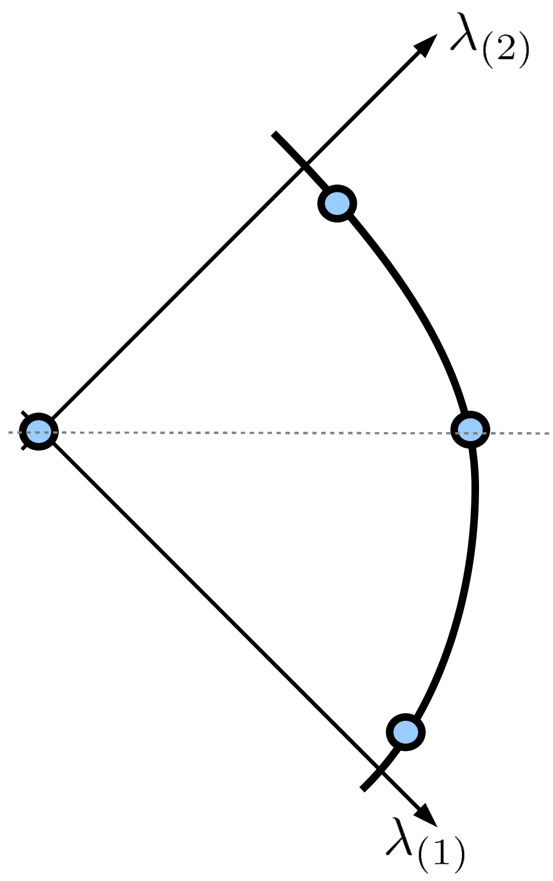
the functions $(A^{(i)}, B^{(i)}, C^{(i)})$ should <u>scale,</u> $\sim \;(x^{\alpha^{(i)}}, x^{\beta^{(i)}}, x^{\frac{p+1}{p}})$



Critical curve along which $(A^{(2)}, B^{(2)}, C^{(2)})$ scale.

Critical curve along which $(A^{(1)}, B^{(1)}, C^{(1)})$ scale.

Shape of the critical curves



At small x, the functions $C^{(1)}, C^{(2)}$ behave

$$C^{(1)} = c_0 + c_1 x + c_2 x^2 - \mathrm{const}(\lambda_{(1)} - \lambda_{(2)}) \cdot x^{\frac{p+1}{p}}$$

$$C^{(2)} = c_0 + c_1 x + c_2 x^2 + \mathrm{const}'(\lambda_{(1)} - \lambda_{(2)}) \cdot x^{\frac{p+1}{p}}$$

