

Integrability and Non-planarity

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- Integrability of the spectral problem of planar $\mathcal{N} = 4$ SYM
- Beyond the planar limit
- Non-planar ABJM theory and integrability
- Non-planar ABJ theory, integrability and parity
- $\mathcal{N} = 4$ SYM with gaugegroup $SO(N)$
- Summary and outlook

The spectral problem of planar $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM, gauge group $SU(N) \longleftrightarrow$ IIB strings on $AdS_5 \times S^5$

$$\underbrace{\lambda = g_{\text{YM}}^2 N}_{\text{loop expansion}}, \quad \underbrace{\frac{1}{N}}_{\text{topological exp.}}, \quad \underbrace{\frac{R^2}{\alpha'}}_{\text{spectrum}} = \sqrt{\lambda}, \quad \underbrace{g_s = \frac{\lambda}{N}}_{\text{interactions}}$$

Local gauge invariant operators \longleftrightarrow string states

Conformal dimensions, $\Delta \longleftrightarrow$ energies of string states

The planar spectral problem of $\mathcal{N} = 4$ SYM: **INTEGRABLE**

Determine $\Delta = \Delta(\lambda)$ for $N \rightarrow \infty$

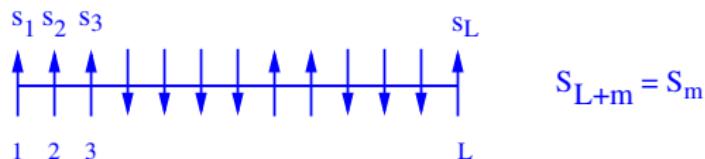
Diagonalize dilatation operator D

Theme of the talk: What happens when we go beyond the planar limit (i.e. N finite)

Integrability of the planar spectral problem

Ex: SU(2) sector, one loop order, $\mathcal{O} = \text{Tr}(ZZZXXXXXZZXXXXZ)$

[Minahan & Zarembo '02]



$$\hat{D} = \frac{\lambda}{2} \sum_{n=1}^L (1 - \bar{\sigma}_n \cdot \bar{\sigma}_{n+1}) = \lambda \sum_{n=1}^L (1 - P_{n,n+1}) \equiv \lambda \sum_{n=1}^L \hat{H}_{n,n+1}$$

Conserved charges: $\exists \hat{Q}_i, \quad i = 1, \dots, L : \quad [\hat{Q}_i, \hat{Q}_j] = 0$

$$\hat{Q}_1 = \sum_n e^{i\hat{P}_n}, \quad \hat{Q}_2 = \hat{D}$$

$$\hat{Q}_3 = \sum_n [\hat{H}_{n,n+1}, \hat{H}_{n+1,n+2}] =$$

A horizontal line with three dots labeled n, n+1, and n+2. Above the line, a blue bracket connects the first two dots, and another blue bracket connects the last two dots. A curved blue arrow above the line connects all three dots.

$$\hat{Q}_m :$$

A horizontal line with six dots. A blue bracket below the line groups the first five dots, and another blue bracket groups the last four dots. A curved blue arrow above the line connects all six dots. The text "m sites" is written below the line.

Beyond one-loop order

Higher orders in λ :

Spin chain with long range interactions

Order λ^n : interactions between $n+1$ nearest neighbours

Still integrable:

\exists conserved charges $Q_i, i = 1, \dots, L$:

at n -loop order: $Q_i = Q_i^0 + \lambda Q_i^1 + \dots + \lambda^n Q_i^n$,

$$[Q_i, Q_j] = \mathcal{O}(\lambda^{n+1}), \quad Q_i^n \text{ of range } (i+n)$$

(Almost) proved to be true at any loop order

Discovery: Observation of otherwise unexplained degeneracies
in the spectrum [Beisert, C.K. & Staudacher '03]

Parity

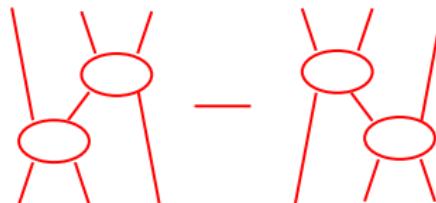
$$\hat{P} \text{Tr}(Z^3 X^2 ZX) = \text{Tr}(XZX^2 Z^3) = \text{Tr}(Z^3 XZX^2), \quad \hat{P}^2 = 1$$

$$[\hat{P}, \hat{H}] = 0, \text{ i.e. eigenstates of } \hat{H} \text{ of definite parity, } P = \pm 1$$

Observation: Pairs of operators with opposite parity but the same energy. Survive loop corrections.

Explanation: The existence of \hat{Q}_3 , i.e. integrability

$$Q_3 = \sum_n [H_{n,n+1}, H_{n+1,n+2}] =$$



$$\{\hat{Q}_3, P\} = 0, \quad [\hat{Q}_3, \hat{H}] = 0$$

The operators in a degenerate pair are connected via \hat{Q}_3 .

Beyond the planar limit

$\mathcal{O} = \text{Tr}(X \dots XZ \dots) \text{Tr}(X \dots XZ \dots) \subset SU(2)$ sector.

[Constable et al '02], [Beisert, C.K., Plefka, Semenoff & Staudacher '02]

$$\begin{aligned}\hat{D} &= -g_{\text{YM}}^2 : \text{Tr}[Z, X][\check{Z}, \check{X}] :, & (\check{Z})_{\alpha\beta} &= \frac{\delta}{\delta Z_{\beta\alpha}} \\ &= \lambda(D_0 + \underbrace{\frac{1}{N} D_+}_{\text{adds a trace}} + \underbrace{\frac{1}{N} D_-}_{\text{removes a trace}})\end{aligned}$$

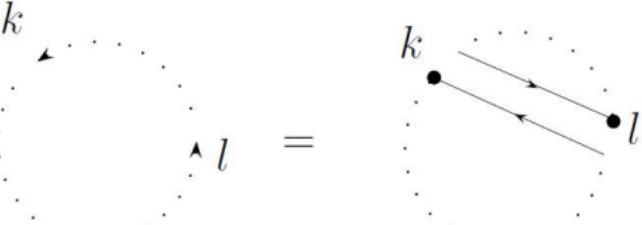
Origin: Quartic interaction between scalars

Example:

$$\begin{aligned}\text{Tr}(ZX\check{Z}\check{X}) \cdot \text{Tr}(XZXZ) \text{Tr}(XZ) &= \text{Tr}(ZX\check{Z}ZXZ) \text{Tr}(XZ) \\ &= N\text{Tr}(ZXXXZ) \text{Tr}(XZ) + \text{Tr}(ZX) \text{Tr}(ZXX) \text{Tr}(XZ) + \text{Tr}(ZXZZZXZ)\end{aligned}$$

The non-planar part of \hat{D}

$$D_+ + D_- = \sum_k \sum_{l \neq k+1} (1 - P_{k,l}) \Sigma_{k+1,l} \equiv \sum_k H_k^{(1)}$$

$$\Sigma_{kl} =$$


$$\Sigma_{kl} =$$


Easy to evaluate

- $D_+ \mathcal{O}, D_- \mathcal{O}$ involves a finite (small) number of operations
- Only diagonalization of finite-dim. matrix

Strategy:

- Consider closed set of operators. Ex: Length 8 with 3 excitations
- Find the planar eigenvalues and eigenstates (can be checked by Bethe eqns.).
- Write down \hat{D} in the basis of planar eigenstates and do perturbation theory in $\frac{1}{N}$.

Lessons learned

- Δ does not always have a well-defined expansion in λ and $\frac{1}{N}$ but D has. (Higher loop effect.)

[Ryzhov '01], [Arutyunov et al. '02] $\left[\begin{array}{c} \text{Bianchi, Kovacs} \\ \text{Rossi, Stanev '02} \end{array} \right]$ $\left[\begin{array}{c} \text{Beisert, C.K.} \\ \text{Staudacher '03} \end{array} \right]$

- Degeneracies between single and double trace states (of equal parity) lead to $\frac{1}{N}$ as opposed to $\frac{1}{N^2}$ corrections.
- Including $\frac{1}{N}$ corrections, degeneracies between parity pairs are lifted, but still $[H, P] = 0$
 \implies absence of Q_3 (and integrability), at least in its previous form $\left[\begin{array}{c} \text{Beisert, C.K.} \\ \text{Staudacher '03} \end{array} \right]$