Critical curves are given by $c_0 = c_1 = 0.$

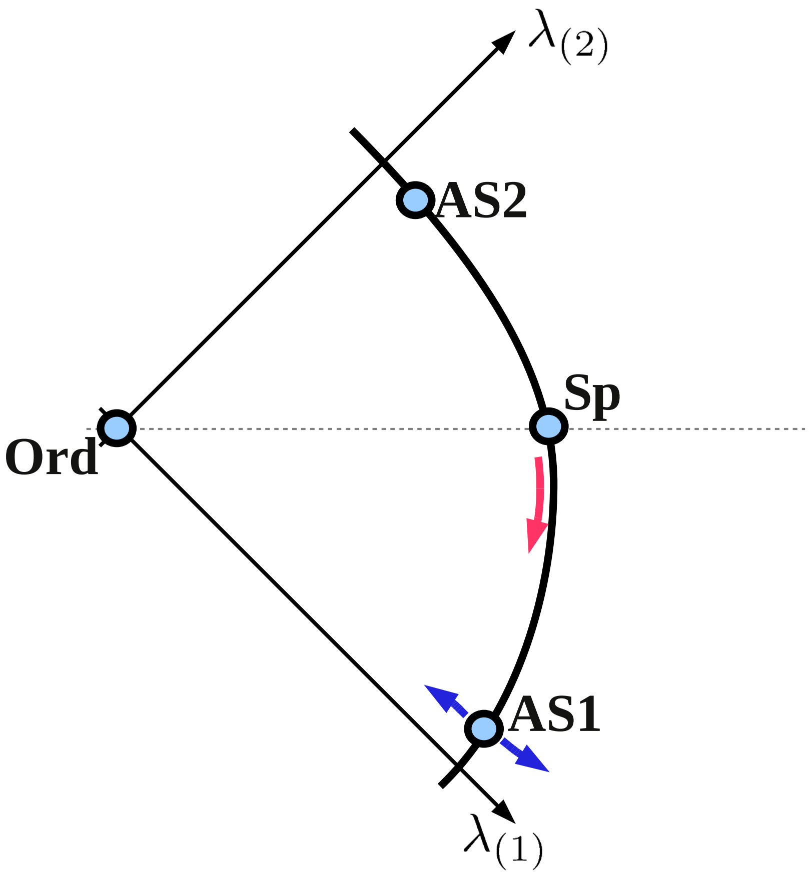
$$\begin{aligned} c_0 &= \lambda_{(1)}\lambda_{(2)}(g_2 H + g_3 H_1) + (\lambda_{(1)} + \lambda_{(2)})(H - \tfrac{1}{2} g_1) \\ &\quad + y - (\lambda_{(1)} - \lambda_{(2)}) \frac{g_1(n_{(1)} - n_{(2)})}{2(2+n)} \end{aligned}$$

$$c_1 = \lambda_{(1)}\lambda_{(2)} g_3 H - 1 - \tfrac{1}{2}(\lambda_{(1)} + \lambda_{(2)}) g_2 + (\lambda_{(1)} - \lambda_{(2)}) \frac{g_2(n_{(1)} - n_{(2)})}{2(2-n)}$$

$$H = \frac{1}{\beta}\left\langle \mathrm{tr}\mathbf{H}(y) \right\rangle, \quad H_1 = \frac{1}{\beta}\left\langle \mathrm{tr}\mathbf{H}(y) X \right\rangle$$

$$V'(x) = g_1 + g_2 x + g_3 x^2$$

Relevant perturbations



* Near **Sp** and along the critical curve to **AS1**,

$$C^{(1)} \sim x^2 + \Delta \cdot x^{\frac{p+1}{p}} + \cdots .$$

* Near **AS1**,

$$C^{(1)} \sim x^{\frac{p+1}{p}} + t_B \cdot x + \cdots .$$

KPZ scaling determines the operators

$$\Delta \Longleftrightarrow \Phi_{(3,3)}, \quad t_B \Longleftrightarrow \Phi_{(1,3)}$$

Spectrum of boundary operators

Let us choose

$$n = 2\cos\frac{\pi}{p}, \quad n_{(1)} = \frac{\sin[\pi(r-1)/p]}{\sin[\pi r/p]}, \quad n_{(2)} = \frac{\sin[\pi(r+1)/p]}{\sin[\pi r/p]},$$

Along the critical curve from **Sp** to **AS1**, the 2pt functions scale as

$$D_L^{(1)}(x) \ \sim \ x^{L+\frac{L-r}{p}} + \cdots ,$$

$$D_L^{(2)}(x) \ \sim \ x^{L+\frac{L+r}{p}} + \cdots .$$

From the KPZ scaling we find

$$S_L^{(1)} \Longleftrightarrow \Phi_{r-L,r}$$

$$S_L^{(2)} \Longleftrightarrow \Phi_{r+L,r}$$

Exact solution for 2pt functions at **AS1**

We compare the solution of the loop equation

$$A^{(1)}(x)B^{(1)}(-x) \;=\; \mu_B - t_B x - W_c(x) - n_{(1)}W_c(-x)$$

with that of Liouville gravity approach

$$S \;=\; \int_\Sigma \left(\mathcal{L}_{\rm minimal} + \partial\phi\bar{\partial}\phi + \mu\mathcal{O}_{(1,1)} + t\mathcal{O}_{(1,3)}\right)$$

$$+\int_{\rm Ord} x\mathcal{O}^B_{(1,1)} + \int_{\text{J-S}} \left(\mu_B\mathcal{O}^B_{(1,1)} + t_B\mathcal{O}^B_{(1,3)}\right)$$