Quantum mechanical perturbation theory

$$\hat{D} = D_0 + \frac{1}{N}(D_+ + D_-)$$

Assume. $D_0\mathcal{O} = E_0\mathcal{O}$, Expand: $E = E_0 + \frac{1}{N}E_1 + \frac{1}{N^2}E_2$

Non-degenerate perturbation theory

$$E_1 = \langle \mathcal{O}|D_- + D_+|\mathcal{O}\rangle = 0$$

$$E_2 = \sum_{\mathcal{K} \neq \mathcal{O}} \frac{\langle \mathcal{O}|D_- + D_+|\mathcal{K}\rangle \langle \mathcal{K}|D_- + D_+|\mathcal{O}\rangle}{E_{\mathcal{O}} - E_{\mathcal{K}}}$$

Degenerate perturbation theory:

Diagonalize the perturbation in the subset of degenerate states
⇒ Energy corrections of order $\frac{1}{N}$ as opposed to $\frac{1}{N^2}$.

Conserved charges beyond the planar limit ?

$$D = D^0 + \frac{1}{N} D^1, \quad Q = Q^0 + \frac{1}{N} Q^1$$

Determine Q^1 such that

$$0 = [D^0, Q^1] + [D^1, Q^0]$$

Q^1 must involve splitting and joining.

A natural guess: $Q^1 = \sum_{n=1}^L [D_n^0, D_{n+1}^1] + [D_n^1, D_{n+1}^0]$
where

$$D_n^0 = 1 - P_{n,n+1}, \quad D_n^1 = \underbrace{\sum_{l \neq n+1} (1 - P_{n,l}) \Sigma_{n,l}}_{\text{extremely non-local}},$$

Does not work

ABJM theory

ABJM theory: 3D $\mathcal{N} = 6$ $U(N)_k \times \overline{U(N)}_{-k}$ superconformal CSM

[Aharony, Bergman, Jafferis & Maldacena '08]

$$\begin{aligned} S = & \int \left\{ \frac{k}{4\pi} \epsilon^{mnp} \text{Tr} \left(A_m \partial_n A_p + \frac{2i}{3} A_m A_n A_p \right) - \frac{k}{4\pi} \epsilon^{mnp} \overline{\text{Tr}} \left(\overline{A}_m \partial_n \overline{A}_p + \frac{2i}{3} \overline{A}_m \overline{A}_n \overline{A}_p \right) \right. \\ & + \frac{1}{2} \overline{\text{Tr}} \left((-D_m Y_I)^\dagger (D^m Y^I) + i \Psi_I^\dagger \not{D} \Psi_I \right) + \frac{1}{2} \text{Tr} \left(-(D_m Y^I) (D^m Y_I^\dagger) + i \Psi_I \not{D} \Psi^{\dagger I} \right) \\ & \left. - \underbrace{V_{fermion}}_{YY\Psi\Psi} - \underbrace{V_{boson}}_{sextic} \right\} \end{aligned}$$

Y^I complex scalars, $I = 1, 2, 3, 4, \in (N, \bar{N})$,

$$A_m : \text{Adj}(U(N)), \quad \overline{A}_m : \text{Adj}(\overline{U(N)})$$

't Hooft expansion: $\lambda = \underbrace{\frac{N}{k}}$,
loop expansion

$\frac{1}{N}$
topological exp.

The AdS_4/CFT_3 correspondence

For $N, k \rightarrow \infty$, $N^{1/5} \ll k \ll N$, ABJM theory dual to

Type IIA string theory on $AdS_4 \times CP^3$

(N units of F_4 flux on AdS_4 , k units of F_2 flux on a $CP^1 \subset CP^3$.)

$$\frac{R^2}{\alpha'} = \sqrt{\lambda},$$

$$g_s = \underbrace{\frac{\lambda^{5/4}}{N}}$$

Invisible in perturbative ABJM theory

Our aim: Determine \hat{D} to order λ^2 and all orders in $\frac{1}{N}$.