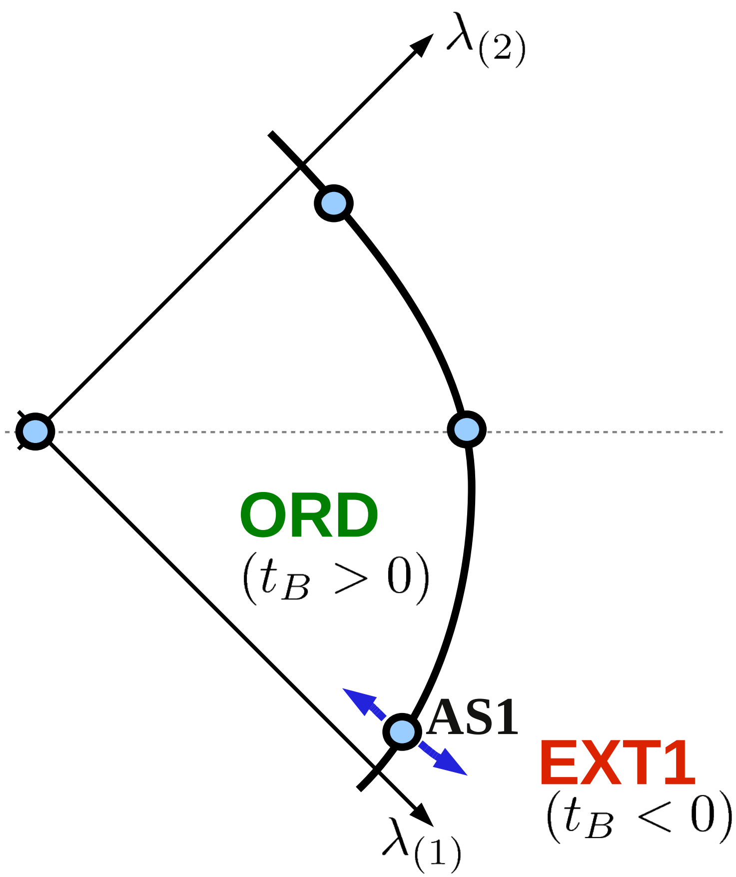
* When $t_B = t = 0,\ \left(x = M\cosh\tau,\ W_c = M^{\frac{p+1}{p}}\cosh\frac{p+1}{p}\tau\right)$

  Loop equation agrees with a recursion relation

  for Liouville Disk 2pt functions

* We also analytically solved the loop equation for $\mu = \mu_B = 0.$

Extraordinary transition



* Loops avoid the boundary in the phase **ORD**.

* Loops avoid the boundary also in the phase **EXT1**, due to a red loop coating the boundary.

The loop equation exhibits the (high-t_B / low-t_B) duality

ORD $^{(r)}$ ⟷ **EXT1** $^{(p+1-r)}$

coated by 1 red loop

Summary

We studied the $O(n)$ model in the dilute phase

with new boundary conditions breaking $O(n)$ to $O(n_{(1)}) \times O(n_{(2)})$

using matrix model.

Loop equations

$\Rightarrow$
- Phase structure
- Spectrum of boundary-changing operators
- Exact disk 2pt amplitudes

Some fundamental quantities remained to be computed,

including

$$H(y) \;=\; \frac{1}{\beta}\left\langle \mathrm{tr}\left(\frac{1}{y - X - \lambda_{(1)}\vec{Y}_{(1)}^{2} - \lambda_{(2)}\vec{Y}_{(2)}^{2}}\right)\right\rangle .$$