Why:

- Compare the structure of $\mathcal{N} = 4$ SYM and ABJM theory beyond the planar limit
- Investigate the integrability properties for finite N
- Look for surprises

Integrability of the planar spectrum of ABJM theory

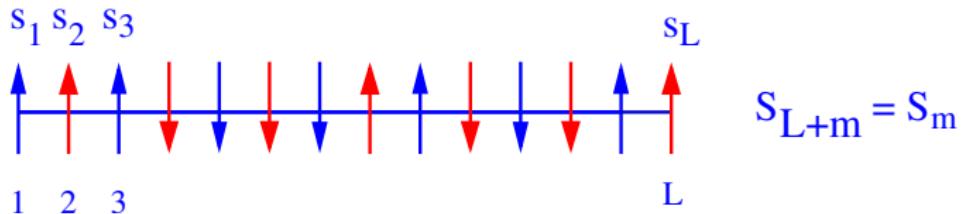
[Minahan & Zarembo '08]

For simplicity restrict to $SU(2) \times SU(2)$ sector:

$$\mathcal{O} = \text{Tr}(Z^A W_B Z^C \dots) \text{Tr}(Z^{A'} W_{B'} Z^{C'} \dots), \quad A, B \in \{1, 2\}$$

$$D_{planar} = \lambda^2 \sum_{I=1}^{2L} (1 - P_{I,I+2})$$

Two Heisenberg magnets, coupled via $\sum_i p_i = 0$ (cyclicity of Tr)



ABJM theory beyond the planar limit

[C.K., Orselli & Zoubos '08]

$$\hat{D} =: V_F^{bos} : = - : \left(\frac{4\pi}{k} \right)^2 \text{Tr} \left[\check{W}_A \check{Z}^B \check{W}_C W_A Z^B W_C - \check{W}_A \check{Z}^B \check{W}_C W_C Z^B W_A + \check{Z}^A \check{W}_B \check{Z}^C Z^A W_B Z^C - \check{Z}^A \check{W}_B \check{Z}^C Z^C W_B Z^A \right] :, \quad A, B, C \in \{1, 2\}$$

Origin: Sextic terms in the potential

Structure of \hat{D} :

$$\hat{D} = \lambda^2 (D_0 + \frac{1}{N} (D_+ + D_-) + \underbrace{\frac{1}{N^2} (D_{++} + D_{--} + D_{00})}_{\text{New type of terms}})$$

New type of correction (in non-degenerate perturbation theory)

$$\delta E_2 = \langle \mathcal{O} | D_{00} | \mathcal{O} \rangle$$

Ex: Length 8 with 2 different excitations

$$\left(\begin{array}{cc|cc} 8 & \frac{8}{N^2} & \frac{16}{N} & \frac{4}{N} & -\frac{8}{N^2} & 0 & 0 \\ \frac{8}{N^2} & 4 - \frac{12}{N^2} & 0 & -\frac{2}{N} & -\frac{4}{N^2} & 0 & 0 \\ \frac{16}{N} & -\frac{8}{N} & 8 & 0 & 0 & 0 & 0 \\ 0 & -\frac{16}{N} & -\frac{8}{N^2} & 6 - \frac{8}{N^2} & -\frac{12}{N} & 0 & 0 \\ 0 & \frac{8}{N^2} & 0 & -\frac{12}{N} & 8 - \frac{8}{N^2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 + \frac{4}{N^2} & \frac{2}{N} \\ 0 & 0 & 0 & 0 & 0 & \frac{8}{N} & 6 + \frac{8}{N^2} \end{array} \right).$$

Notice: Parity is still a good quantum number: $[\hat{D}, \hat{P}] = 0$

Planar parity pairs: $(\mathcal{O}_2, \mathcal{O}_6)$ and $(\mathcal{O}_4, \mathcal{O}_7)$.

Energy corrections

$$\delta E_{6,7} = \frac{6}{N^2} \mp \left(\sqrt{1 + \frac{20}{N^2} + \frac{4}{N^4}} - 1 \right), \quad \delta E_{1,3} = \mp \frac{16}{N}.$$

$$\delta E_2 = -\frac{28}{N^2}, \quad \delta E_4 = -\frac{64}{N^2}, \quad \delta E_5 = \frac{64}{N^2},$$

Notice: All degeneracies are lifted \implies absence of Q_3 (in its previous form)

ABJ-theory, Integrability and Parity

ABJ theory: $3D$, $\mathcal{N} = 6$ $U(N)_k \times \overline{U(M)}_{-k}$ superconformal CSM

[Aharony, Bergman & Jafferis '08]

Difference from before: $Y^I, \Psi_I \in (N, \overline{M})$ bi-fundamental
 $A_m \in \text{Adj}(U(N)), \overline{A}_m \in \text{Adj}(\overline{U(M)})$

't Hooft expansion: $\lambda = \frac{N}{k}, \overline{\lambda} = \frac{M}{k}, \frac{1}{N}, \frac{1}{M}$.

Dual string theory for $N^{1/5} \ll k \ll N, |M - N| < k$:

Type IIA string theory on $AdS_4 \times CP^3$ with fluxes +
NSNS 2-form B_2 with non-trivial holonomy

$$b_2 = \frac{1}{2\pi} \int_{CP^1 \subset CP^3} B_2 = \frac{|M - N|}{k}$$

Breaks world sheet parity

ABJ theory and Integrability

Planar dilatation operator derived at two loops

[Bak & Rey '08], [Minahan, Schulgin & Zarembo '09]

Only difference: $\lambda^2 \rightarrow \lambda\bar{\lambda}$: Still integrable
 No parity breaking effects

Question: What happens beyond the planar limit?

\hat{D} can be derived using effective vertices

[Caputa, C.K. & Zoubos '09]

$$\begin{aligned}\hat{D} &= :V_{Boson}^F: \\ &= \lambda\bar{\lambda}(D_0 + \frac{1}{\mathcal{M}}(D_+ + D_-) + \frac{1}{\mathcal{M}^2}(D_{++} + D_{--} + D_{00})),\end{aligned}$$

where $\frac{1}{\mathcal{M}} = \frac{1}{N}$ or $\frac{1}{M}$ and $\frac{1}{\mathcal{M}^2} = \frac{1}{M^2}$ or $\frac{1}{N^2}$ or $\frac{1}{MN}$.

Ex: Length 8 with 2 excitations

$$\left(\begin{array}{cccccc|cc} 8 & \frac{8}{MN} & \frac{8}{N} + \frac{8}{M} & \frac{2}{N} + \frac{2}{M} & -\frac{8}{MN} & 0 & \frac{2}{M} - \frac{2}{N} \\ \frac{8}{MN} & 4 - \frac{12}{MN} & 0 & -\frac{1}{N} - \frac{1}{M} & -\frac{8}{MN} & 0 & \frac{1}{N} - \frac{1}{M} \\ \frac{8}{N} + \frac{8}{M} & -\frac{4}{N} - \frac{4}{M} & 8 & 0 & 0 & \frac{4}{M} - \frac{4}{N} & 0 \\ 0 & -\frac{8}{N} - \frac{8}{M} & -\frac{8}{MN} & 6 - \frac{8}{MN} & -\frac{6}{N} - \frac{6}{M} & \frac{4}{M} - \frac{4}{N} & 0 \\ 0 & \frac{8}{MN} & 0 & -\frac{6}{N} - \frac{6}{M} & 8 - \frac{8}{MN} & 0 & \frac{6}{N} - \frac{6}{M} \\ \hline 0 & 0 & 0 & \frac{1}{M} - \frac{1}{N} & 0 & 4 + \frac{4}{MN} & \frac{1}{N} + \frac{1}{M} \\ 0 & 0 & 0 & 0 & \frac{2}{N} - \frac{2}{M} & \frac{4}{N} + \frac{4}{M} & 6 + \frac{8}{MN} \end{array} \right).$$

Observations

- Parity no longer conserved
- Energy corrections symmetric in M and N
- All degeneracies lifted for all values of M and N

Other gauge groups

$\mathcal{N} = 4$ SYM, gauge group $SO(N) \longleftrightarrow$ IIB strings on $AdS_5 \times RP^5$
[Witten '98]

$RP^5 = S^5/Z_2$, $(z^i \equiv -z^i)$, orientifold

Planar spectral problem \subset planar spectral problem for $SU(N)$

Parity is gauged:

$$X^T = -X \implies \hat{P}\text{Tr}(X_{i_1} \dots X_{i_L}) = (-1)^L \text{Tr}(X_{i_1} \dots X_{i_L})$$

New $\frac{1}{N}$ -effects not involving splitting and joining

Feynman diags w/ cross-caps \longleftrightarrow non-orientable world sheets

$\frac{1}{N}$ effects for gauge group $SO(N)$

Restrict to $SU(2)$ sector: $\mathcal{O} = \text{Tr}(X \dots XZ \dots) \text{Tr}(X \dots XZ \dots)$

$$\begin{aligned}\hat{D} &= -g_{YM}^2 \text{Tr}[Z, X][\check{Z}, \check{X}], \quad (\check{Z})_{\alpha\beta} Z_{\gamma\epsilon} = \frac{1}{\sqrt{2}}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\epsilon}) \\ &= \lambda(D_0 + \frac{1}{N}D_+ + \frac{1}{N}\tilde{D}_- + \underbrace{\frac{1}{N}D_{flip}}_{\text{Acts inside a trace}})\end{aligned}$$

$$\overbrace{D_{flip} \cdot \text{Tr}(XWZY)} = \text{Tr}(XZW^TY) + \text{Tr}(XZYW^T) - \text{Tr}(XW^TYZ) - \text{Tr}(XYW^TZ)$$

Energy corrections generically of order $\frac{1}{N}$: $E_1 = \langle \mathcal{O} | D_{flip} | \mathcal{O} \rangle$

Search for integrability with gauge group $SO(N)$

- No degenerate parity pairs (parity is gauged).
- Degeneracy between single and multiple trace states lifted by $\frac{1}{N}$ -corrections.
- Considering only the perturbation D_{flip}
(restrict to single trace states, not degenerate with multi-trace states)

- Try to construct conserved charges $Q = Q^0 + \frac{1}{N} Q^1$

$$0 = [D_0, Q^1] + [D_{flip}, Q^0], \quad \text{does not work}$$

- Try to look for perturbed Bethe equations

Considering only D_{flip}

Two excitation states: $O_p^J = \text{Tr}(XZ^p X Z^{J-p})$, J even

Planar eigenstates: $D_0|n^J\rangle = E_n^0|n^J\rangle$

$$|n^J\rangle = \frac{1}{J+1} \sum_{p=0}^J \cos\left(\frac{\pi n(2p+1)}{J+1}\right) O_p^J, \quad 0 \leq n \leq \frac{J}{2}$$

$$E_n^0 = 8 \sin^2\left(\frac{\pi n}{J+1}\right)$$

Non-planar correction: $E_n = E_n^0 + \frac{1}{N} E_n^{flip}$ (prediction for strings)

$$E_n^{flip} = \langle n^J | D_{flip} | n^J \rangle$$

$$= \underbrace{2 \sin^2\left(\frac{\pi n}{J+1}\right)}$$

correction of disp. rel.?

$$-\frac{1}{J+1} \left\{ 4 \tan^2\left(\frac{\pi n}{J+1}\right) - \tan^2\left(\frac{2\pi n}{J+1}\right) - \cos\left(\frac{2\pi n}{J+1}\right) \right\}$$

correction of momenta?

E_n^{flip} from a perturbed Bethe ansatz?

Bethe eqn. for length L and M excitations

$$e^{ip_k L} = \prod_{m \neq k}^M \frac{u_k - u_m + \frac{i}{2}}{u_k - u_m - \frac{i}{2}}, \quad \text{where} \quad e^{ip} = \frac{x(u + \frac{i}{2})}{x(u - \frac{i}{2})}$$

Dispersion relation: $E = 16 \sin^2\left(\frac{p}{2}\right) + \delta E(p)$

Parametrizing $x(u) = u(1 - \frac{1}{N}f(u))$ we find

$$f(u + \frac{i}{2}) - f(u - \frac{i}{2}) = \frac{u^2 + \frac{1}{4}}{2u^3} - u - \frac{(u^2 + \frac{1}{4})^2}{u(u^2 - \frac{1}{4})}$$

No solution

- No sign of integrability beyond the planar limit (yet?)
- Need to rethink the concept of integrability when going beyond the planar limit